## ③ Directed Global Markov Property

For any $A, B, C \subseteq V$ $\quad X_A \perp\!\!\!\perp X_C \mid X_B$ if
A is d-separated by B from C

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$\forall$ undirected, simple (nonrepeating) paths $a_1 \dots c$, $a_1 \in A$, $c \in C$
the path is blocked by B.

Path $a_1 \dots c$ is blocked by B if $\exists$ 3 nodes

**Intuition**

| | | |
|---|---|---|
| a .... ① → ② → ③ .... c | & $2 \in B$ | $x_1 \perp\!\!\!\perp x_3 \mid x_2$ |
| a .... ① ← ② ← ③ .... c | & $2 \in B$ | $x_1 \perp\!\!\!\perp x_3 \mid x_2$ |
| a .... ① ← ② → ③ .... c | & $2 \in B$ | $x_1 \perp\!\!\!\perp x_3 \mid x_2$ |
| a .... ① → ② ← ③ .... c | & $2 \notin B$ | $x_1 \perp\!\!\!\perp x_3$ |

Claim (3, global) $\overset{(PDF)}{\Longrightarrow}$ (1, Ordered) $\Longrightarrow$ (2, Local)

(chain rule + local MP)

## ② Directed |Local| Markov Property

$$X_i \perp\!\!\!\perp X_{nd(i) \setminus \pi(i)} \mid \pi(i)$$

$nd(i)$ = nondescendents of i
set of nodes j where $\nexists$ directed path $i, \dots, j$

$x_1$ $\emptyset$
$x_2$ $\emptyset$
$x_3 \perp\!\!\!\perp x_1, x_4, x_5 \mid x_2$

## ① Directed Ordered Markov Property

Given a DAG $G = (V, E)$ & $(1, \dots, n)$

$$X_i \perp\!\!\!\perp X_{Pr_i \setminus \pi(i)} \mid X_{\pi(i)}$$

$Pr(i)$ = nodes before i

WTS $\underset{\text{Ordered}}{P(X)} \Rightarrow \underset{\text{Local}}{P(X)}$

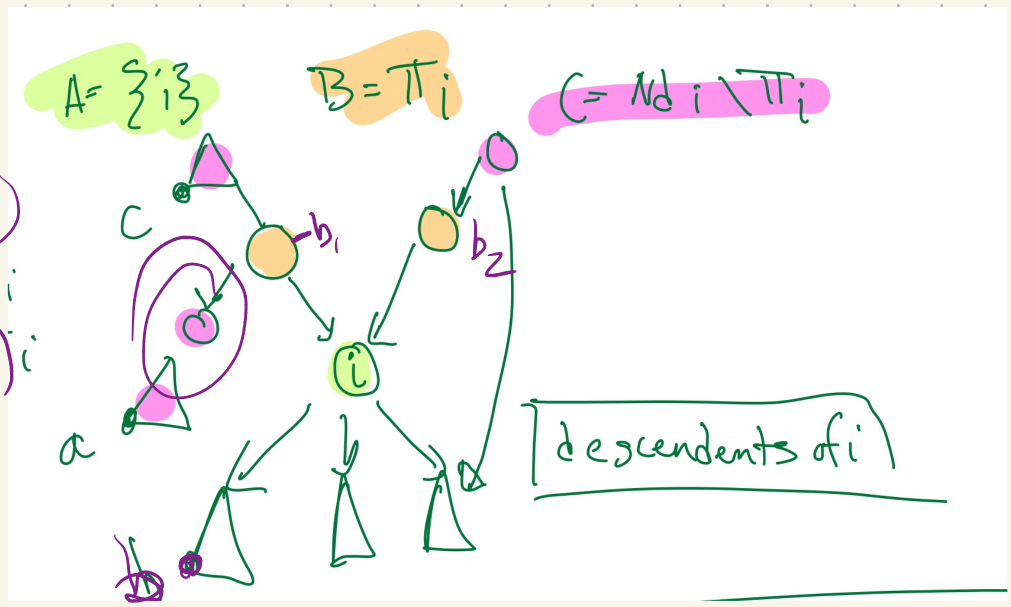
$X_i \perp\!\!\!\perp X_{pr(i)\setminus\pi(i)} \mid X_{\pi(i)}$ $\qquad$ $X_i \perp\!\!\!\perp X_{nd(i)\setminus\pi(i)} \mid \pi(i)$

$c \perp\!\!\!\perp i \mid \pi(i)$

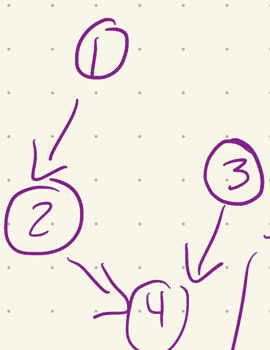
$b \not\perp\!\!\!\perp i \mid \pi(i)$

$a \perp i \mid \pi(i)$

[Whether $i <_o a$ or $a <_i$ ]




Let $I(G)$ set of all $\overset{\text{cond}}{\text{independence}}$ implied by $G=(V,E)$




$I(G) = \{ X_3 \perp X_1, X_2$

$X_4 \perp\!\!\!\perp X_1 \mid X_2$

Let $I(D)$ = set of all $\perp\!\!\!\perp$ implied by factorization of $D$

Consider $Z_1 \perp Z_2 \sim \text{Ber}(1/3)$

$X_1 = Z_1 \quad X_2 = Z_1 \quad X_3 = Z_2 \quad X_4 = X_2 \overset{\text{XOR}}{\oplus} X_3$

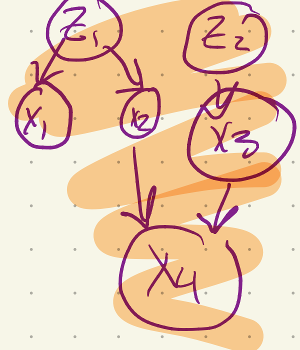
$I(P) = \{ X_1 \cancel{\perp} X_2$

$X_1 \perp X_3$

$X_2 \perp X_3$

$X_4 \perp\!\!\!\perp X_1 \mid X_2$

$X_1 \perp\!\!\!\perp X_3 \mid X_2$

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(1) $\exists\ P(X)$ s.t. no DAG perfectly captures $\perp\!\!\!\perp$ $\quad I(G) \neq I(P)$

$X_1 = X_2 \qquad X_1 \perp\!\!\!\perp X_3 \mid X_2$

$X_3 = X_1 + Z \qquad X_2 \perp\!\!\!\perp X_3 \mid X_1$

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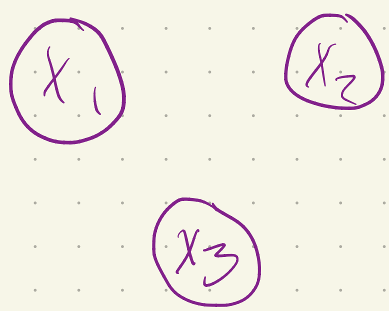


Exercise: Why not?

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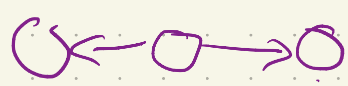
② $\forall G, \exists P$ s.t. $I(G) = I(P)$

③ $\exists\ G_1 \neq G_2$ s.t. $I(G_1) = I(G_2)$



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**Undirected GMs** aka **Markov RF's**

Given an undirected Graph $G = (V, E)$

- A clique $C \subseteq V$ is fully connected $\forall i, j \in C \quad (i, j) \in E$
- A clique $C$ is maximal if $\forall j \notin C \quad C \cup \{j\}$ is not a clique

The set of all cliques for a graph is denoted $\mathcal{C}$

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**Def** An UGM on $G = (V, E)$ $|V| = n$ is a family of distributions on $X = [X_1, \ldots, X_n]$ that factorizes as

$$P(X) = \frac{1}{Z} \prod_{c \in \mathcal{C}} f_c(X_c)$$

$f_c : \mathcal{X}^{|c|} \to \mathbb{R}^+$ is called a <u>factor</u>

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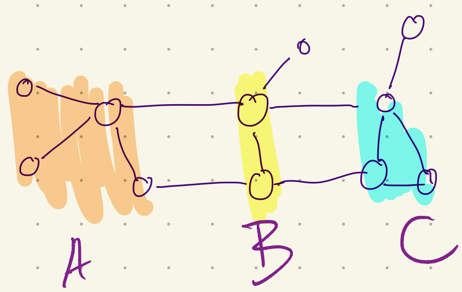
**Def [Graph Separation]**

$B \subseteq V$ separates $A$ & $C \subseteq V$ if $\forall$ paths from $a \in A$ to $c \in C$ $(\forall a, c)$ the path passes thru $B$




**Def** Global Markov Property (G)

A distribution $P(X)$ satisfies (G) if $\forall$ $A, B, C \subseteq V$ if $B$ separates $A$ from $C$

$\Rightarrow$ $X_A \perp\!\!\!\perp X_C \mid X_B$.

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Factorization $\Rightarrow$ (G)