Inference! On GMs → Efficient Algorithm

Inference problems Given a MRF $G=(V,E)$, $P(x)=\prod_{c\in\mathcal{C}} f_c(x_c)$

(1) Calculate $P(x_1)$ marginals

(2) Calculate $\underset{x}{\operatorname{argmax}}\, P(x)$

Or $\underset{x_2}{\operatorname{argmax}}\, P(x_2 \mid x_3)$

(3) Calculate Z

(normalization term / partition fn)

(4) Sample from $P(X)$

Brute Force – baby's first "algorithm" for marginals

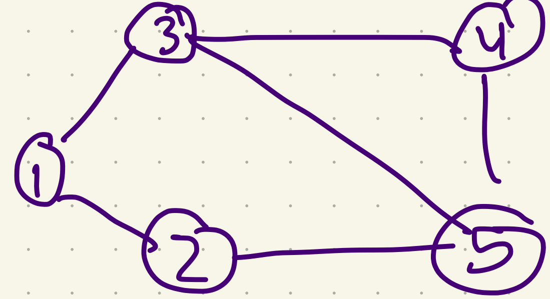
(1) Enumerate all values x_1 could take $(|X|)$

(2) Sum over all other $n-1$ variables $(|X|^{n-1})$

Complexity: $O(|X|^n)$.

$$P(X_1 = x_1) = \sum_{x_2 \ldots x_n} P(x_1, x_2, \ldots, x_n)$$

Elimination Algorithm



$$P(X) = \frac{1}{Z} f_{12}(x_1 x_2) f_{13}(x_1 x_3) f_{25}(x_2 x_5) f_{345}(x_3, x_4, x_5)$$

Goal: Compute $P(x_1)$ (Brute force $O(|X|^5)$)

$$P(x_1) \propto \sum_{x_2, x_3 x_4, x_5} f_{12}(x_1 x_2) f_{13}(x_1 x_3) f_{25}(x_2 x_5) f_{345}(x_3, x_4, x_5)$$

Eliminate variables in order $(5, 4, 3, 2)$

(5)

$$\propto \sum_{x_2, x_3, x_4} f_{12}(x_1 x_2) f_{13}(x_{13}) \underbrace{\boxed{\sum_{x_5} f_{25}(x_2 x_5) f_{345}(x_3 x_4 x_5)}}_{m_5(x_2, x_3, x_4)}$$

num x_4 → $|X| \cdot |X|^3$ → summing x_2, x_3, x_4

$$\propto \sum_{x_2 x_3} f_{12}(x_1, x_2) f_{13}(x_1, x_3) \underbrace{\boxed{\sum_{x_4} m_5(x_2 x_3 x_4)}}_{m_4(x_2\ x_3)}$$

e

$|X| \cdot |X|^2$

$$\alpha \sum_{x_2} f_{12}(x_1, x_2) \underbrace{\sum_{x_3} f_{13}(x_1, x_3) \cdot m_4(x_2, x_3)}_{m_3(x_1,\, x_2)}$$

$$|\mathcal{X}| \cdot |\mathcal{X}|^2$$

$$\alpha \underbrace{\sum_{x_2} f_{12}(x_{12})\, m_3(x_1, x_2)}_{m_2(x_1)}$$

$$|\mathcal{X}| \cdot |\mathcal{X}|$$

Total CC = $O(|\mathcal{X}|^4)$

To complute $P(X_1 = x_1) \ \forall x_1 \in \mathcal{X}$

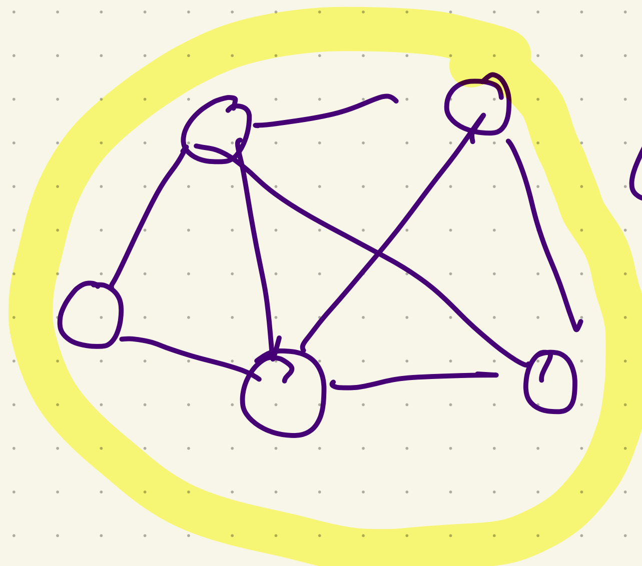
<u>Def</u> A reconstituted graph $G' = (V, \tilde{E})$ of $G = (V, E)$ and a variable elimination ordering $P = (P_1, \dots, P_n) = \sigma_1 \dots \sigma_n$

Remove P_1 and let $E_1 = \{ (i, j)$ where i & j were neighbors of σ_1 in $G \}$

$\vdots$

P_n

$\tilde{E} = E \cup E_1 \cup \dots E_n$



$G = (V, E_{n-1})$

$E \cup E_1$

Has a clique of size 4 $\Rightarrow$ $|X|^4$

<u>Def</u> The <u>tree-width</u> of a graph

$\min_{\text{all ordering}} \{$ max size of a clique in $G' = (V, \tilde{E})$ the reconstituted graph $\}$

Elimination Alg for MAP $\underset{x}{\operatorname{argmax}}\ P(x) = x^*$

Q: Can we use our marginal alg?

$\underset{x_1}{\operatorname{argmax}}\ P_1(x_1) \stackrel{?}{=} x_1^*$

(No! but sort of ☺)

marginal dist over x_1

$$\underset{x_1}{\operatorname{argmax}} \sum_{x_2, \dots, x_n} P(x_1, \dots, x_n) \neq \underset{x_1}{\operatorname{argmax}} \max_{x_2, \dots, x_n} P(x_1, \dots, x_n)$$

Ordering $(5, 4, 3, 2, 1)$

$$\underset{x_1 \dots x_5}{\operatorname{argmax}}\ P(x_1, \dots, x_5) \propto \underset{x_1 \dots x_5}{\operatorname{argmax}}\ f_{12} f_{13} f_{25} f_{345}$$

$$\underset{x_1 \dots x_4}{\operatorname{argmax}}\ f_{12} f_{13} \boxed{\max_{x_5} f_{25} f_{345}}$$

$m_5^*(x_2, x_3, x_4)$

$\boxed{x_5^*(x_2, x_3, x_4)}$

$$\propto \operatorname*{argmax}_{x_1 x_2 x_3} f_{12} f_{13} \underbrace{\max_{x_4} m_5^*(x_2, x_3, x_4)}_{x_4^*(x_2,x_3)} \quad m_4^*(x_2,x_3)$$

$$\propto \operatorname*{argmax}_{x_1, x_2} f_{12} \underbrace{\max_{x_3} \boxed{f_{13}\, m_4^*(x_2, x_3)}}_{x_3^*(x_1,x_2)} \quad m_3^*(x_1,x_2)$$

$$\propto \operatorname*{argmax}_{x_1} \underbrace{\max_{x_2} \boxed{f_{12}\, m_3^*(x_1, x_2)}}_{x_2^*(x_1)} \quad m_2^*(x_1)$$

$$m_2 = \begin{bmatrix} \\ \\ \end{bmatrix} \in \{0,1\}^{|X|} \to i$$

"Probability of the most probable $x_2 \ldots x_n \mid X_1 = x_i$"

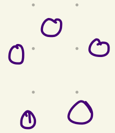
$$P(X_1 \ldots x_n) = P(x_2 \ldots x_n \mid x_1) \cdot P(x_1)$$

CC : again $O(|X|^4)$

Our first ~~for~~ formal measure of graph complexity which directly captures CC of inference

1

small
simple
'not dense'



$E=\emptyset$

$Tw = 1$

$(n|X|)$

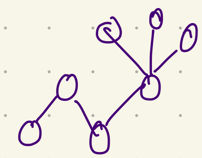
r-deg 0

Chain

2

$O(|X|^2)$

2

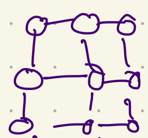


tree

2

$O(|X|^2)$

$\boxed{n-1}$



2-d grids

$\sqrt{n}$

$O(|X|^{\sqrt{n}})$

4

3-d grid

$n^{2/3}$

$O(|X|^{n^{2/3}})$

6

Erdös-
Reyni
Graph
(random)

$p = \frac{c}{n}$

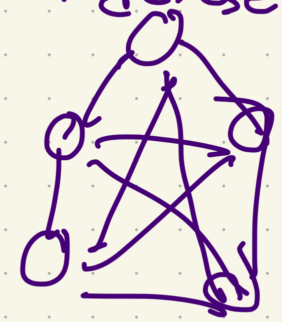
$Tw = n$

$O(|X|^n)$

max degree

c

large
complex
"dense"



1

0

max