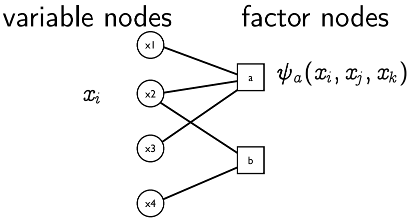
5. Density evolution

Probabilistic analysis of message passing algorithms



- consider factor graph model $G = (V, F, E)$ and

$$\mu(x) = \frac{1}{Z} \prod_{a \in F} \psi_a(x_{\partial a}) \prod_{i \in V} \psi_i(x_i)$$

- sum-product algorithm and max-product algorithms are instances of *message-passing algorithms*
  - discrete $x_i \in \mathcal{X}$
  - two sets of messages $\{\nu_{i \to a}(x_i)\}$ and $\{\tilde{\nu}_{a \to i}(x_i)\}$
  - update:

$$\nu_{i \to a}^{(t+1)} = F_{i \to a}(\{\tilde{\nu}_{b \to i}^{(t)} : b \in \partial i \setminus a\})$$
$$\tilde{\nu}_{a \to i}^{(t)} = G_{a \to i}(\{\nu_{j \to a}^{(t)} : j \in \partial a \setminus i\})$$

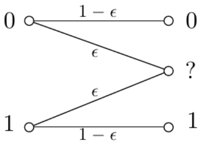
- assumptions for probabilistic analysis
    - a *random graph* is a graph $G = (V, F, E)$ where E is drawn randomly from a set of possible graphs
      e.g., Erdös-Renyi graph, random regular graph
    - asymptotic analysis: in the limit $n \to \infty$

- *density evolution* is used in
    - analyzing channel codes
    - analyzing solution space of XORSAT
    - analyzing a message-passing algorithm for crowdsourcing
    - analyzing belief propagation for community detection
    - etc.

Example: channel coding

- sending messages through a noisy *channel*

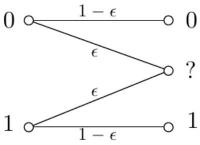
- channel is defined by $\mathbb{P}_{Y|X}(y|x)$
- Binary Erasure Channel (BEC)
  - input $x_i \in \{0, 1\}$, output $y_i \in \{0, 1, *\}$

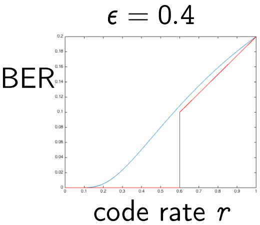


- goal: estimate $\widehat{x}_1, \dots, \widehat{x}_n$ given $y_1, \dots, y_n$
- performance metric: average bit error probability

$$P_{\text{error}} \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}(x_i \neq \widehat{x}_i)$$

Message length vs. block length

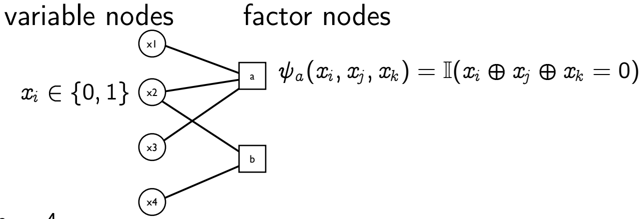




- no coding: $(01001) \Rightarrow (01*0*)$
  - message $k = 5$ bits, block length $n = 5$
    $\Rightarrow$ rate of this code $r \triangleq k/n = 1$, delay is one
  - $P_{\text{error}} = \epsilon/2$
- repetition code: $(000111000000111) \Rightarrow (0**1*10*0***111)$
  - $k = 5$, $n = 15$
  - rate $r = 1/3$ and $P_{\text{error}} = \epsilon^3/2$, delay is 3
  - in general, $P_{\text{error}} = \epsilon^{1/r}/2 > 0$ (unless rate is zero)
- information theory
  - *capacity* of a BEC is $1 - \epsilon$
  - there exists a code such that $\lim_{n\to\infty} P_{\text{error}} = 0$ with rate $r < 1 - \epsilon$
  - using the BEC n times, one can reliably send $k = (1-\epsilon)n$ bits of messages

Modern coding theory

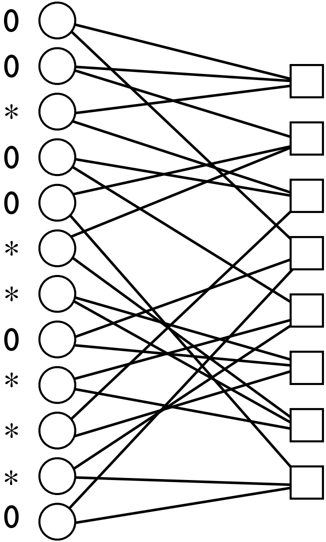
- modern codes = iterative decoding (belief propagation)
  - Turbo code
  - Low-Density Parity Check (LDPC) code
  - Polar code
  - etc.
- LDPC code is defined by a factor graph model



  - block length $n = 4$
  - number of factors $m = 2$
  - allowed messages $= \{0000, 0111, 1010, 1101\}$
  - message size $k \triangleq \log_2(\#\text{ of allowed messages}) = 2$ $(k = n - m)$
  - rate $r \triangleq k/n = 1/2$
  - received $y = (0 * 1 *)$, then $\widehat{x} = (0111)$
  - received $y = (0 * * *)$, then ?

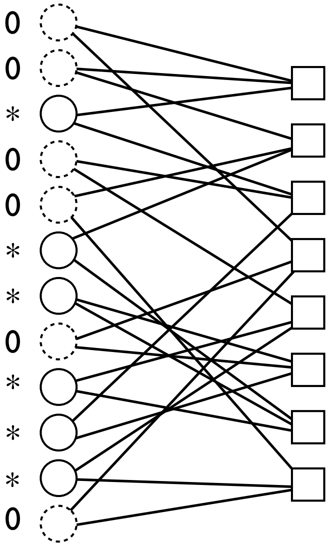
Peeling decoder is equivalent to sum-product for BEC

without loss of generality, suppose all 0's sent



Peeling decoder is equivalent to sum-product for BEC

without loss of generality, suppose all 0's sent



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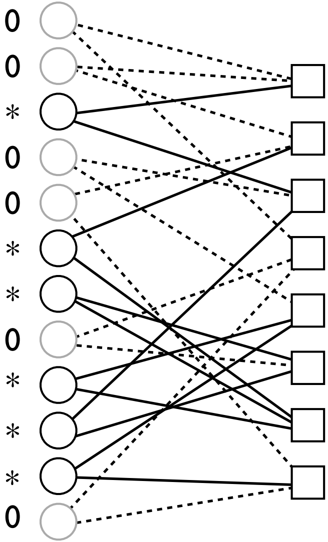

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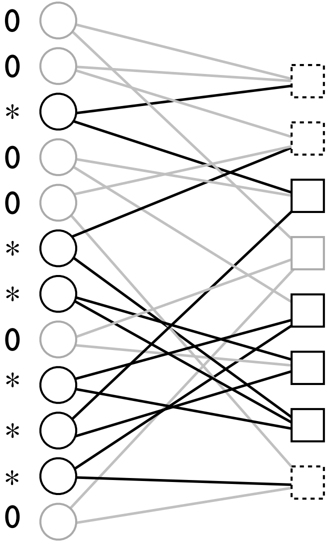

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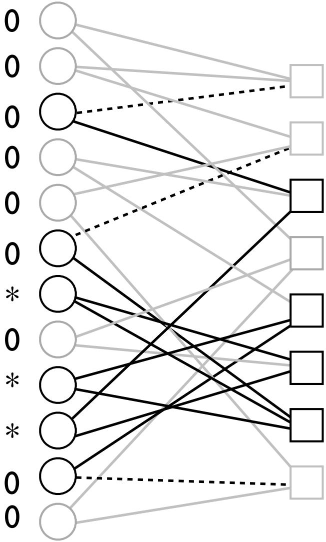

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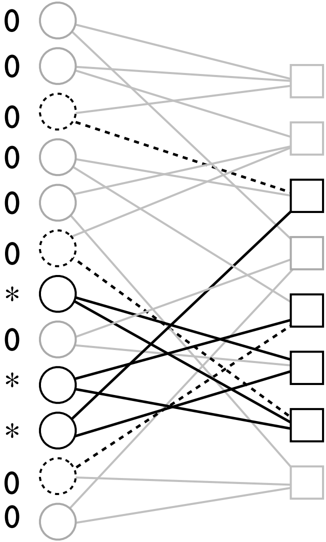

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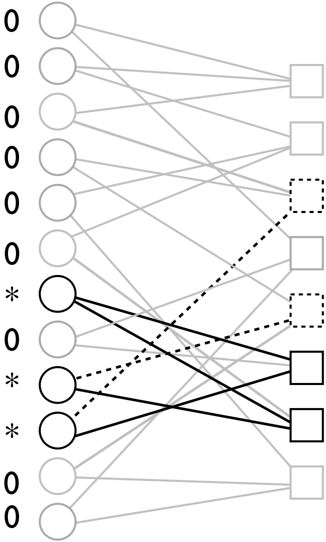

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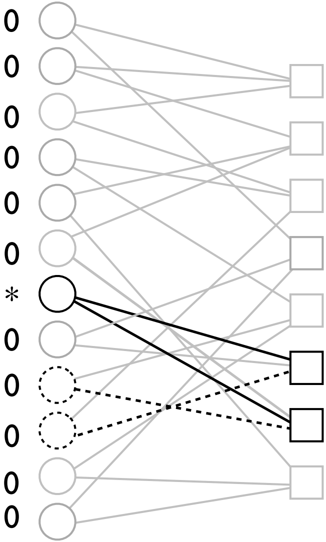

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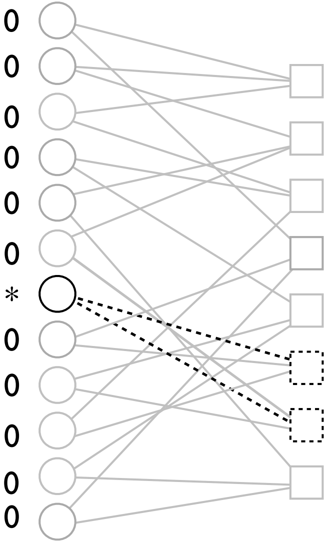


Modern coding theory

- decoding using belief propagation

$$\mu_y(x) = \frac{1}{Z} \prod_{i \in V} \mathbb{P}_{Y|X}(y_i|x_i) \prod_{a \in F} \mathbb{I}(\oplus x_{\partial a} = 0)$$

- use (parallel) sum-product algorithm to find $\mu(x_i)$ and let

$$\widehat{x}_i = \arg\max \mu(x_i)$$

  - minimizes bit error rate

Decoding by sum-product algorithm

- Directly applying parallel sum-product algorithm
  $$\nu_{i\to a}^{(t+1)}(x_i) = \mathbb{P}(y_i|x_i)\prod_{b\in\partial i\setminus\{a\}}\tilde{\nu}_{b\to i}^{(t)}(x_i)$$
  $$\tilde{\nu}_{a\to i}^{(t+1)} = \sum_{x_{\partial a\setminus\{i\}}}\prod_{j\in\partial a\setminus\{i\}}\nu_{j\to a}(x_j)\mathbb{I}(\oplus x_{\partial a}=0)$$
- Notice that all $\nu, \tilde{\nu}$'s can take only one of the following three values:
  $$\nu_{i\to a}(x_i) \in \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1/2\\1/2\end{bmatrix} \right\}$$
  hence, we will map these vectors to symbols $\{0, 1, *\}$
- because (proof by induction)
  - initially,
  $$\tilde{\nu}_{a\to i}^{(0)}(x_i) = \begin{bmatrix}1/2\\1/2\end{bmatrix}, \quad \nu_{i\to a}^{(1)}(x_i) = \begin{cases} \begin{bmatrix}1\\0\end{bmatrix} & \text{if } y_i = 0 \\ \begin{bmatrix}0\\1\end{bmatrix} & \text{if } y_i = 1 \\ \begin{bmatrix}1/2\\1/2\end{bmatrix} & \text{if } y_i = * \end{cases}$$

- recursively, assuming the input messages up to t are one of the three types,

$$\tilde{\nu}_{a\to i}^{(t+1)}(x_i) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if all other bits are determined and add up to 0} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if all other bits are determined and add up to 1} \\ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} & \text{if there is at least one bit that is not determined} \end{cases}$$

$$\nu_{i\to a}^{(t+1)}(x_i) = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if at least one of the input message is } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if at least one of the input message is } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} & \text{if all input messages are } \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \end{cases}$$

- consequently, the messages only take those three values
- we will denote those three types of messages as $0, 1$, and $*$, meaning determined to be 0 or 1, or not determined.

- (simplified) Parallel sum-product for BEC
  - $\nu^{(t)}_{i\to a} \in \{0, 1, *\}$ our belief about x_i
  - $\tilde{\nu}^{(t)}_{a\to i} \in \{0, 1, *\}$ our belief about x_i
  - at iteration 0: $\nu^{(0)}_{i\to a} = y_i$
  - at iteration t:

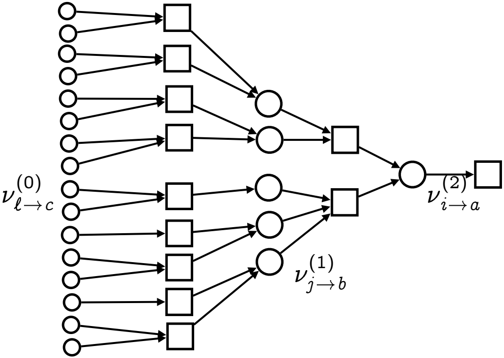
$$\tilde{\nu}^{(t)}_{a\to i} = \begin{cases} * & \text{if any of the incoming messages is a } * \\ \oplus x_{\partial a \setminus i} & \text{otherwise} \end{cases}$$

$$\nu^{(t)}_{i\to a} = \begin{cases} * & \text{if all of the incoming messages are } * \\ x_{b\to i} & \text{otherwise} \end{cases}$$

- this is equivalent to the peeling decoder

Probabilistic analysis: density evolution

- an LDPC code is defined by a graph G
- probabilistic analysis: we want to predict the performance of a given LDPC code G
- to this end, we use **density evolution** on the **computation tree**
- if G is **locally tree like** up to depth k, and if we run sum-product algorithm for k iterations, then the resulting message $\nu_{i\to a}^{(k)}$ is fully described by the computation tree for the message $\nu_{i\to a}^{(k)}$:



- however, it is not always possible to apply density evolution
- a few assumptions
  - sparse random graph construction
    (e.g. random (ℓ, r)-regular graph from the configuration model)
  - asymptotic analysis:
    in the limit $n \to \infty$ but finite number of iterations t
- why do we need these assumptions?
  - it is difficult to analyze one particular graph, so we resort to the expected performance where the expectation also take into account the randomness in the graph generation
  - random sparse graphs are locally tree-like
    - if we consider random (d, d)-regular graphs, the expected number of 2-cycles is $(\frac{1}{n} + \cdots + \frac{d-1}{n}) \times n$, which is small compared to the number of edges

Probabilistic analysis: density evolution

- locally-tree like structure ensures that the incoming messages are independent
- formally, as $n \to \infty$ local neighborhood of a node converges in probability to a random tree

$$\mathbb{P}(\lim_{n\to\infty} \text{depth } k \text{ neighborhood of a random } i \text{ is a tree}) = 1$$

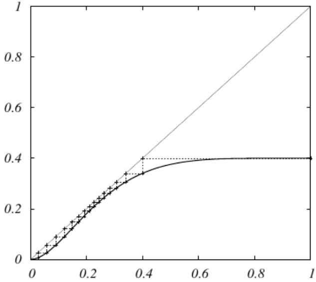
- **density evolution** for (ℓ, r)-regular graph
  - $z_t \in [0,1]$ be the probability a randomly chosen message from $\{\nu_{i\to a}^{(t)}\}$ is an erasure
  - $w_t \in [0,1]$ be the probability a randomly chosen message from $\{\tilde{\nu}_{a\to i}^{(t)}\}$ is an erasure
  - in the limit $n \to \infty$, they satisfy the *density evolution equations*

$$\begin{aligned} w_t &= 1 - (1 - z_{t-1})^{r-1} \\ z_t &= \epsilon w_t^{\ell-1} \end{aligned}$$

$$z_t \quad = \quad \epsilon(1-(1-z_{t-1})^{r-1})^{\ell-1}$$

with initial condition $z_0 = \epsilon$

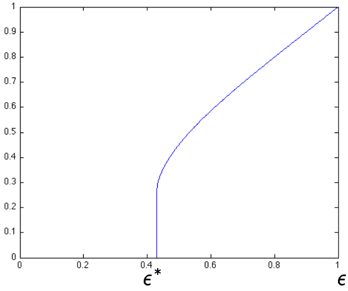
- density evolution for (3,6) code with $\epsilon = 0.4$(left) and 0.45(right)



- rate of this code = 0.5, threshold $\epsilon^* \simeq$ 0.4xxx,
- this simple code achieves rate less than the capacity $= 1 - \epsilon$
- $P_{\text{error}}(t) = \lim_{n\to\infty} P_{\text{error}}(n, t)$
- analyze $\lim_{t\to\infty} \lim_{n\to\infty} P_{\text{error}}(n, t)$, is this what we want?

for a given value of ϵ, we can **numerically** run the density evolution, since it is an evolution of a scalar value, which gives

bit error rate of $(3, 6)$-codes

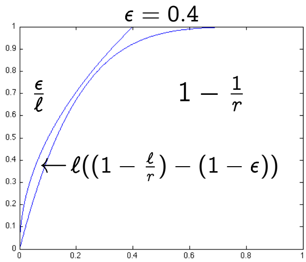


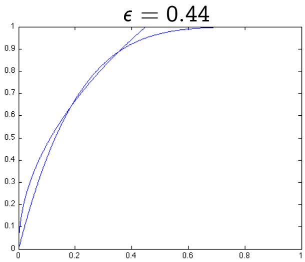
how do we find ϵ^*?

$$z_t \quad = \quad \epsilon(1-(1-z_{t-1})^{r-1})^{\ell-1}$$

let's change the equation to

$$\left(\frac{z_t}{\epsilon}\right)^{1/(\ell-1)} \quad = \quad 1-(1-z_{t-1})^{r-1}$$





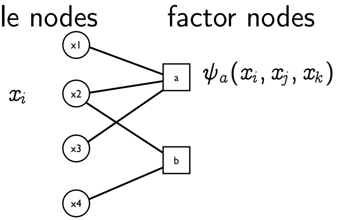
for a given ϵ, if there is no overlap, then achieve zero error probability

$$\int_0^1 \epsilon y^{\ell-1} dy = \frac{\epsilon}{\ell}\,, \quad \int_0^1 (1-(1-x)^{r-1})dx = 1-\frac{1}{r}$$

rate of the code $= 1-\frac{\ell}{r}$ vs. capacity $= 1-\epsilon$

extend this analysis to construct capacity achieving *tornado codes*

density evolution for general message passing algorithms



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  - update:

$$\nu_{i \to a}^{(t+1)} = F_{i \to a}(\{\tilde{\nu}_{b \to i}^{(t)} : b \in \partial i \setminus a\})$$
$$\tilde{\nu}_{a \to i}^{(t)} = G_{a \to i}(\{\nu_{j \to a}^{(t)} : j \in \partial a \setminus i\})$$

  - density evolution equation

$$z^{(t+1)} = F(w_1^{(t)}, \ldots, w_{\ell-1}^{(t)})$$
$$w^{(t)} = G(z_1^{(t)}, \ldots, z_{k-1}^{(t)})$$

- formally, as $n \to \infty$ a randomly chosen message from $\{\nu_{i\to a}^{(t)}\}$ converge in probability to $z^{(t)}$
- who cares about random graphs?
- who cares about asymptotics?

alphabet $x_i \in \mathcal{X}$	messages $\nu_{i\to a} \in \mathcal{Y}$	density $\mathcal{Z}$
discrete $\{0,1\}$	discrete $\{0,1,*\}$	continuous $\mathbb{R}$
discrete	continuous $\mathbb{R}^{\lvert\mathcal{X}\rvert-1}$	distribution over $\mathbb{R}^{\lvert\mathcal{X}\rvert-1}$
continuous $\mathbb{R}$	distribution over $\mathbb{R}$	dist. over dist. over $\mathbb{R}$

- how do we compute evolution of distributions?
  - quantization
  - Gaussian approximation
  - *population dynamics*: represent the density using 'samples'