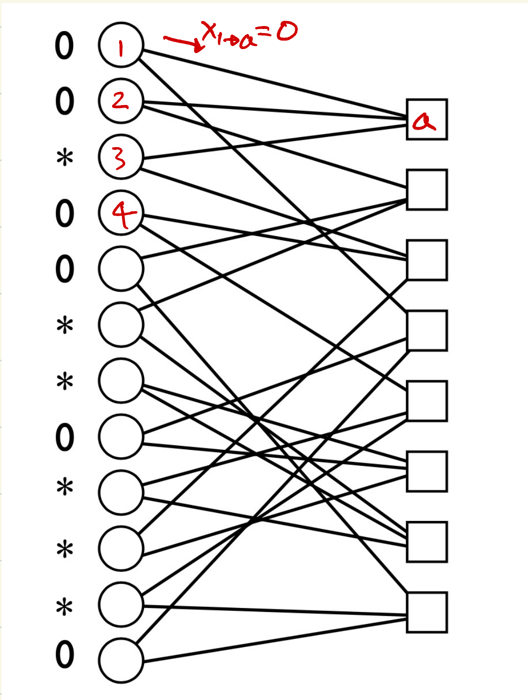
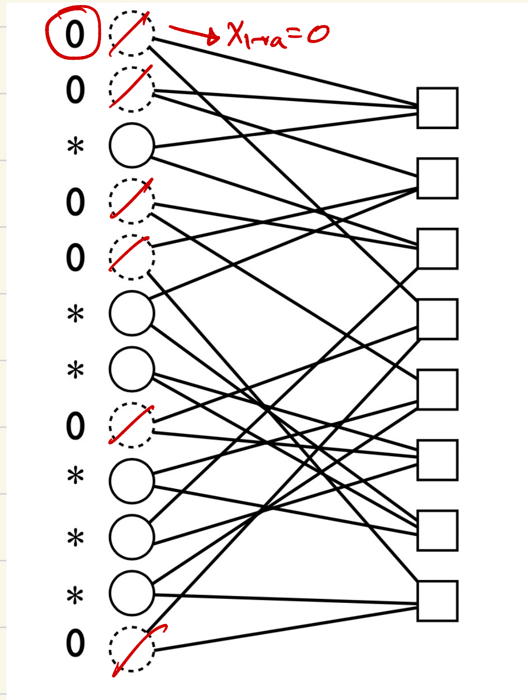
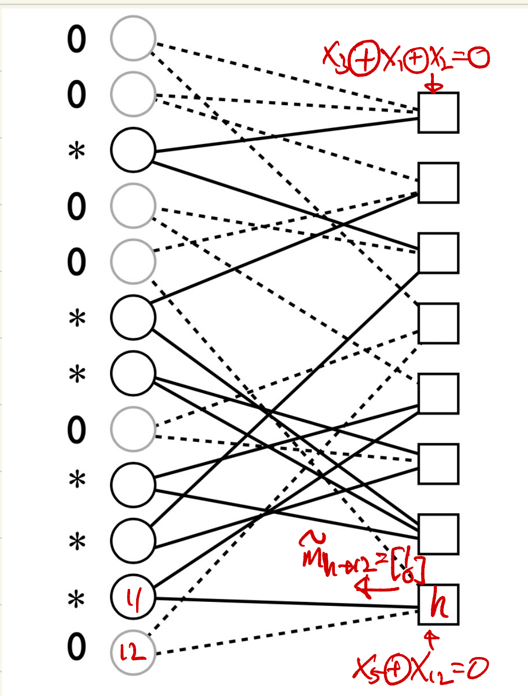
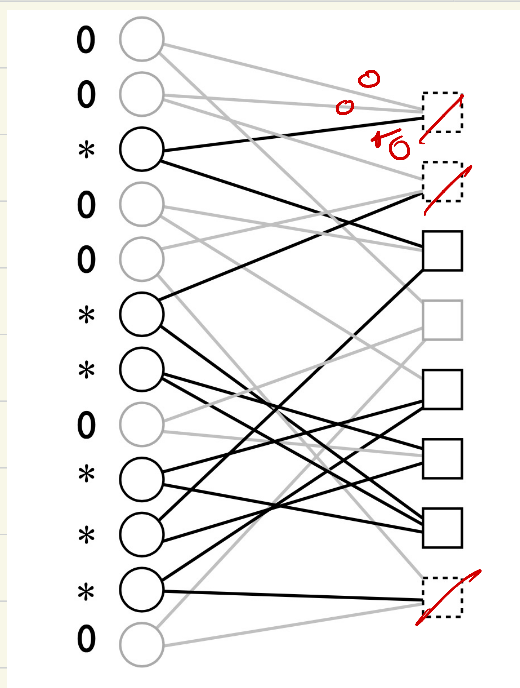
Without loss of generality, suppose all 0's were sent, Peel nodes/edges whose messages are $\begin{bmatrix}0\\1\end{bmatrix}$ or $\begin{bmatrix}1\\0\end{bmatrix}$.



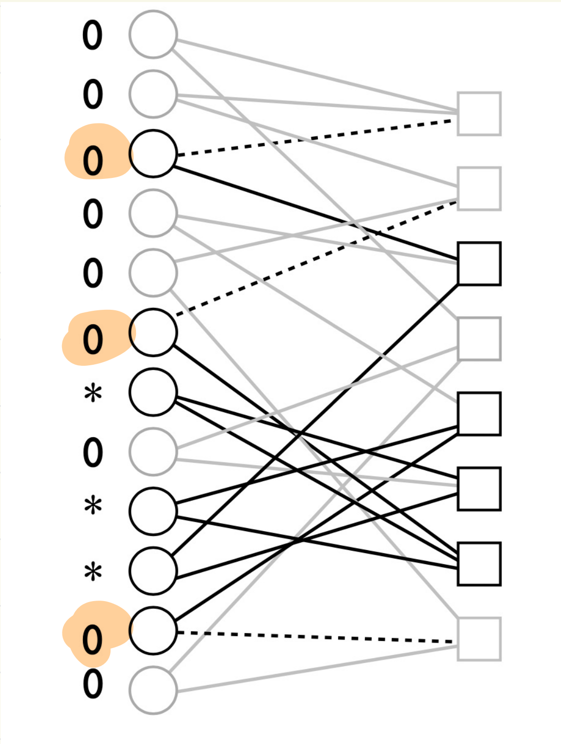


$x_1=0$





* factor nodes with 1 remaining edge can be decoded.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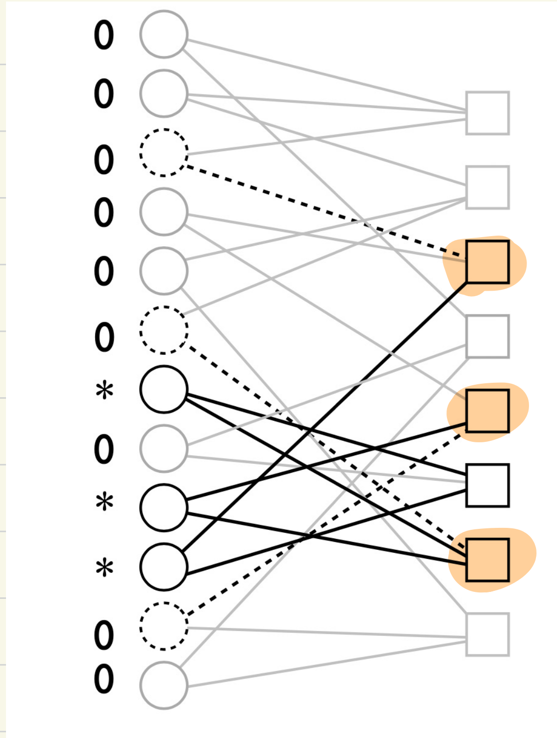
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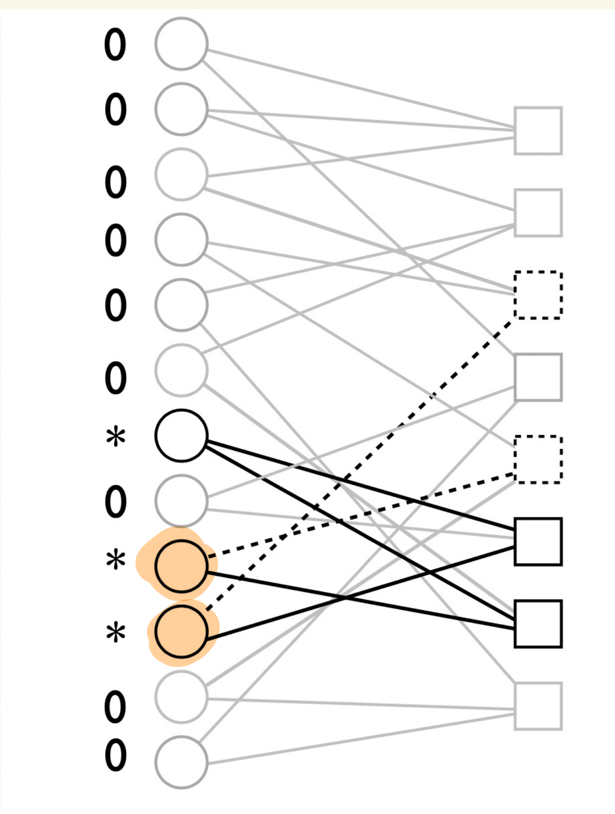
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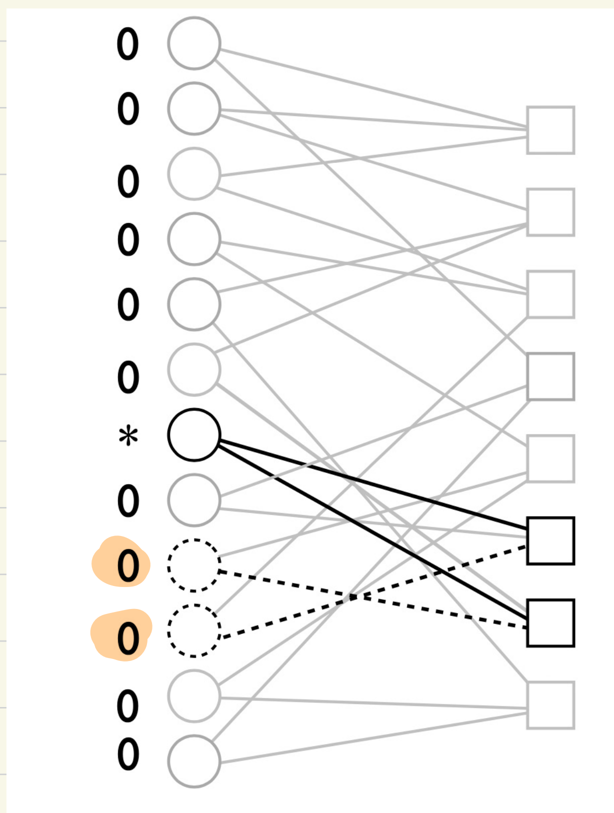
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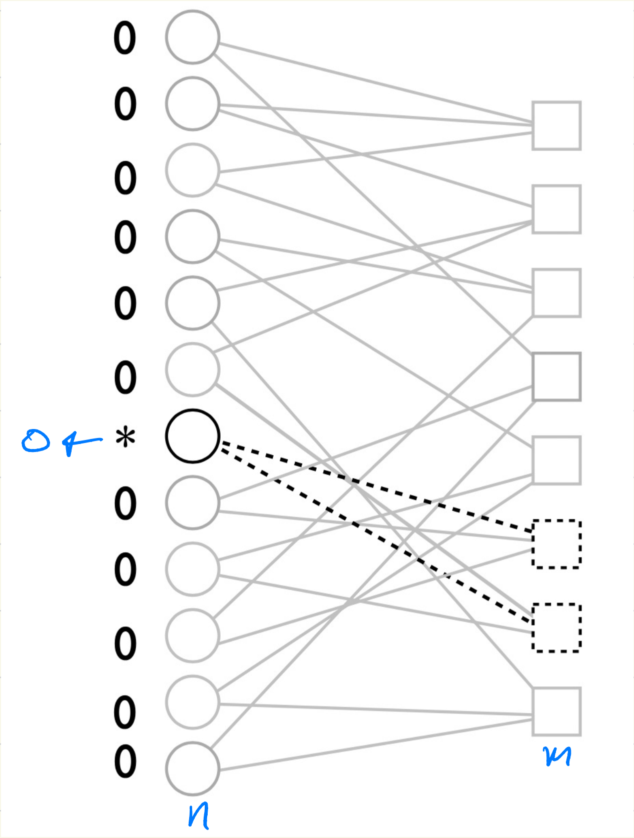


Density Evolution estimates the performance of B.P.

① Graph is drawn from R.G. (n, m, l, r)

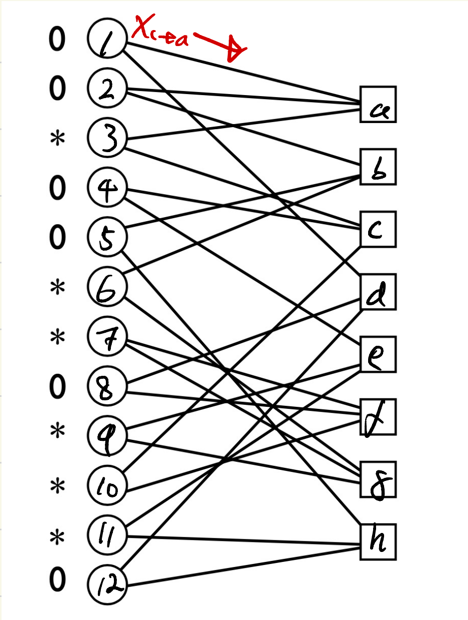
② asymptotically $n \uparrow \infty$, $m \uparrow \infty$, fixed l & r.

* strategy : suppose graph is independently being drawn at random as you go through BP.

: Justify it is accurate under ① & ②.

* Density Evolution

Density of the messages $\{m_{i\to a}\}_{(i,a)\in E}$
$\{\tilde{m}_{a\to i}\}_{(a,i)\in E}$.



for $t=1$.

$$\left(\begin{array}{c} X^{(1)}_{1\to a} = 0 \\ \vdots \\ X^{(1)}_{3\to a} = * \end{array} \right)$$

BEC (ε).

$$Y_i = \begin{cases} 0 & \text{w.p. } 1-\varepsilon \\ * & \varepsilon \end{cases}$$

Set Histogram, $|E|$

$1-\xi$ $\quad$ $\xi^{(1)} \approx \varepsilon$

$0 \qquad *$

$\xi^{(1)} \triangleq \hat{\mathbb{P}}(X^{(1)}_{i\to a} = *) \in [0,1]$

Density

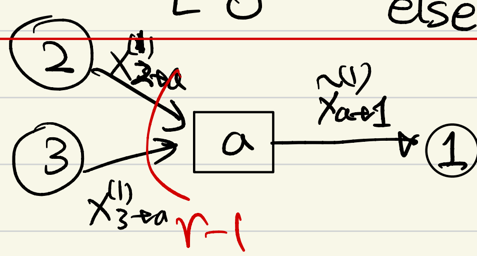
$$\frac{n\cdot\varepsilon\cdot l}{nl}$$

$\cdot\ \mathbb{E}[\xi^{(1)}] = \varepsilon$

$\cdot\ \lim_{n\to\infty} \xi^{(1)} \to \varepsilon$

for $t=1$, $\quad \tilde{X}^{(1)}_{a\to i} \in \{0, *, 1\}$ $\quad$ Just for this example when all zeros codeword was sent.

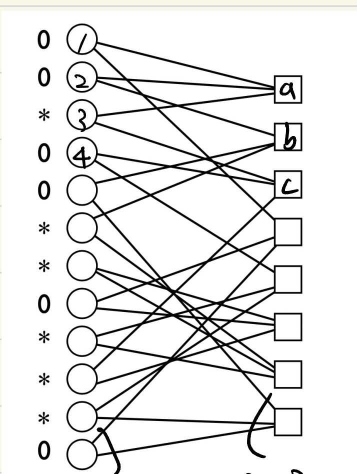
$$\boxed{\tilde{X}^{(1)}_{a\to 1} = \begin{cases} * & \text{if } X_{2\to a} = * \text{ OR } X_{3\to a} = * \\ 0 & \text{else} \end{cases}}$$



$X_{2\to a} = *$, $X_{3\to a} = 0$

know $X_2 \oplus X_3 \oplus X_1 = 0$

$\quad\quad ∥ \quad ∥$
$\quad\quad 0 \quad 0$



$l=2 \qquad r=3$

?

Statistics

$\mathbb{P}(X^{(1)}_{3\to a} = *) = \xi^{(1)} \longrightarrow \mathbb{P}(\hat{X}^{(1)}_{a\to 1} = *) =$

(a) $\hat{\xi}^{(1)} = 1 - (1-\xi^{(1)})^{r-1}$

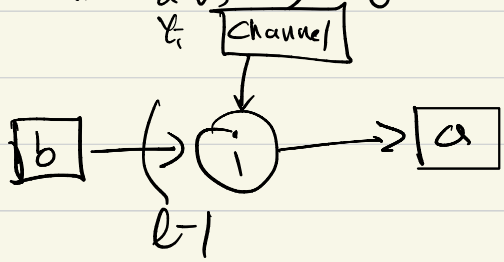
$t=2$ $m^{(2)}_{i\to a}$

(b)

$\mathbb{P}(\hat{X}^{(1)}_{a\to j}=*) = \tilde{\delta}^{(1)} \longrightarrow$ what is $\mathbb{P}(X^{(2)}_{i\to a}=*) = \varepsilon\cdot\left(\tilde{\delta}^{(1)}\right)^{\ell-1}$

$\parallel\nabla$

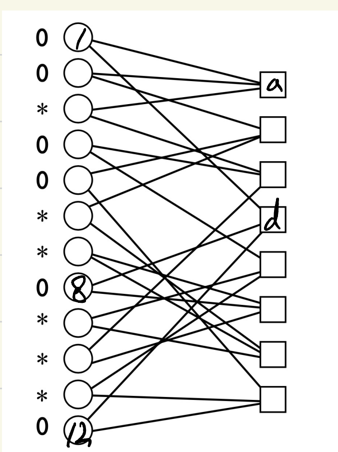
$\delta^{(2)}$



$$X^{(2)}_{i\to a} = \begin{cases} * & \text{if } Y_i=*, \text{ all incoming } * \\ 0 & \text{else} \end{cases}$$

$\mathbb{P}(Y_i=*)=\varepsilon$

$X_i \longrightarrow a$

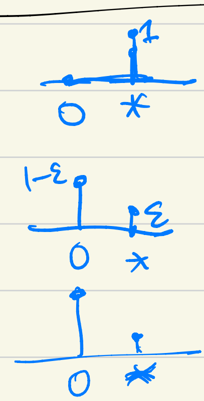


$$X_{i\to a} = \begin{cases} * & \text{if all incoming } * \\ X_{b\to i} & \text{otherwise} \end{cases}$$

$$\hat{X}_{a\to i} = \begin{cases} * & \text{at least one incoming } * \\ \oplus X_{\partial a\setminus i\to i} & \text{otherwise.} \end{cases}$$

BP

iterations



$$\delta^{(t+1)} = \varepsilon\cdot\left(\tilde{\delta}^{(t)}\right)^{\ell-1}$$

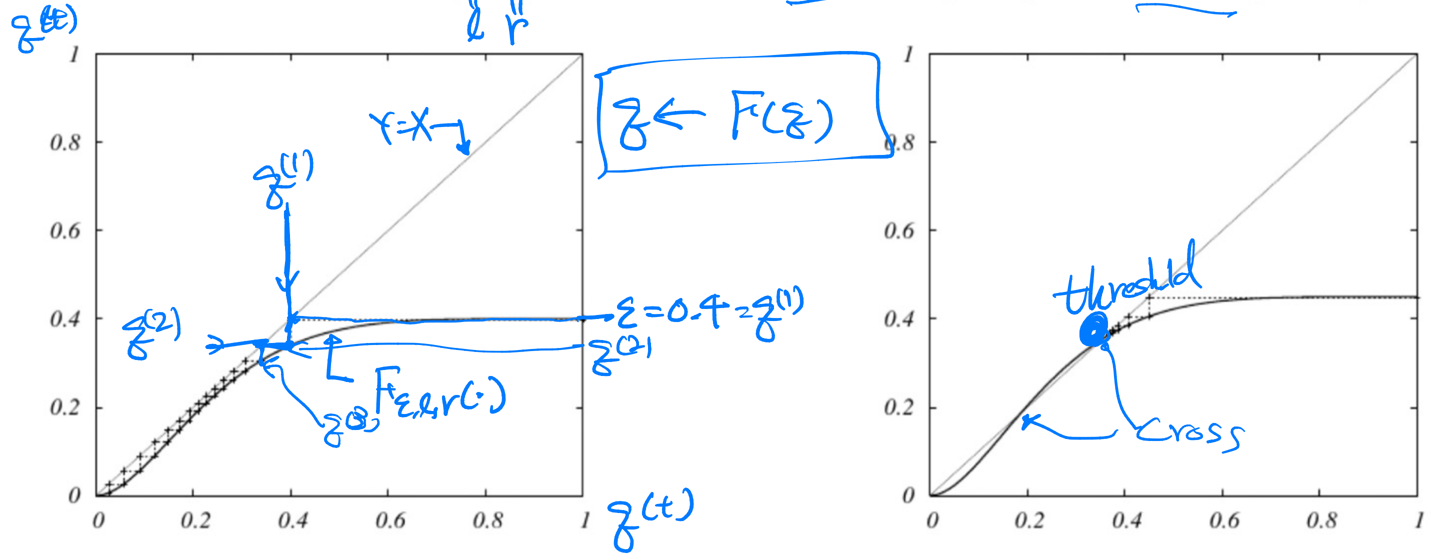
$$\tilde{\delta}^{(t+1)} = 1-\left(1-\delta^{(t+1)}\right)^{r-1}$$

$$\boxed{\delta^{(t+1)} = \varepsilon\left(1-\left(1-\delta^{(t)}\right)^{r-1}\right)^{\ell-1}}$$

$$\xi = F_{\varepsilon,l,r}(\xi) \longleftrightarrow \xi \leftarrow \varepsilon\,(1-(1-\xi)^{r-1})^{l-1}$$

density evolution for (3,6) code with $\epsilon = 0.4$(left) and 0.45(right)

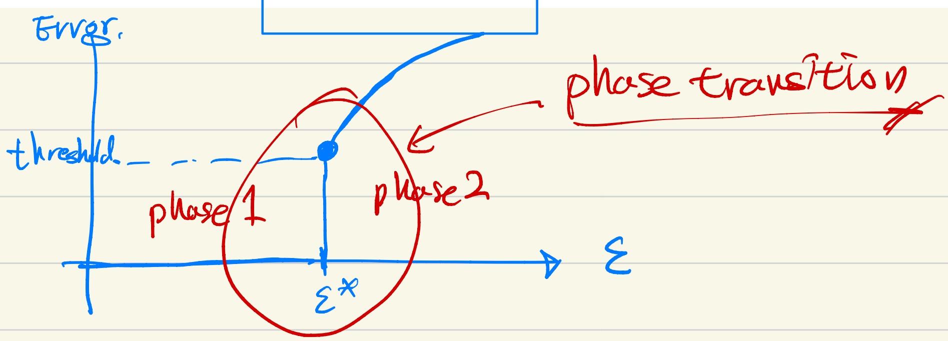
$l \quad r$



rate of this code = 0.5, threshold $\boxed{\epsilon^* \simeq 0.4xxx,}$

EXIT chart.

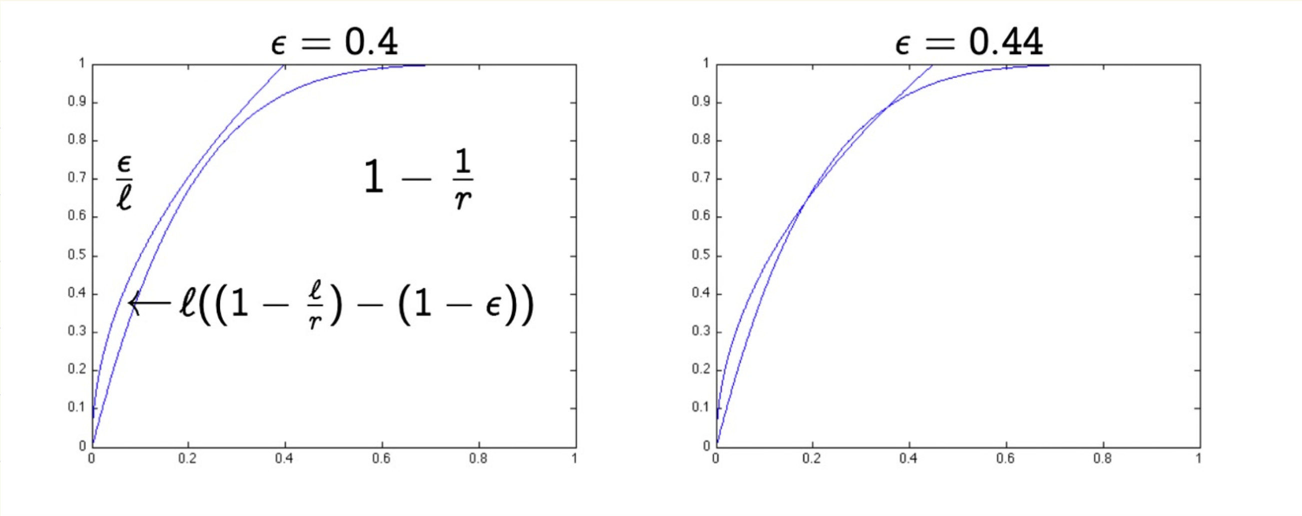
Numerically Calculate.



How were we able to do D.E.?

simple ⟶ complex.

	Density of messages $\xi^{(t)} \in$ Vector	Cannot be solved numerically $\boxed{\xi^{(t)} \in \text{functions}}$	Cannot be computed. function of functions.
Algorithm updating messages:	Message $m_{i\to a} \in$ Discrete	$m_{i\to a} \in$ Vector.	$m_{i\to a} \in$ Functions.
	Variable $X_i \in$ Discrete	$X_i \in$ Discrete.	$X_i \in$ Continuous.
example	(BEC, LDPC	(BSC, LDPC	Continuous $P(x_i)$



$\epsilon = 0.4$
$\frac{\epsilon}{\ell}$
$1 - \frac{1}{r}$
$\leftarrow \ell((1 - \frac{\ell}{r}) - (1 - \epsilon))$
$\epsilon = 0.44$

Gaussian Graphical Models.

So far, we learned $P(x_1^n)$, $x_i \in \mathcal{X}$

- any factor $f_a(x_1, \cdots, x_e)$ can be written as a table $|\mathcal{X}|^e$
- algorithms only used $+$, $\times$, and look-up table.

Now, we want Continuous distributions in $\mathbb{R}^n$.

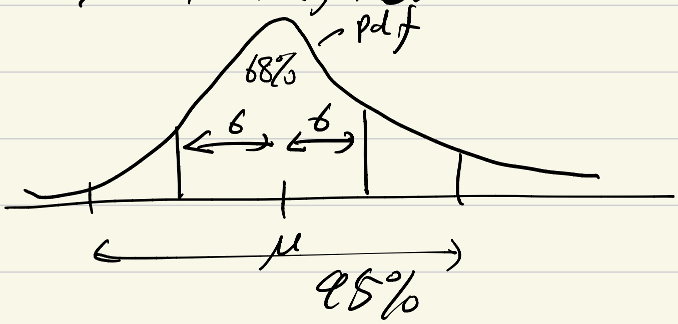
- we need parametric family to [store factors / compute message.

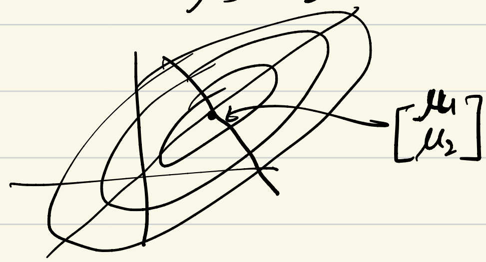
Def. $X = (X_1, \cdots, X_n)$ is Gaussian $N(\mu, \Sigma)$ if

$$\underbrace{P(x)}_{\text{p.d.f}} = \frac{1}{(2\pi)^{n/2} \underbrace{|\Sigma|}_{\text{determinant}}^{1/2}} \cdot \exp\left(-\frac{1}{2}(x-\mu)^T \underbrace{\Sigma^{-1}}_{\text{P.D. } \Sigma}(x-\mu)\right)$$

where $\mu = \mathbb{E}[X]$, $\Sigma = Cov(X) = \mathbb{E}[(X-\mu)(X-\mu)^T]$

ex> $X \sim N(\mu, \sigma)$





Def. Covariance Form

$X \sim N(\mu, \Sigma)$ $\Leftrightarrow P(x) \propto \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$

$$\exp\left(-\frac{1}{2}x^T\Sigma^{-1}x + \mu^T\Sigma^{-1}x\right)$$

Information Form

$X \sim N^{-1}(h, J) \leftrightarrow P(x) \propto \exp\left(-\frac{1}{2}x^T J x + h^T x\right)$.

$$\boxed{J = \Sigma^{-1}, \quad h = \Sigma^{-1}\mu}$$

Remark ① Marginalization is easy with $N(\mu, \Sigma)$
② Conditioning is easy with $N^{-1}(h, J)$

① Marginalization

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] \right) \qquad X_1 \in \mathbb{R}^{n_1},\ X_2 \in \mathbb{R}^{n_2}$$

$$P(X_1) = \int P(X_1, X_2)\, dx_2$$

$$X_1 \sim N(\mu_1, \Sigma_{11})$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N^{-1}\left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \left[\begin{array}{c|c} J_{11} & J_{12} \\ \hline J_{21} & J_{22} \end{array}\right] \right) \qquad X_1 \in \mathbb{R},\ X_2 \in \mathbb{R}^{n-1}$$

$$X_1 \sim N^{-1}\left(h_1 - J_{12} J_{22}^{-1} h_2,\ J_{11} - J_{12} J_{22}^{-1} J_{21} \right)$$

Complicated $\rightarrow$ $O(n^3)$ to invert J_{22}.

② Conditioning

$$P(X_1 | X_2) = N\left(\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (X_2 - \mu_2),\ \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21} \right)$$

Complex $\rightarrow$ Σ_{22}^{-1} takes $O(n^3)$ time.

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N^{-1}\left(\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}, \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \right)$$

$$P(X_1 | X_2)$$
$$\propto \exp\left(-\frac{1}{2} [X_1^T\ X_2^T] \begin{bmatrix} J_{11} & J_{12} \\ J_{22} & J_{21} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [h_1^T\ h_2^T] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right)$$
$$\propto \exp\left(-\frac{1}{2} X_1^T J_{11} X_1 + \underbrace{\left[h_1^T - \frac{1}{2} X_2^T J_{21} - \frac{1}{2} X_2^T J_{12}^T \right]}_{(h_1 - J_{12} X_2)^T} \cdot X_1 + c \right)$$

$$N^{-1}(h_1 - J_{12} X_2,\ J_{11})$$