Q. How good is BP ? $|\mathcal{X}| < \infty$

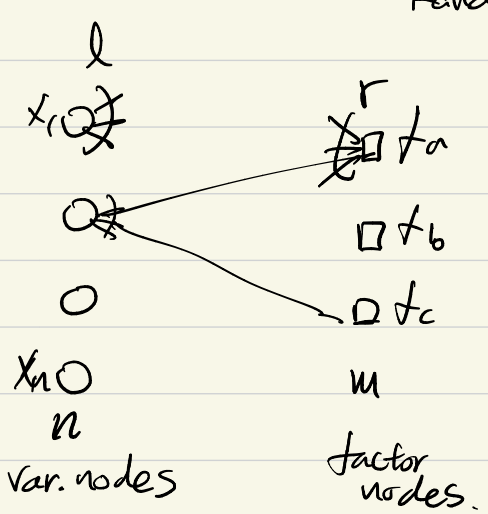
what is known
- If G is tree → exact.
- If G has 1 loop → Converges (o) [Weiss, 2000]
  may not be exact
- Maximum Weight Matching → BP is exact [Bayati Shah Sharma 2005]
  (Computationally easy)
- In the limit of large graph G,
  Density Evolution provides asymptotic performance estimate.

- Compressed Sensing
- LDPC decoding ✓
- Crowd sourcing [Karger Shah Oh 2011]
- Community detection Stochastic Block Models

* Density Evolution assumes:
  Factor Graph model on Random Graphs.

Def. Random Graph $G_{n,m,\ell,r}$ ($n, m, \ell, r \in \mathbb{Z}$)

Random graph G is drawn uniformly over all graphs that have var node degree ℓ and factor node degree r.

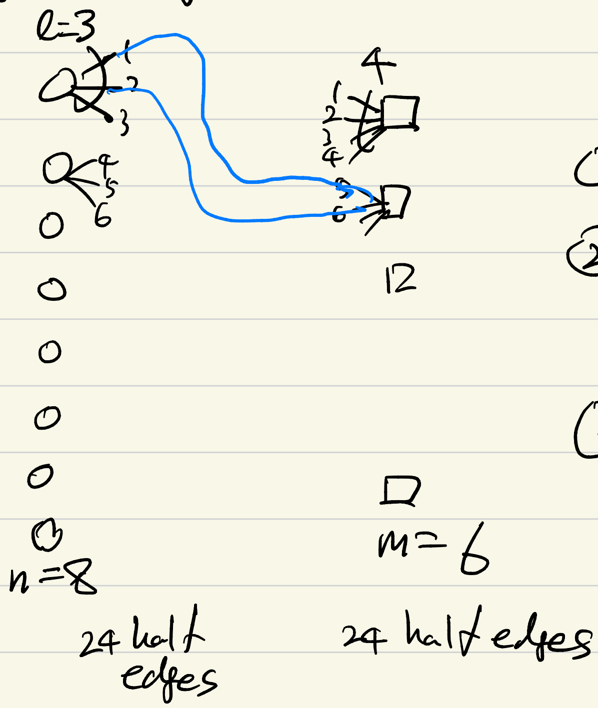


- Given $(\overbrace{n, m, l, r})$, how do you generate such graph?

$|E| = n \cdot l = m \cdot r$

Def. Configuration Model. $(l=3,\ r=4)$

$n = 8,\ m = 6$



① random permutation $\pi : |E| \to |E|$

② match half edge w.r.t π.

$\pi_1 = 5$

③ fix double edges.

* When you do have control over designing F.G model.

Example> LDPC codes.

Factor Graph Model:

assume Binary Erasure Channel. $BEC(\varepsilon)$

$X_i \in \{0, 1\}$ $\qquad$ $Y_i \in \{0, 1, *\}$

$\mathbb{P}(X|Y) \propto \mathbb{P}(Y, X) \propto \mathbb{P}(Y|X) \cdot \mathbb{P}(X)$

$\mathbb{P}(Y_i = * | X_i) = \varepsilon$

$\mathbb{P}(Y_i = X_i | X_i) = 1 - \varepsilon$.

$$\mathbb{P}_Y(X) = \frac{1}{Z} \prod_{i=1}^{n} \mathbb{P}(Y_i | X_i) \prod_{a \in F} \mathbb{I}\left(\oplus X_{\partial a} = 0\right)$$

$\uparrow$ channel. $\qquad$ $\uparrow$ binary XOR LDPC parity check.

Q. $\mathbb{P}(X_i | Y) \leftarrow$ B.P.

Marginals.

$$P(x) = \frac{1}{Z} \prod_i \mathbb{P}(Y_i \mid X_i) \cdot \prod_a \mathbb{I}(\oplus X_{\partial a} = 0)$$

* Belief Propagation LDPC code = F.G.

$$m_{i\to a}(X_i) = \mathbb{P}(Y_i \mid X_i) \cdot \prod_{b \in \partial i \setminus \{a\}} \tilde{m}_{b\to i}(X_i) ,$$

$$\mathbb{P}(Y_i \mid X_i) \begin{cases} \varepsilon & Y_i = * \\ 1-\varepsilon & X_i = Y_i \\ 0 & Y_i = \neg X_i \end{cases}$$

↑ binary negation

$$\tilde{m}_{a\to i}(X_i) = \sum_{X_{\partial a \setminus \{i\}}} \left\{ \prod_{j \in \partial a \setminus \{i\}} m_{j\to a}(X_j) \right\} \mathbb{I}(\oplus X_{\partial a} = 0)$$

claim: $m_{i\to a}(X_i)$, $\tilde{m}_{a\to i}(X_i)$ ($X_{i\to a}$) $\in \left\{ \begin{matrix} X_i=0 \\ X_i=1 \end{matrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \right\}$ → the messages are also discrete

$X_{i\to a} = 0$, $X_{i\to a} = 1$, $X_{i\to a} = *$

↑ BEC not BSC

proof: by induction

initially: $\tilde{m}^{(0)}_{b\to i}(X_i) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

$$m^{(1)}_{i\to a}(X_i) = \begin{cases} \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix} \propto \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} & \text{if } Y_i = * \\ \begin{bmatrix} 1-\varepsilon \\ 0 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 0 \end{bmatrix} & Y_i = 0 \\ \begin{bmatrix} 0 \\ 1-\varepsilon \end{bmatrix} \propto \begin{bmatrix} 0 \\ 1 \end{bmatrix} & Y_i = 1 \end{cases}$$

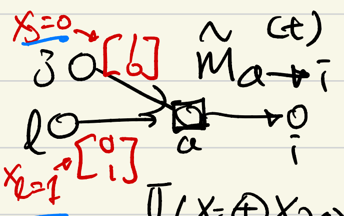
$X_{i\to a} = *$

$X_{i\to a} = 0$

$X_{i\to a} = 1$

Induction: ① $\tilde{m}^{(t)}_{a\to i}$

Case 1: all incoming messages are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$$\tilde{m}^{(t)}_{a\to i} = \begin{bmatrix} m(X_j=0)\, m(X_l=0) + m(X_j=1)\, m(X_l=1) \\ m(X_j=0)\, m(X_l=1) + m(X_j=1)\, m(X_l=0) \end{bmatrix}$$

$\mathbb{I}(X_i \oplus X_{\partial a \setminus \{i\}} = 0)$

1, if 1

0 if 0

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Case 2: If there is at least one incoming message that is $\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, then

$$\tilde{m}^{(t)}_{a\to i} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

$$\tilde{X}^{(t)}_{a\to i} = \begin{cases} \oplus X_{\partial a \setminus \{i\} \to i} \\ * & \text{if } \exists \text{ incoming message that is } * \end{cases}$$

$\tilde{m}^{(t)}_{i\to a}(x_i)$ $\quad \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1/2\\1/2\end{bmatrix} \right\}$

② Update $m^{(t)}_{i\to a}(x_i) = \left(\prod_{b\in\partial i\setminus\{a\}} \tilde{m}^{(t)}_{b\to i}(x_i) \right) \times \mathbb{P}(y_i|x_i)$

Case 1: if all incoming is $\begin{bmatrix}1/2\\1/2\end{bmatrix}$ including $\mathbb{P}(y_i|x_i) = \begin{bmatrix}\varepsilon\\\varepsilon\end{bmatrix}$

$$m^{(t)}_{i\to a}(x_i) = \begin{bmatrix}1/2\\1/2\end{bmatrix} \propto \begin{bmatrix}1/2\times 1/2\times\cdots\\1/2\times 1/2\times\cdots\end{bmatrix}$$

Case 2: if any incoming is $\begin{bmatrix}0\\1\end{bmatrix}$

$$m^{(t)}_{i\to a}(x_i) = \tilde{m}^{(t)}_{b\to i}(x_i) = \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\times 1/2\times 1/2\cdots\\1\times 1/2\times 1/2\cdots\end{bmatrix}$$

we proved the claim $\quad m^{(t)}_{i\to a}, \tilde{m}^{(t)}_{a\to i} \in \left\{ \begin{bmatrix}0\\1\end{bmatrix} \begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}1/2\\1/2\end{bmatrix} \right\}$

⇒ We can re-write the BP update rule for this LDPC

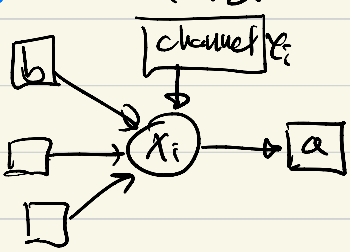
$x^{(t)}_{i\to a} \in \{0,1,*\}$, $\tilde{x}^{(t)}_{a\to i} \in \{0,1,*\}$

Initialize $\tilde{x}^{(0)}_{a\to i} = *$

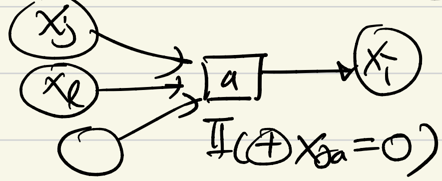
Update:

$$x^{(t+1)}_{i\to a} = \begin{cases} * & \text{, if all of incoming is } * \\ \tilde{x}^{(t)}_{b\to i} & \text{, otherwise} \end{cases}$$

you are guaranteed no conflict.

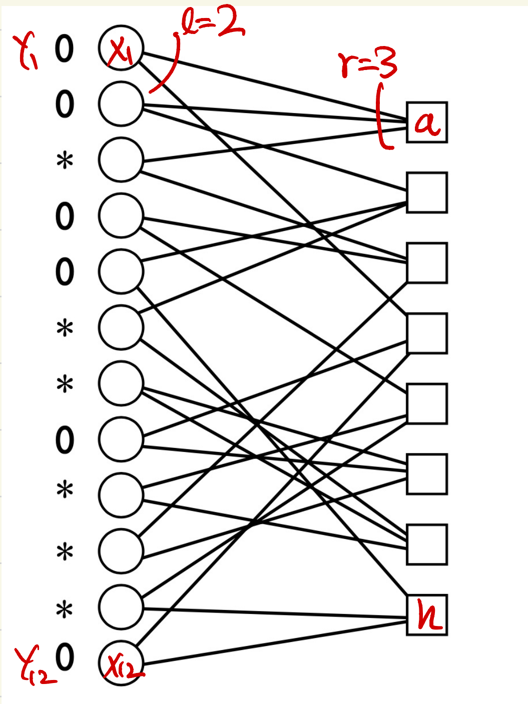


$$\tilde{x}^{(t+1)}_{a\to i} = \begin{cases} \bigoplus x^{(t+1)}_{\partial a\setminus\{i\}} & \text{, if all incoming is 0 or 1} \\ * & \text{, otherwise} \end{cases}$$

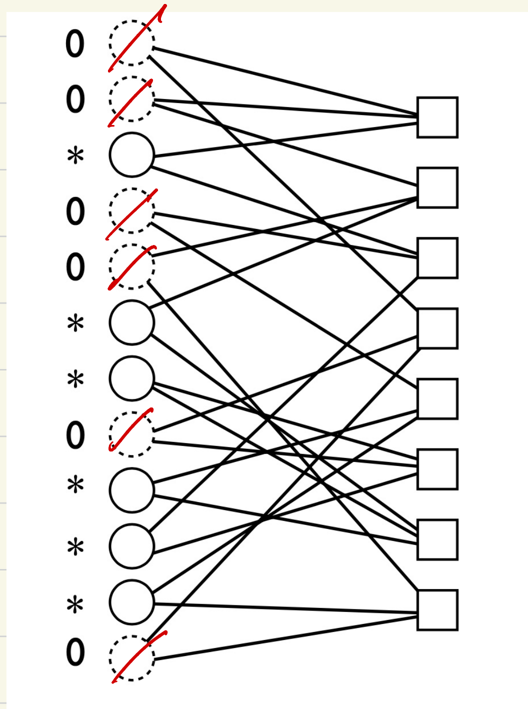


⇒ This version of BP is Peeling Decoder

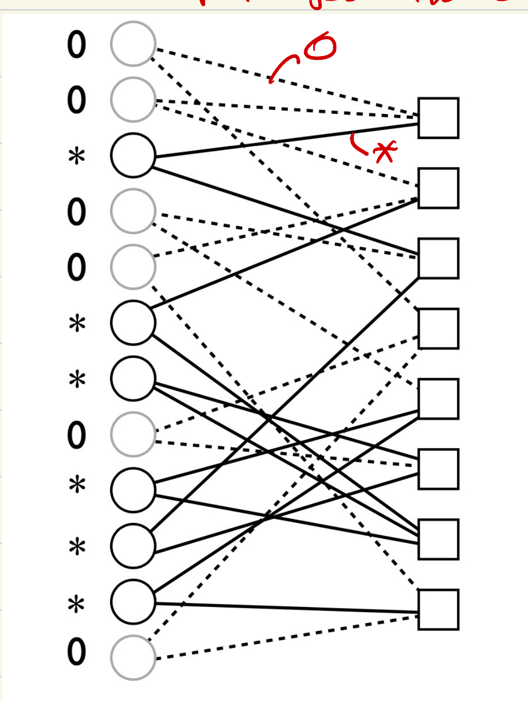
w.l.o.g. $X=0$ was sent



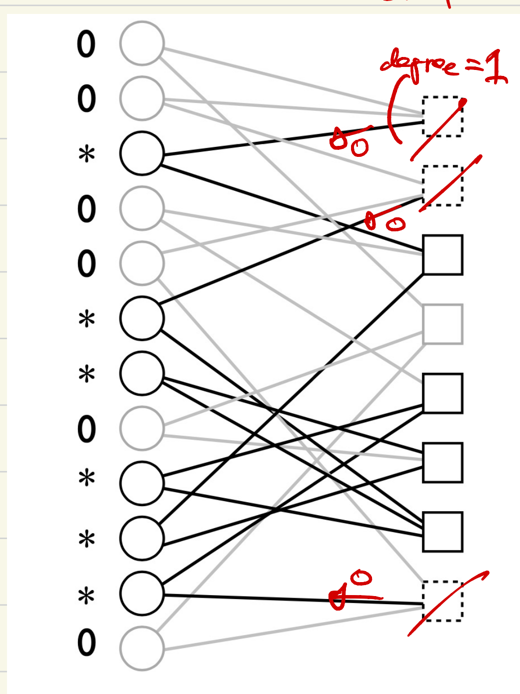
$X^{(1)}_{i\to a} = 0$

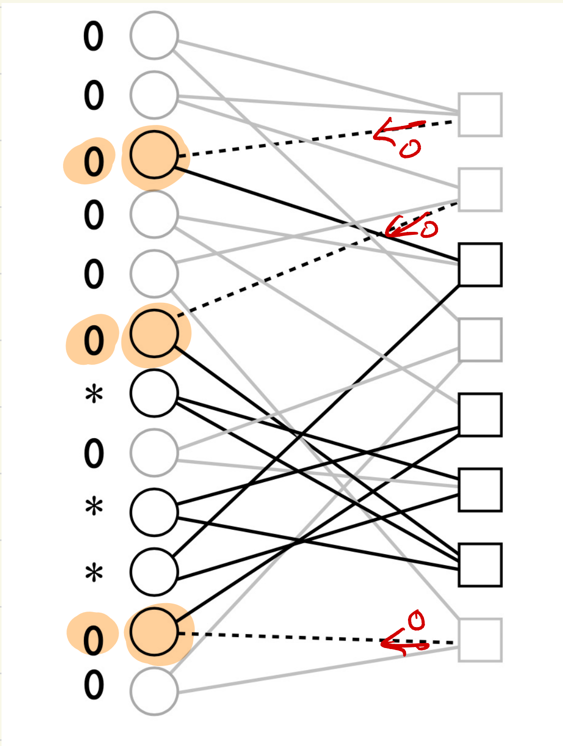


peel edges $X^{(1)}_{i\to a} = 0$



$$\tilde{X}^{(2)}_{a\to i} = \begin{cases} 0 & \text{if all incom'g } 0 \\ * & \text{otherwise} \end{cases}$$

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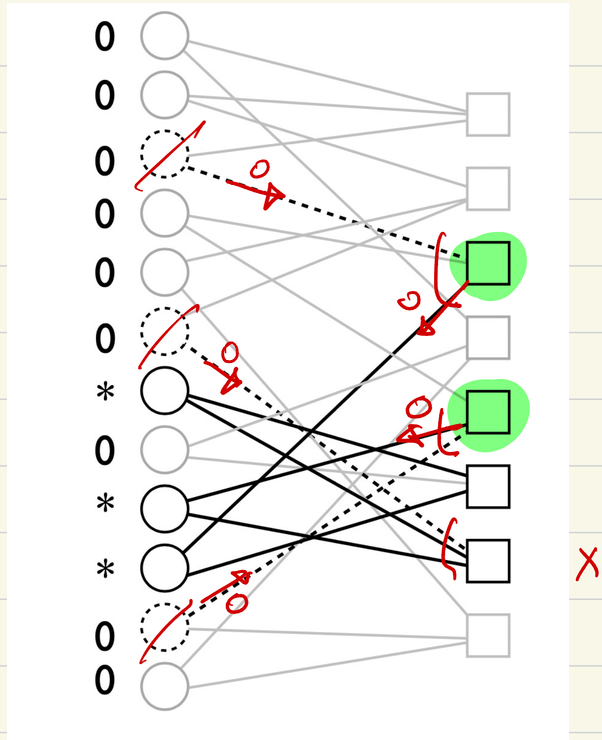
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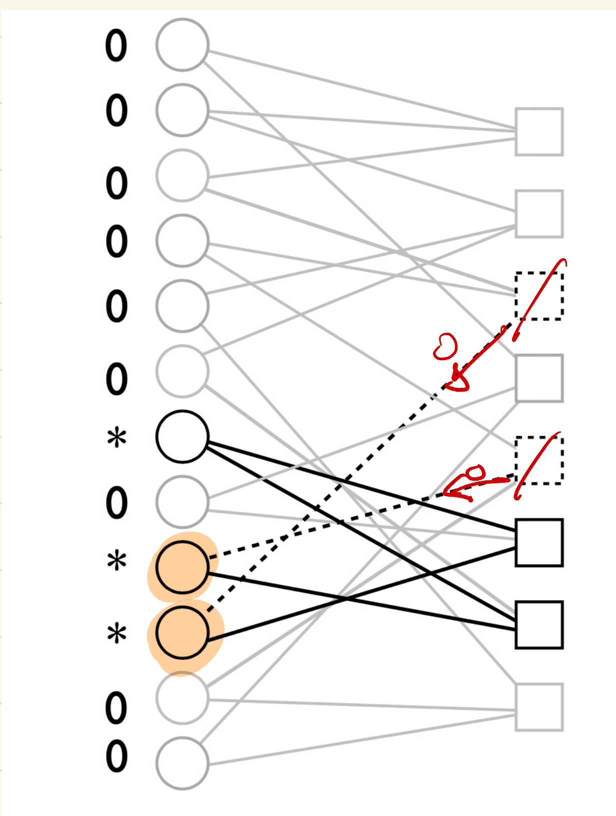
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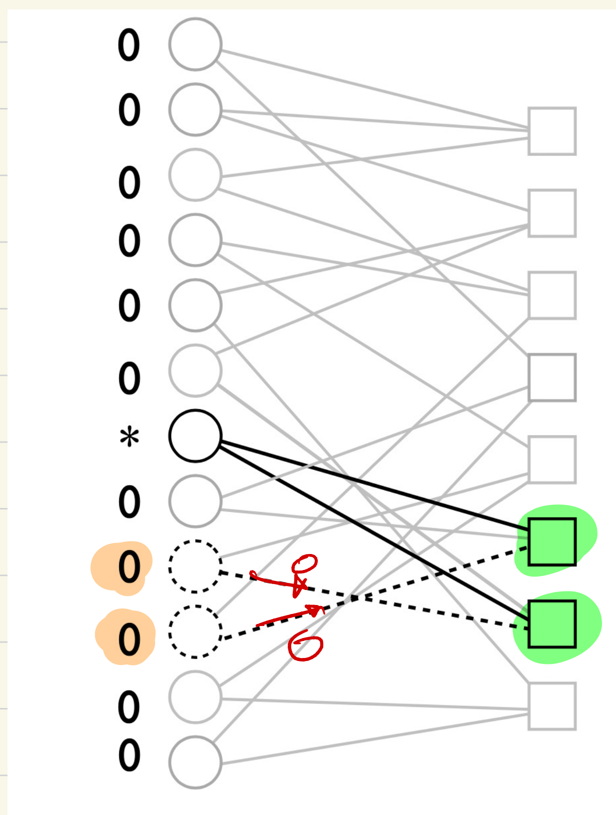
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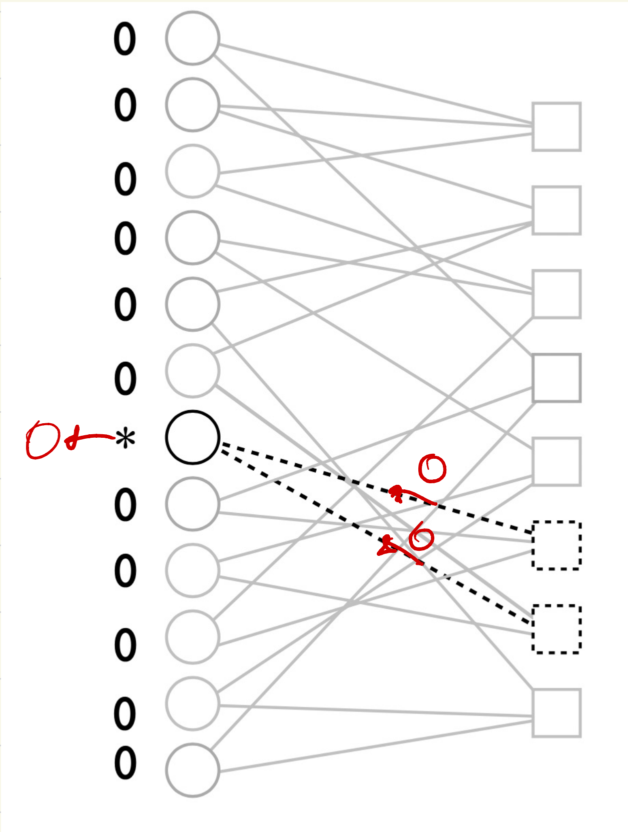


* Design Goal. Given, m, n

give me a graph s.t. error probability is small.

↑

Density Evolution tells you how to design.

Strategy under Random Factor Graph. with $n, m \to \infty$

$G(n, m, \ell, r)$, $BEC(\varepsilon)$.

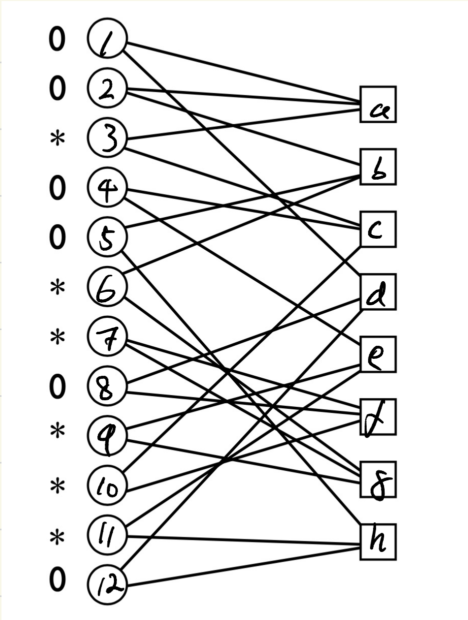
(↳ designed)

① suppose we update BP as follows.

repeat:

- update $\{m^{(t)}_{i \to a}\}$
- Draw a fresh Random Graph $G(n, m, \ell, r)$
- update $\{\hat{m}^{(t)}_{a \to i}\}$
- Draw a fresh Random Graph $G(n, m, \ell, r)$
- $t = t+1$

② Justify this is accurate if
- F.G. is random.
- $m, n \uparrow \infty$.



$\{0, *\}$

$\downarrow$

$$\begin{pmatrix} X^{(1)}_{1\to a} = 0 \\ X^{(1)}_{1\to d} = 0 \\ X^{(1)}_{2\to a} = 0 \\ X^{(1)}_{2\to b} = 0 \\ X^{(1)}_{3\to a} = * \\ X^{(1)}_{3\to c} = * \\ \vdots \end{pmatrix}$$

$|E|$

Histogram ↔ population density
of

$\{X^{(1)}_{i\to a}\} \to \underline{\rho^{(1)} \in [0,1]} = \mathbb{P}(X^{(1)}_{i\to a} = *)$

↑ empirical distirion.