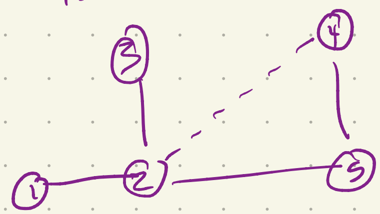
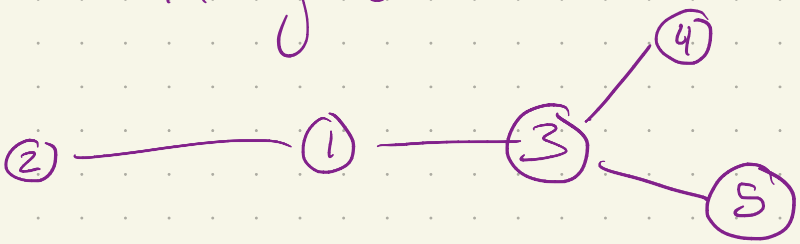
Elimination on A tree

Def A tree $G=(V,E)$ is a graph which is connected & has $|V| = |E|+1$



First eliminating 5 is unnecessarily costly

Idea: Alway eliminate leaves



Elim ordering

$(4, \underline{5}, \underline{3}, 2, 1)$

$m_{4\to 3}(x_3)$ "Message"

$$P(x_1) \propto \sum_{x_2, x_3, x_5} f_{12} f_{13} f_{35} \sum_{x_4} f_{34}$$

$m_{5\to 3}(x_3)$

$$\propto \sum_{x_2 x_3} f_{12} f_{13}\, m_{4\to 3}(x_3) \Big) \sum_{x_5} f_{35}(x_3, x_5) \Big)$$

$$\propto \sum_{x_2} f_{12} \sum_{x_3} f_{13}\, m_{4\to 3}\, m_{5\to 3} \qquad m_{3\to 1}$$

$$\propto m_{3\to 1} \sum_{x_2} f_{12} \qquad \propto m_{3\to 1}\, m_{2\to 1} \qquad m_{2\to 1}$$

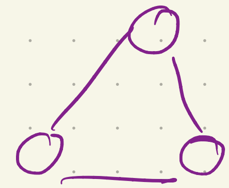
$\to m_1(x_1)$

$$\boxed{P(X_1 = x_1) = \frac{m_1(x_1)}{\sum_{x' \in \mathcal{X}} m_1(x')}}$$

Message passing on an (undirected graph) $G=(V,E)$

- Only apply this to pairwise MRFs

$$P(x) = \frac{1}{Z}\prod_{(i,j)\in E} f_{ij}(x_i, x_j)$$



- Exact for trees, Apx for other pairwise MRFs (HWR)

Sum-Prod Alg

Input : $G=(V,E)$, $f_{ij}(x_i, x_j)$, $\mathcal{X}$, T = # iterations

Out : $\{\hat{P}_i(x_i)\}_{i=1}^{n}$

Initialize $m_{i\to j}(x_j)$, $m_{j\to i}(x_i)$ to Random vectors $\in \mathbb{R}^{|\mathcal{X}|}$ $\forall (i,j)\in E$.

- For $t = 1, \dots, T$

$\forall (i,j)\in E$ & $(j,i)\in E$

Update $m_{i\to j}(x_j) = \sum_{x_i} f_{ij}(x_i, x_j) \prod_{\ell\in\partial i\setminus\{j\}} m_{\ell\to i}(x_i)$

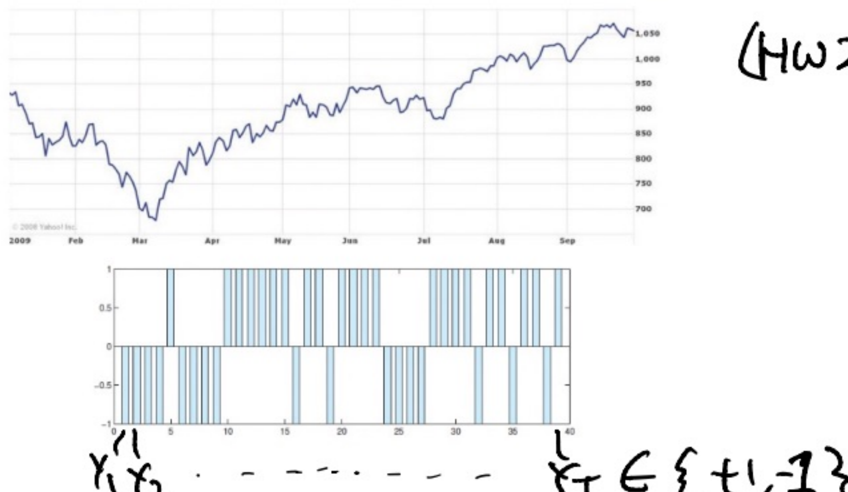
$\forall i \in V$

$m_i(x_i) = \prod_{\ell\in\partial i} m_{\ell\to i}(x_i)$

$$\hat{P}_i(x_i) = \frac{m_i(x_i)}{\sum_{x'} m_i(x')}$$

When G is a tree, $T \geq \mathrm{diam}(G)$, $\hat{P}(x_i) = P(x_i)$.
= length of longest path.

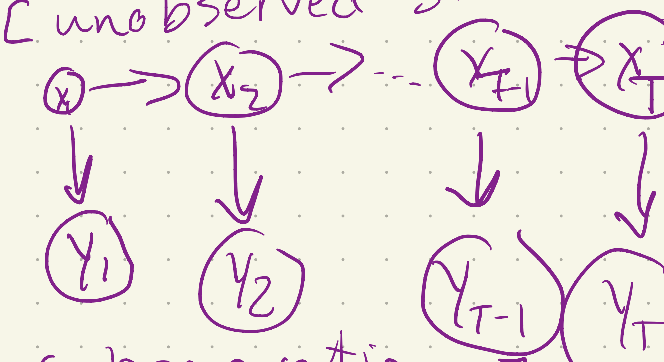
HMM



(HW2)

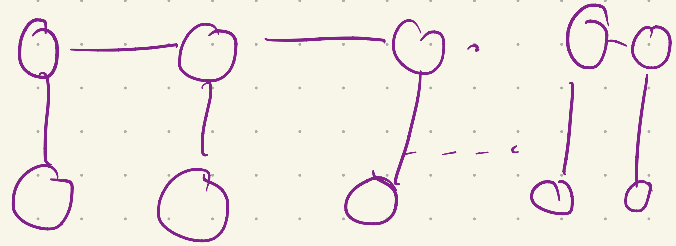
- S&P 500 index over a period of time
- For each week, measure the price movement relative to the previous week: $+1$ indicates up and -1 indicates down
- a hidden Markov model in which x_t denotes the economic state (good or bad) of week t and y_t denotes the price movement (up or down)
- $x_{t+1} = x_t$ with probability 0.8
- $\mathbb{P}_{Y_t|X_t}(y_t = +1 | x_t = \text{'good'}) = \mathbb{P}_{Y_t|X_t}(y_t = -1 | x_t = \text{'bad'}) = q$

[unobserved state]



[observations]

Moralize



$$P(x) \propto \prod_{i \in T-1} f_{i,i+1}(x_i, x_{i+1}) =$$

Factor graphs (A new class of graphical models

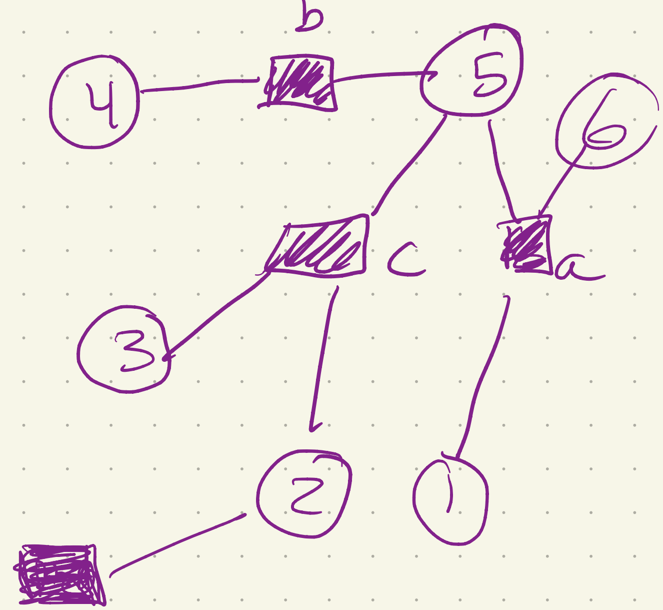
Def A factor graph $G = (V \cup F, E)$

V: variable nodes

F: factor nodes

A probability dist. factorizes accd to a factor graph G if

$$P(x) = \frac{1}{Z} \prod_{a \in F} f_a(x_{j,a})$$



- Factor graphs are a generalization of MRFs
- FG have global Markovian structure

A - B - C sep in $G = (V \cup F, E)$

$$\Rightarrow X_A \perp\!\!\!\perp X_C \mid X_B$$

Belief Propagation

$E = \{(i,a) \mid i \in V,\ a \in F\}$

Input: $G = (V \cup F, E),\ T$

Output $\{\hat{P}(x_i)\}_{i \in V}$

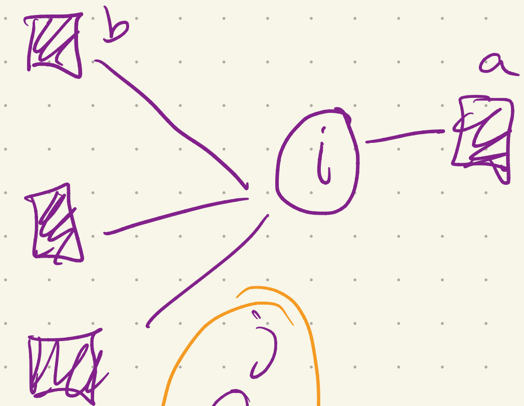
Initialize $\{(m_{i\to a},\ m_{a\to i})\}_{(a,i)\in E}$ → to random

$[\ ] \qquad [\ ] \in \mathbb{R}^{|X|}$

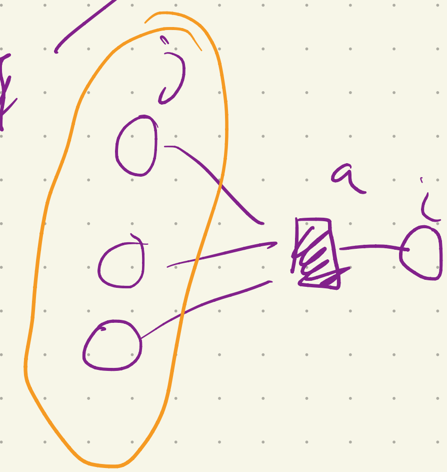
Repeat for $t = 1 \ldots T$:

Update $m_{i\to a}$:

$$m_{i\to a}(x_i) = \prod_{b \in \delta i \setminus \{a\}} \tilde{m}_{b\to i}(x_i)$$



$\tilde{m}_{a\to i}$:



$$\underbrace{\overline{\tilde{m}_{a\to i}(x_i)}}_{\text{"}P(x_i)\text{"}} = \underbrace{\sum_{j \in \delta a \setminus \{i\}} f_a(x_{\delta a})}_{P(x_i \mid x_{j_1}, x_{j_2}, \ldots x_k)} \underbrace{\prod_{j' \in \delta a \setminus \{i\}} m_{j'\to a}(x_{j'})}_{P(x_{j_1}),\ P(x_{j_2}) \ldots}$$

→ Priors