Undirected GM / Markov Random Fields

A distribution $P(X)$ factorizes wrt $G=(V,E)$

if $P(X) \propto \prod_{c\in\mathcal{C}} f_c(X_c)$

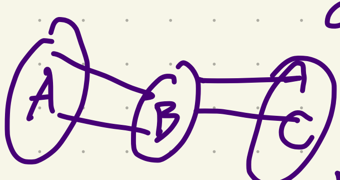
$\propto$ is proportional to

$\mathcal{C}$ → set of all cliques or, equivalently, all maximal cliques

A normalization term can convert this into a probabity distribution.

Graph Separation: We say A+C are separated by B if $\forall$ A − B − C (

any path from any $a\in A$ to any $c \in C$ must pass through $b \in B$



[Global Markov Property (G)]

A distribution $P(X)$ is consistent w G if $\forall$ A − B − C sets $A, B, C \subseteq V$

$$X_A \perp\!\!\!\perp X_C \mid X_B$$

Local markov Property (L)

$P(X)$ satisfies (L) wrt. $G=(V,E)$ if

$$X_i \perp\!\!\!\perp X_{V \setminus \{i\} \cup \delta_i} \mid X_{\delta_i}, \quad \delta_i = \{j \mid (i,j) \in E\}$$

$\forall i \in V$

Pairwise Property (P)

$P(X)$ satisfies (P) wrt graph G if

$$\forall (i,j) \notin E \quad X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}}$$

Claim $(G) \underset{(1)}{\Rightarrow} (L) \underset{(2)}{\Rightarrow} (P)$

Why $(G) \Rightarrow (L)$

(G) for $P(X)$, Graph G: $\forall$ sets A, B, C s.t. $A - B - C$, we ↗ assume have that $X_A \perp\!\!\!\perp X_C \mid X_B$

$? \Rightarrow (L)$: $X_i \perp\!\!\!\perp X_{V \setminus \{i\} \cup \delta_i} \mid X_{\delta_i}$ want to show $\forall i$

Proof why? $A = \{i\}$ $B = S_i$ $C = V \setminus (\{i\} \cup S_i)$

$A - B - C$ (b/c def of the sets)

$X_A \perp\!\!\!\perp X_C \mid X_B$

$X_i \perp\!\!\!\perp X_{V \setminus \{i\} \cup S_i} \mid X_{S_i}$ (L)! $\square$

Claim (L) $\Rightarrow$ (P)

$P(X)$ satisfies (P) if $\forall (i,j) \notin E$ $X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}}$

Lemma [Data Processing inequalities]

Suppose we have a graph s.t. $X - Z - Y - h(Y)$

Then we know $X \perp\!\!\!\perp Y \mid Z$. and

(DP1) • $X \perp\!\!\!\perp h(Y) \mid Z$ $\forall$ functions $h(\cdot)$

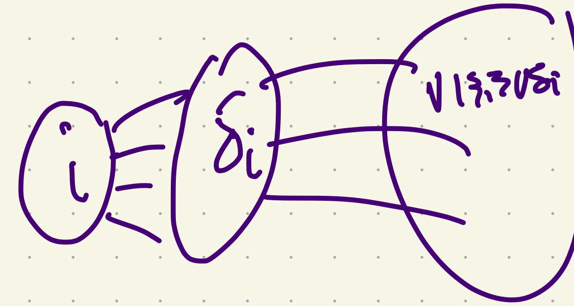
(Postprocessing doesn't break independences]

(DP2) • $X \perp\!\!\!\perp Y \mid Z, h(Y)$

Take arbitrary $(i,j) \notin E$. Want to show

$$X_{\{i\}} \perp\!\!\!\perp X_{\{j\}} \mid X_{V \setminus \{i,j\}}$$

we know: $X_i \perp\!\!\!\perp X_{V \setminus (\{i\} \cup S_i)} \mid X_{S_i}$ $\longrightarrow$



(that $(i,j) \notin E \Rightarrow j \notin S_i$)

DP2 $\Rightarrow$ $X_i \perp\!\!\!\perp X_{V \setminus (\{i\} \cup S_i)} \mid X_{S_i} \cup X_{V \setminus \{i,j\}}$

DP1 $\Rightarrow$ $X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}}$ $\qquad X_{V \setminus \{i,j\}}$

$\square$

Claim $(P) \Rightarrow (G)$ when $P(x) > 0$ $\forall x \in \mathcal{X}^n$.

Lemma Intersection Lemma

If $P(x) > 0$ $\forall x \in \mathcal{X}$

(1) $X_A \perp\!\!\!\perp X_B \mid X_C, X_D$

And

(2) $X_A \perp\!\!\!\perp X_C \mid X_B, X_D$

$\Rightarrow$ $X_A \perp\!\!\!\perp X_B, X_C \mid X_D$

A B

C D

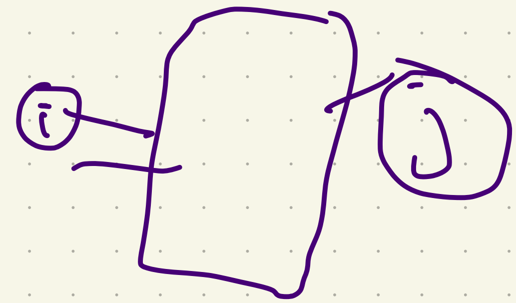
Proof of Claim 1

we know (P) $\forall (i,j) \notin E \quad X_i \perp\!\!\!\perp X_j \mid X_{V \setminus \{i,j\}}$

Want to show $\forall$ A-B-C $\quad X_A \perp\!\!\!\perp X_C \mid X_B$

By induction on $|B| = n-2, n-3, \ldots 2$

(1) Start w $|B| = n-2$



$A = \{i\}$

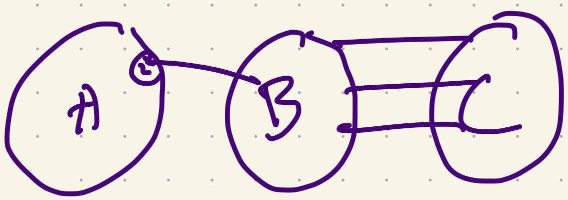
$C = \{j\}$

$$\hookrightarrow \underset{= X_i}{X_A} \perp\!\!\!\perp \underset{= X_j}{X_C} \mid X_{V \setminus \{i,j\}}$$

(2) Want to show (G) for $|B| = s-1 < n-2$

Assume (G holds $\forall |B| \geq s$

Fix $|A| \geq 2$ (WLOG), assumption A-B-C



$A \setminus \{i\} - B \cup \{i\} - C$

$\Rightarrow X_{A \setminus \{i\}} \perp\!\!\!\perp X_C \mid X_{B \cup \{i\}}$

$\{i\} - B \cup A \setminus \{i\} - C$

[Why?]

$$(IH) \Rightarrow X_i \perp\!\!\!\perp X_C \mid X_{B \cup (A \setminus \{i\})}$$

$\square$