Computer-Aided Reasoning for Software

CSE507

A Survey of Theory Solvers

Today

Last lecture

- Introduction to Satisfiability Modulo Theories (SMT)

Today

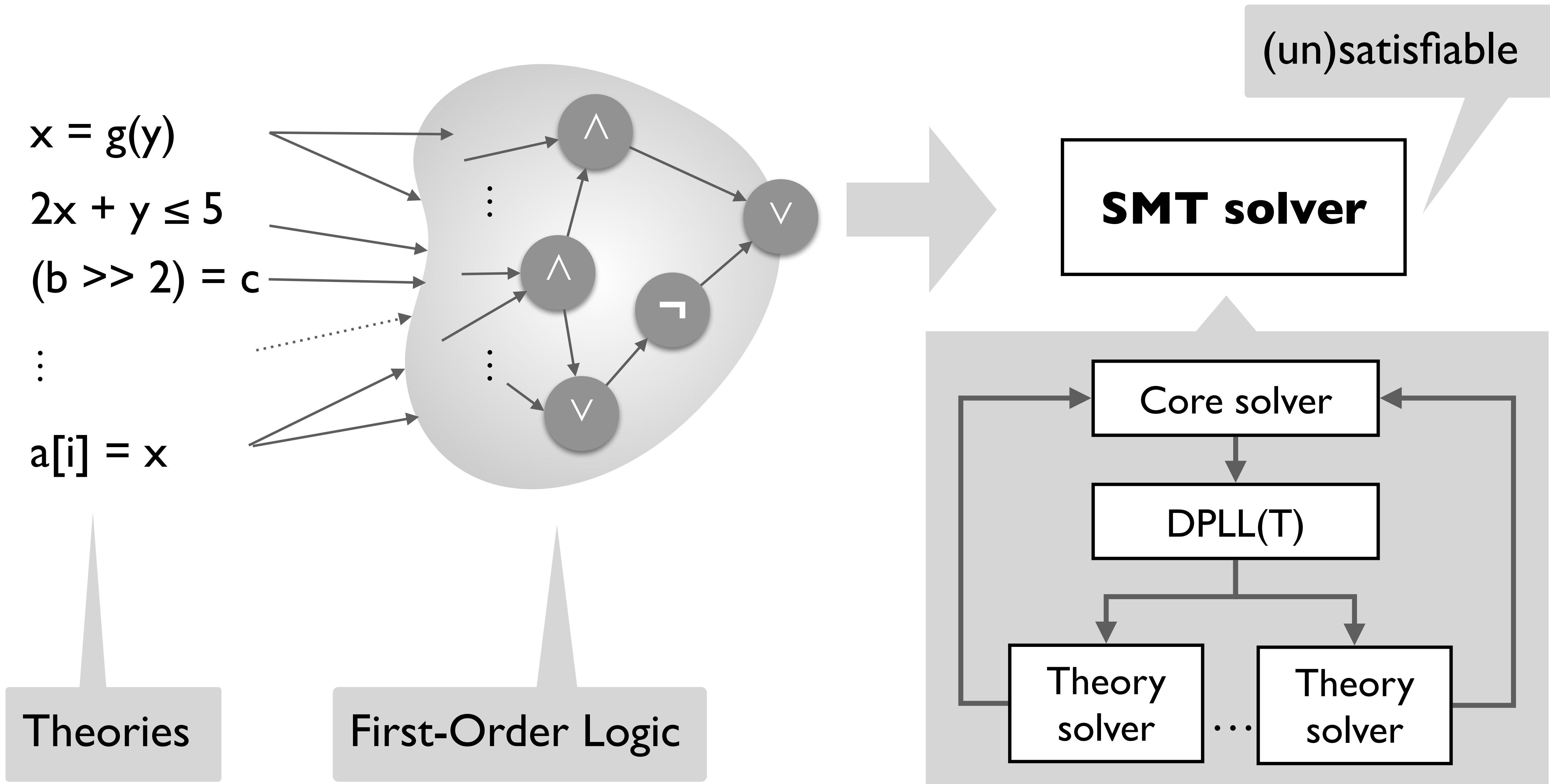
- A quick survey of theory solvers
- An in-depth look at the core theory solver (theory of equality and uninterpreted functions)

Reminders

- HW1 due tonight.
- Project proposal due next week. Find a partner and start brainstorming if you haven't already!

topics

Recall: Satisfiability Modulo Theories (SMT)



A brief survey of common theory solvers

$x = g(y)$

Core solver

$2.3x + y \leq 5$

Theory solver

$2i + j \leq 5$

Theory solver

$(b >> 2) = c$

Theory solver

$a[i] = x$

Theory solver

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2.3x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Bitvectors

$a[i] = x$

Arrays

- **Conjunctions** of linear constraints over R
  - Can be decided in polynomial time, but in practice solved with the **General Simplex** method (worst case exponential)
  - Can also be decided with **Fourier-Motzkin** elimination (exponential)

A brief survey of common theory solvers

x = g(y)

Equality and UF

2.3x + y ≤ 5

Linear Real Arithmetic

2i + j ≤ 5

Linear Integer Arithmetic

(b >> 2) = c

Bitvectors

a[i] = x

Arrays

- **Conjunctions** of linear constraints over Z
  - **Branch-and-cut** (based on Simplex)
  - **Omega Test** (extension of Fourier-Motzkin)
- **Small-Domain Encoding** used for arbitrary combinations of linear constraints over Z
- NP-complete

A brief survey of common theory solvers

x = g(y)

Equality and UF

2.3x + y ≤ 5

Linear Real Arithmetic

2i + j ≤ 5

Linear Integer Arithmetic

(b >> 2) = c

Bitvectors

a[i] = x

Arrays

- **Arbitrary combination** of constraints over bitvectors
  - **Bit blasting** (reduction to SAT)
  - NP-complete

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2.3x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Bitvectors

$a[i] = x$

Arrays

- **Conjunctions** of constraints over read/write terms in the theory of arrays
  - Reduce to $T_=$ satisfiability
  - NP-complete (because the reduction introduces disjunctions)

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

- **Conjunctions** of equality constraints over uninterpreted functions
  - **Congruence closure**
  - Polynomial time

$2.3x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Bitvectors

$a[i] = x$

Arrays

Theory of equality and UF ($T_=$)

Signature (all symbols)

- $\{=, a, b, c, \ldots, f, g, \ldots, \cancel{p}, \cancel{q}, \ldots\}$

Axioms

- reflexivity: $\forall x.\ x = x$
- symmetry: $\forall x, y.\ x = y \rightarrow y = x$
- transitivity: $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \leq i \leq n}\, x_i = y_i) \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \leq i \leq n}\, x_i = y_i) \rightarrow p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)$

Replace predicates with equality constraints over functions:

- introduce a fresh constant T
- for each predicate p, introduce a fresh function f_p
- $p(x_1, \ldots, x_n) \rightsquigarrow f_p(x_1, \ldots, x_n) = T$

Is a conjunction of $T_=$ literals satisfiable?

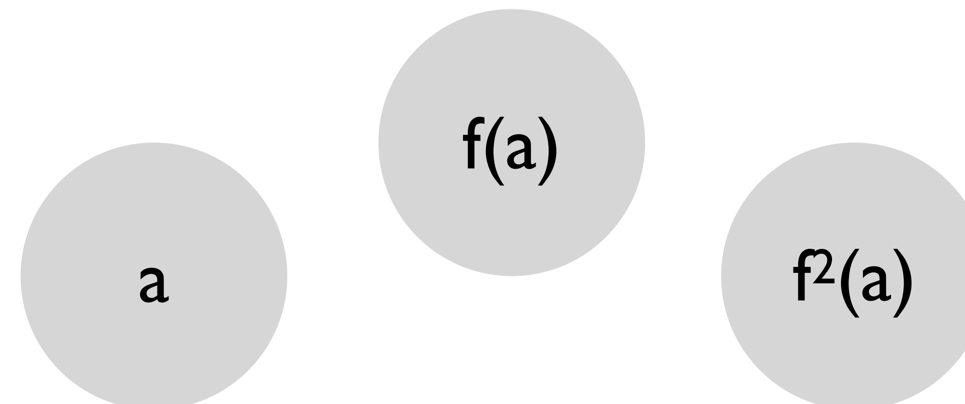
$$f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$$

Is a conjunction of $T_=$ literals satisfiable?

$$f^3(a) = a \land f^5(a) = a \land f(a) \neq a$$

Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**



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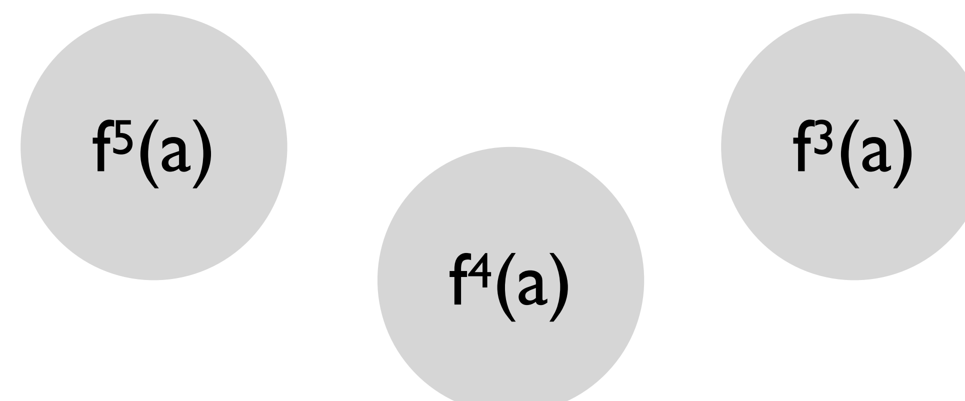

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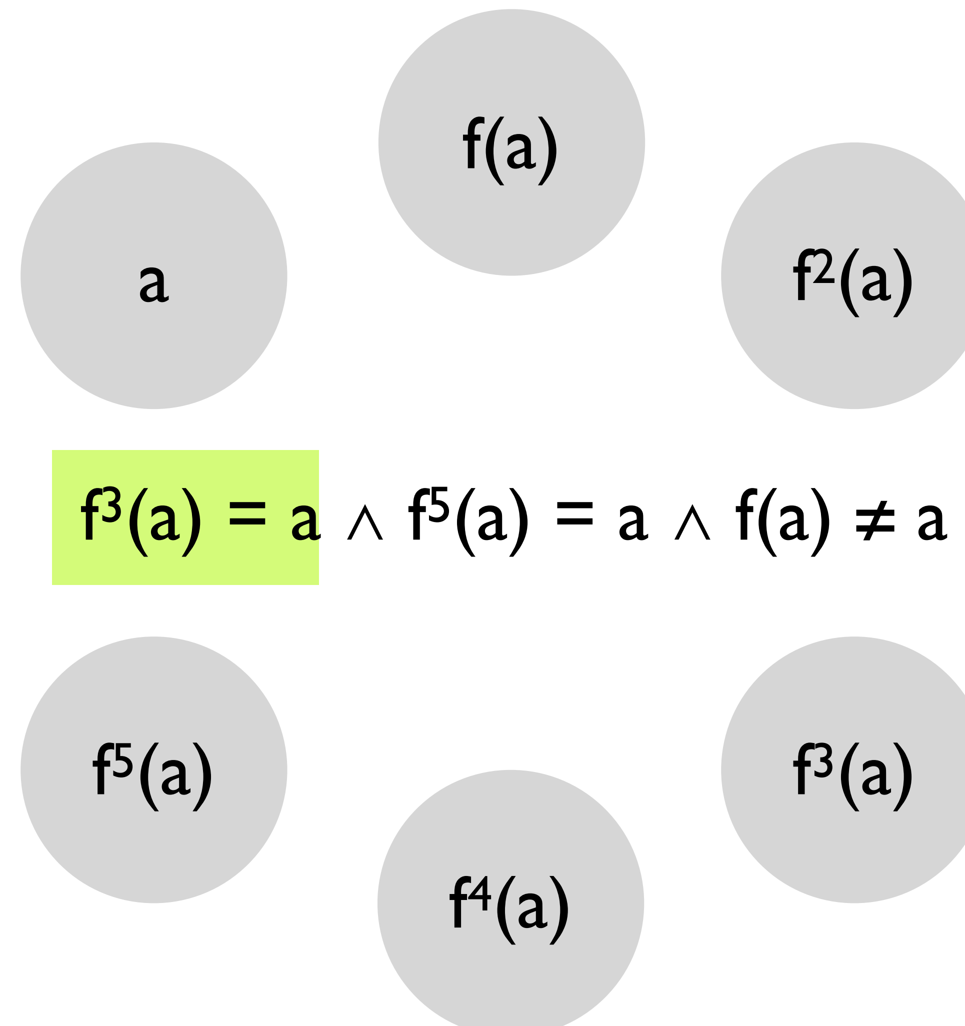

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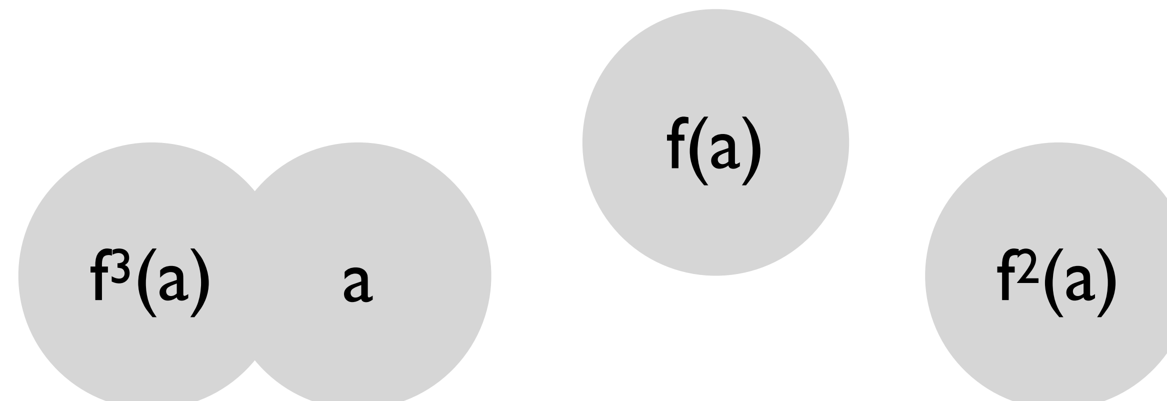

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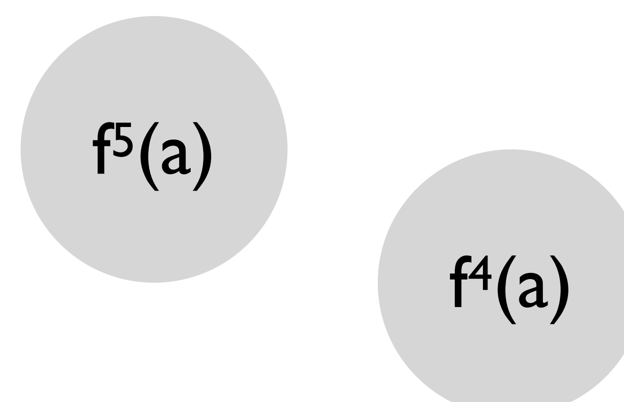

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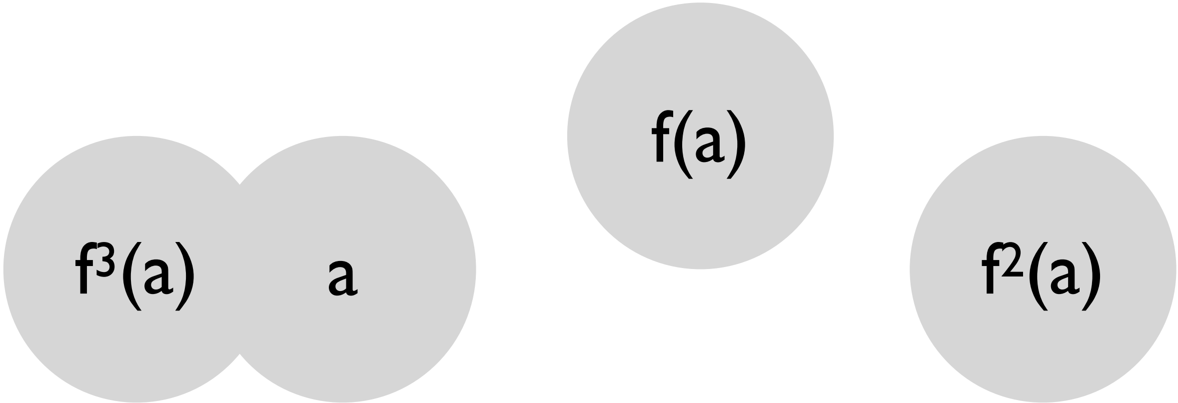

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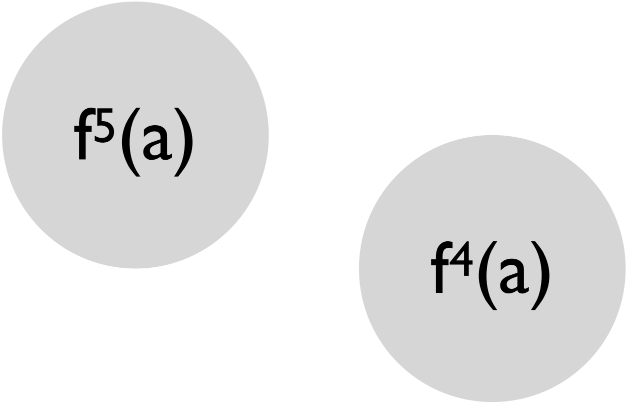

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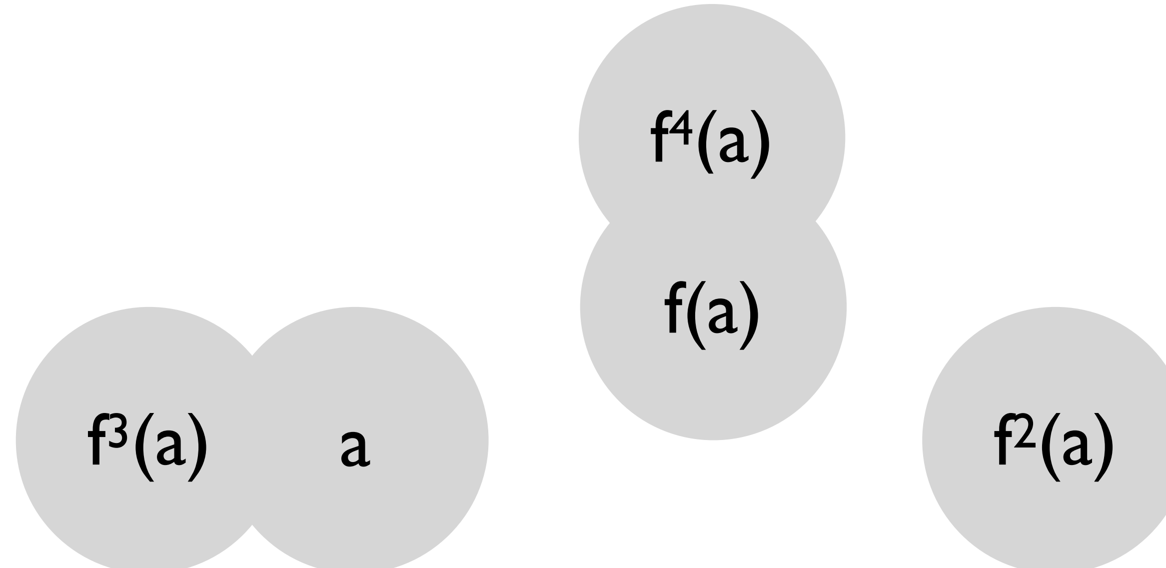

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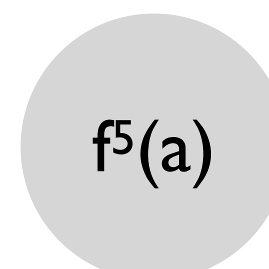

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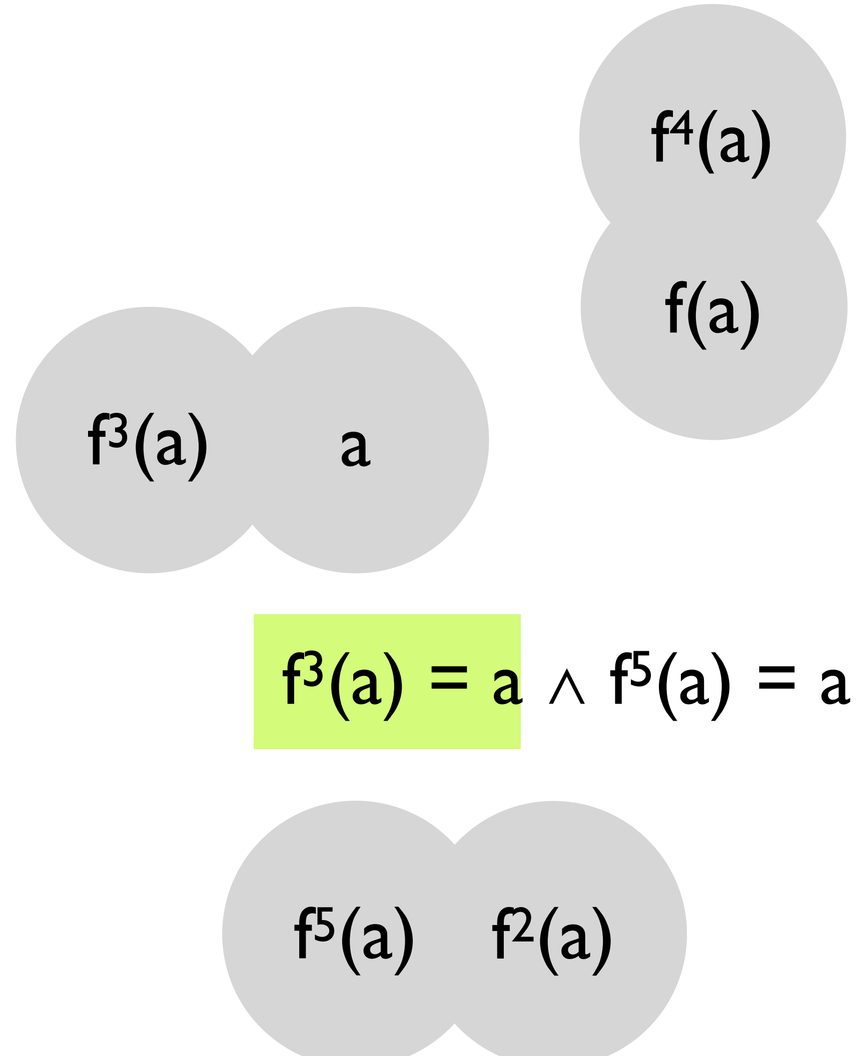

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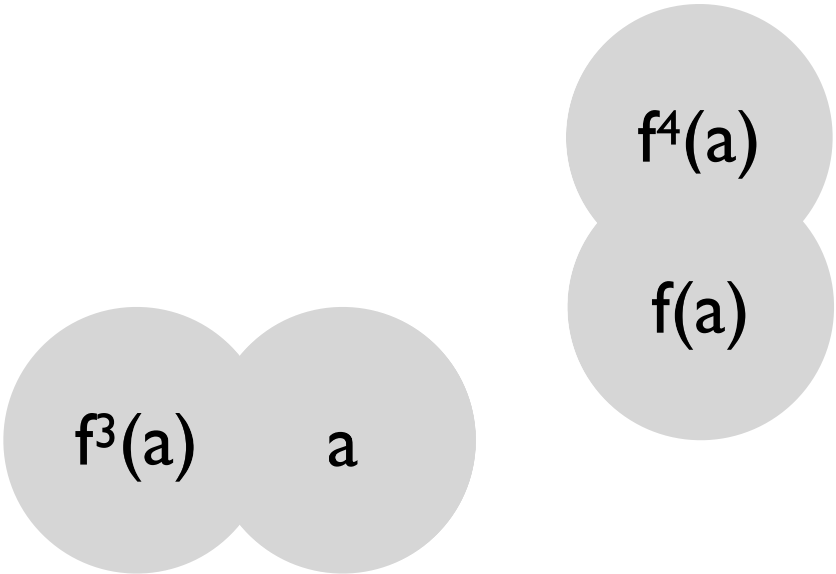

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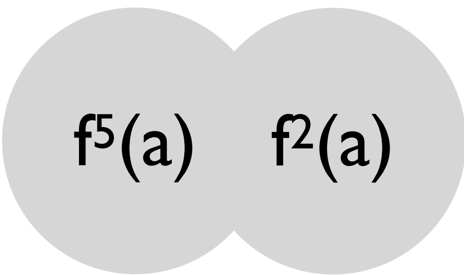

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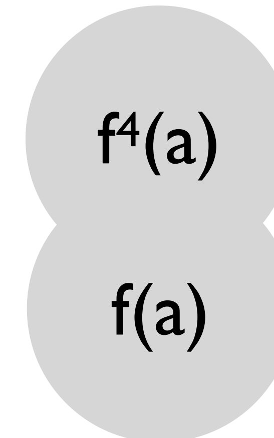

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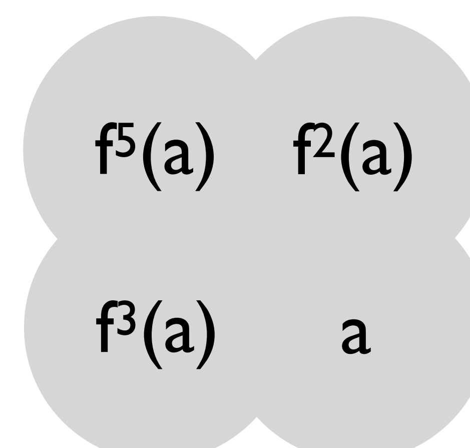

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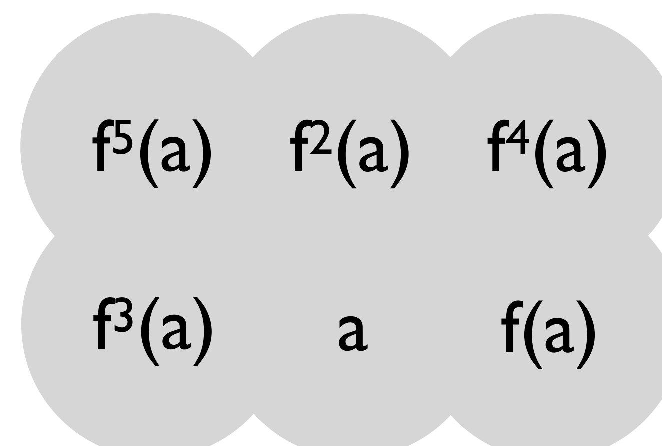

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- Otherwise, output SAT

UNSAT

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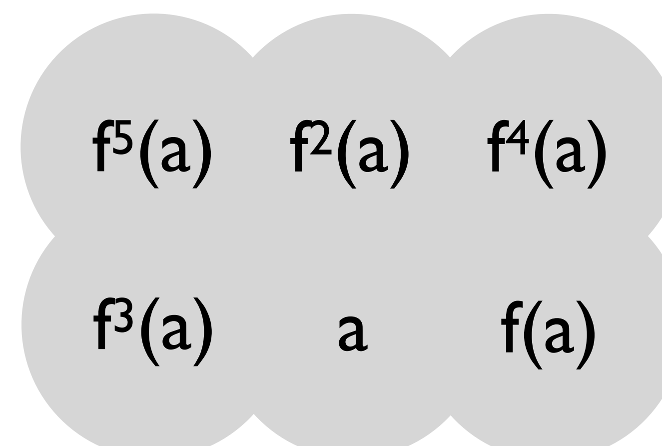

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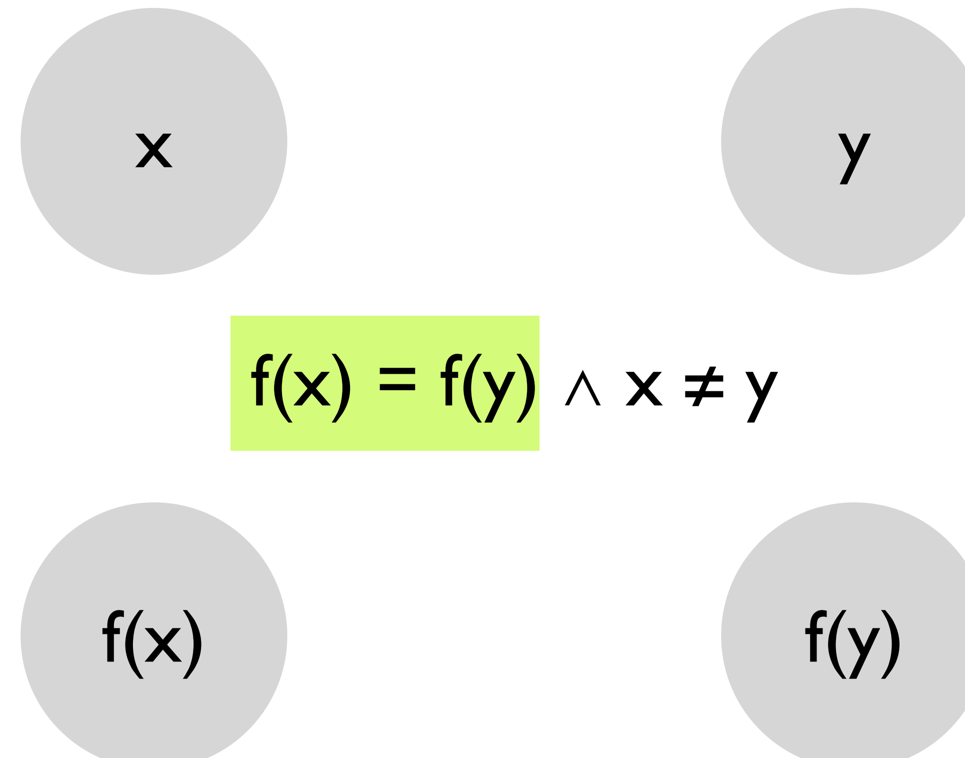

$f(x) = f(y) \wedge x \neq y$

Congruence closure algorithm: another example

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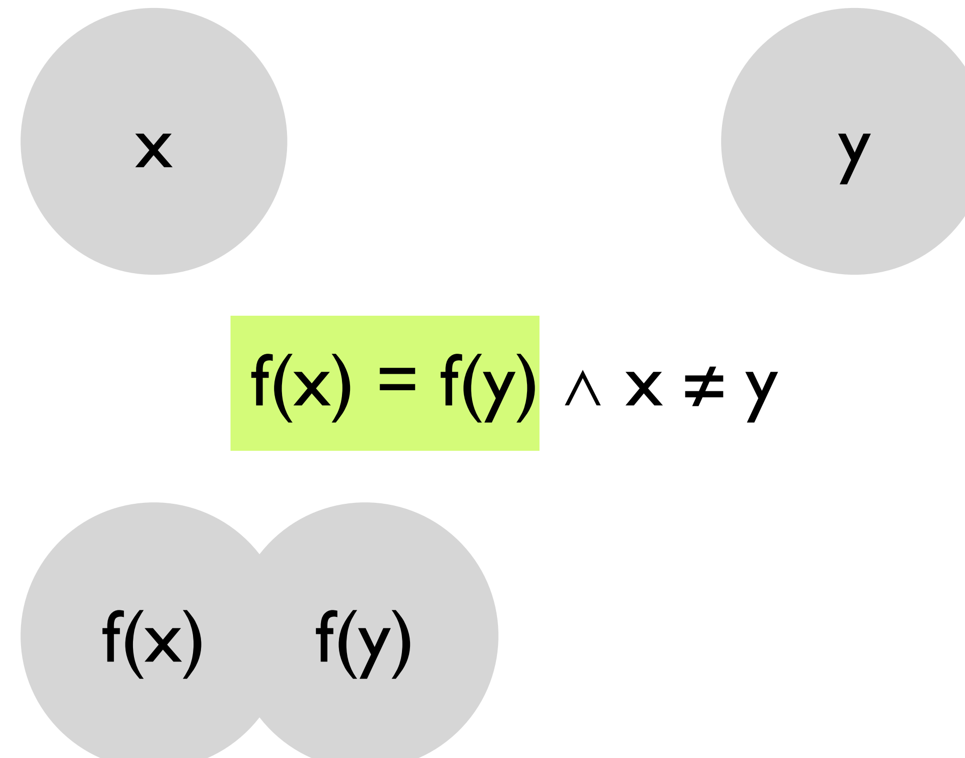

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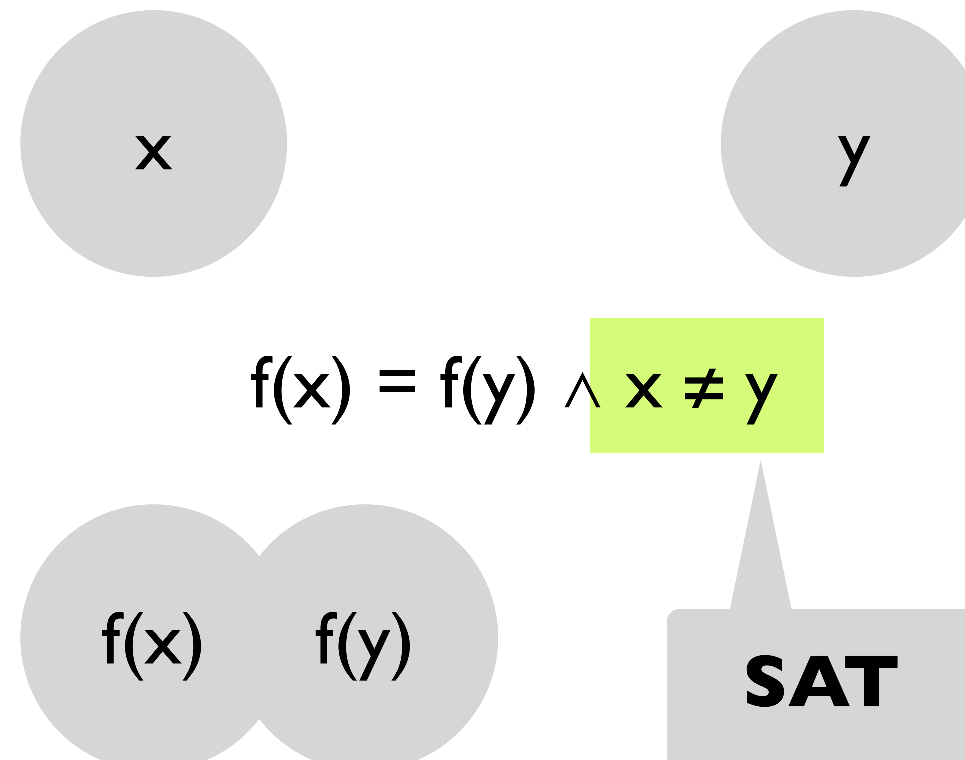

Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

An equivalence relation R is a **congruence relation** if for every n-ary function f

$$\forall \overline{x}, \overline{y}.\ \bigwedge R(x_i, y_i) \rightarrow R(f(\overline{x}), f(\overline{y}))$$

The **equivalence class** of an element $s \in S$ under an equivalence relation R:

$$\{\ s' \in S \mid R(s, s')\ \}$$

An equivalence class is called a **congruence class** if R is a congruence relation.

What is the equivalence class of 9 under $\equiv_3$?

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

What is the equivalence closure of $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle\}$?

$$R^E = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle \\ \langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle, \\ \langle a, c\rangle, \langle c, a\rangle\}$$

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

The **congruence closure** R^C of a binary relation R is the smallest congruence relation that contains R.

The congruence closure algorithm computes the congruence closure of the equality relation over terms asserted by a conjunctive quantifier-free formula in $T_=$.

Congruence closure algorithm: data structure

$$f(a, b) = a \land f(f(a, b), b) \neq a$$

Congruence closure algorithm: data structure

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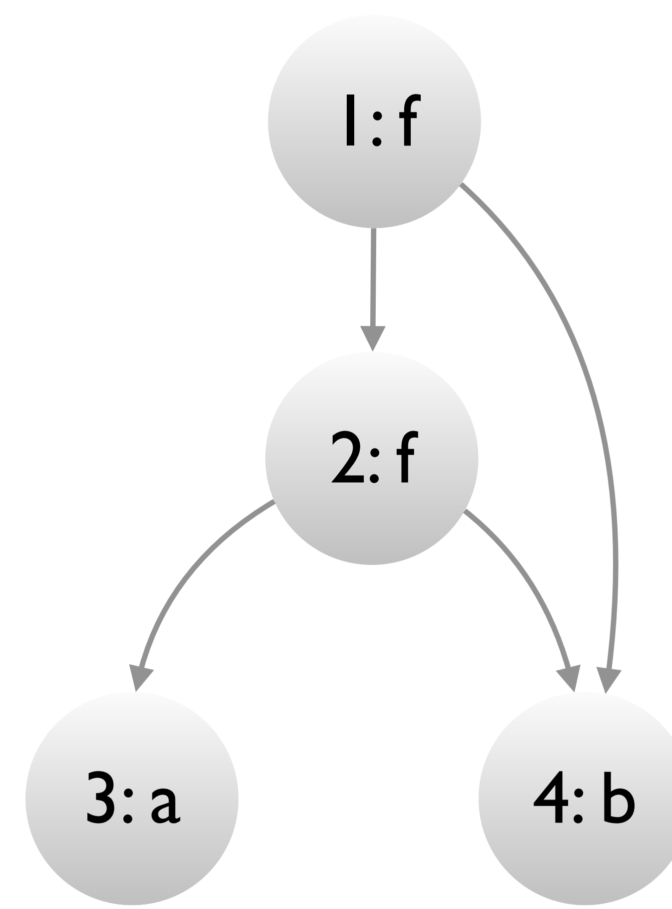
- Represent subterms with a DAG



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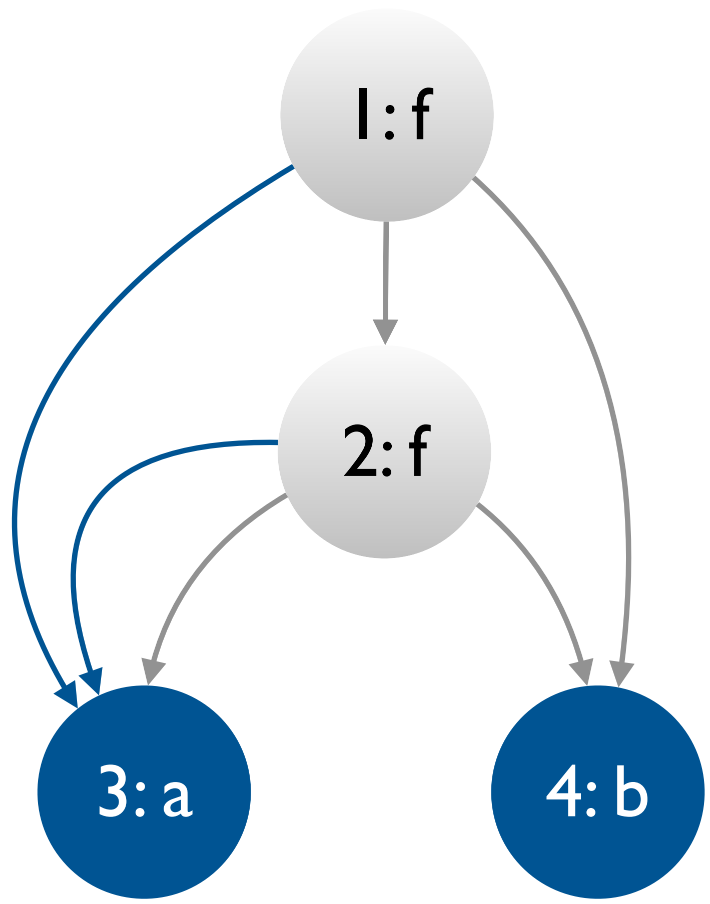
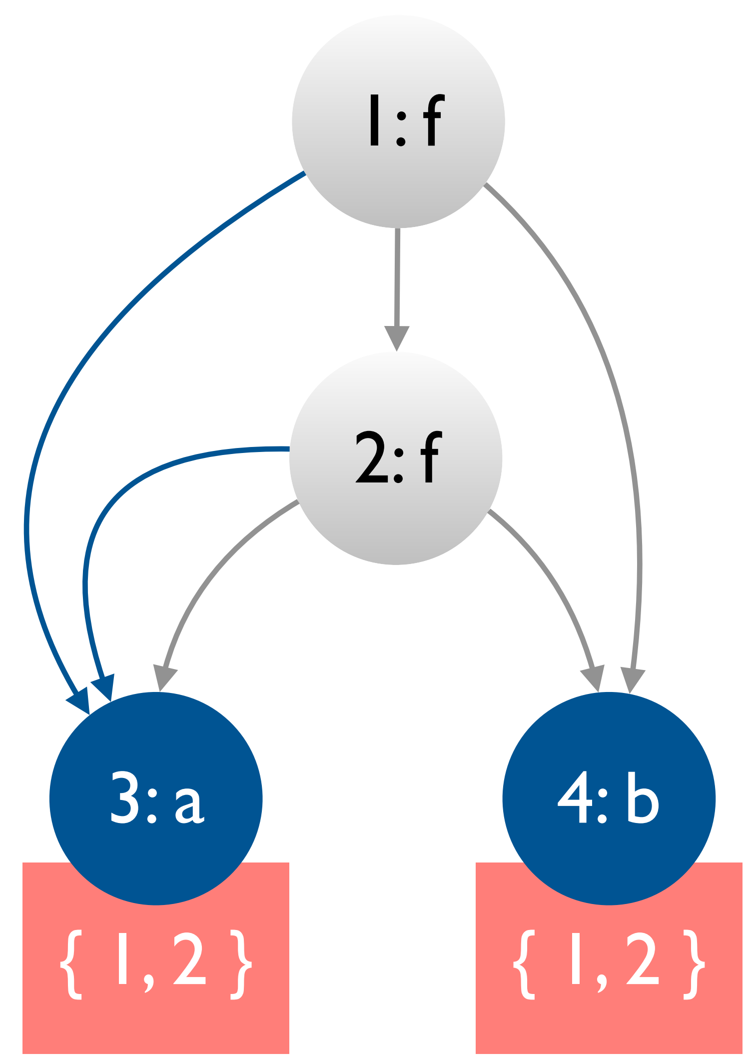
- Represent subterms with a DAG
- Each node has a **find** pointer to another node in its congruence class (or to itself if it is the **representative**)



Congruence closure algorithm: data structure

- Represent subterms with a DAG
- Each node has a **find** pointer to another node in its congruence class (or to itself if it is the **representative**)
- Each representative has a **ccp** field that stores all parents of all nodes in its congruence class.

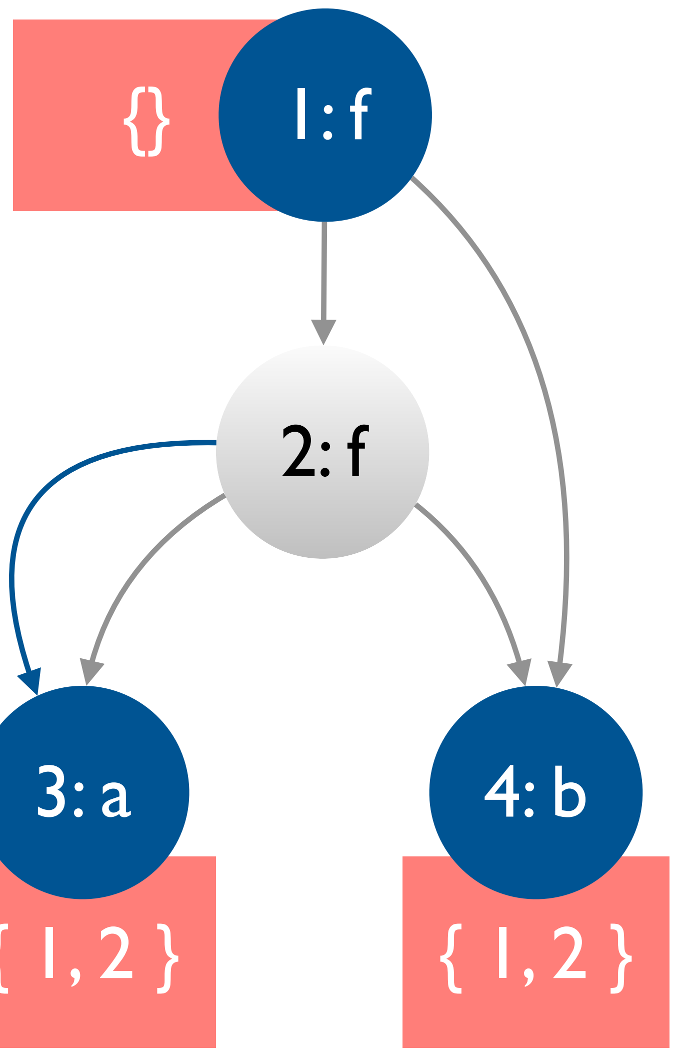
$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.
- UNION combines equivalence classes for nodes i_1 and i_2:
  - $n_1, n_2 \leftarrow \text{FIND}(i_1), \text{FIND}(i_2)$
  - $n_1.\text{find} \leftarrow n_2$
  - $n_2.\text{ccp} \leftarrow n_1.\text{ccp} \cup n_2.\text{ccp}$
  - $n_1.\text{ccp} \leftarrow \varnothing$

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



What is UNION(1, 2)?

Congruence closure algorithm: union-find

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  - $n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)
  - n_1.find $\leftarrow n_2$
  - n_2.ccp $\leftarrow n_1$.ccp $\cup$ n_2.ccp
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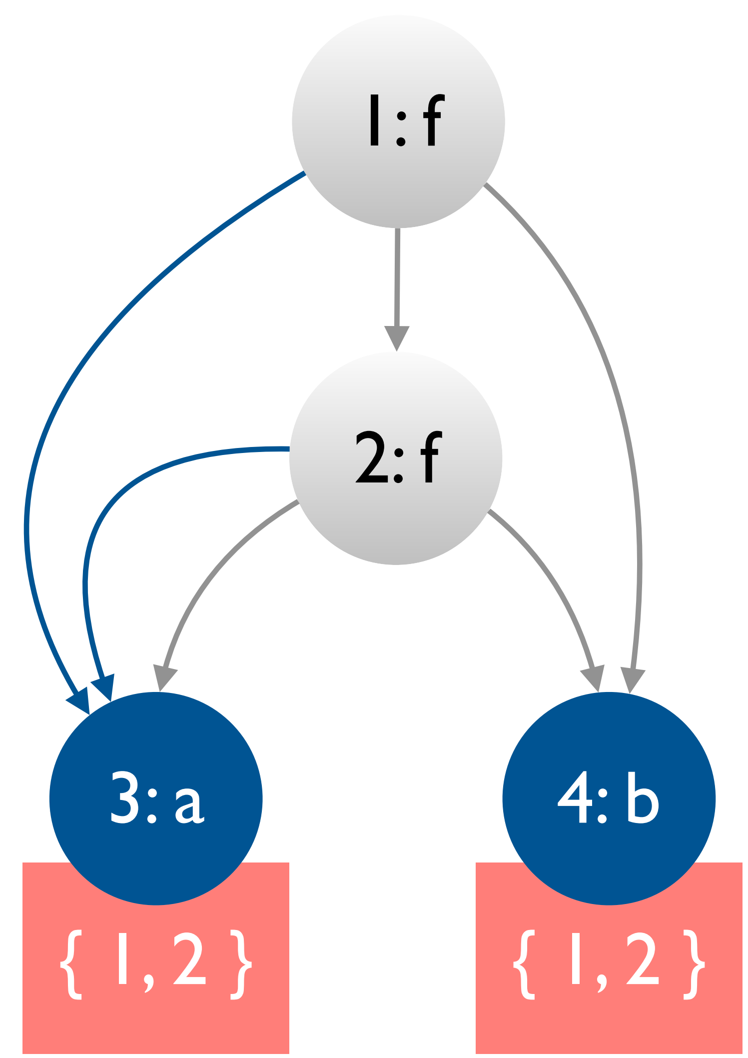


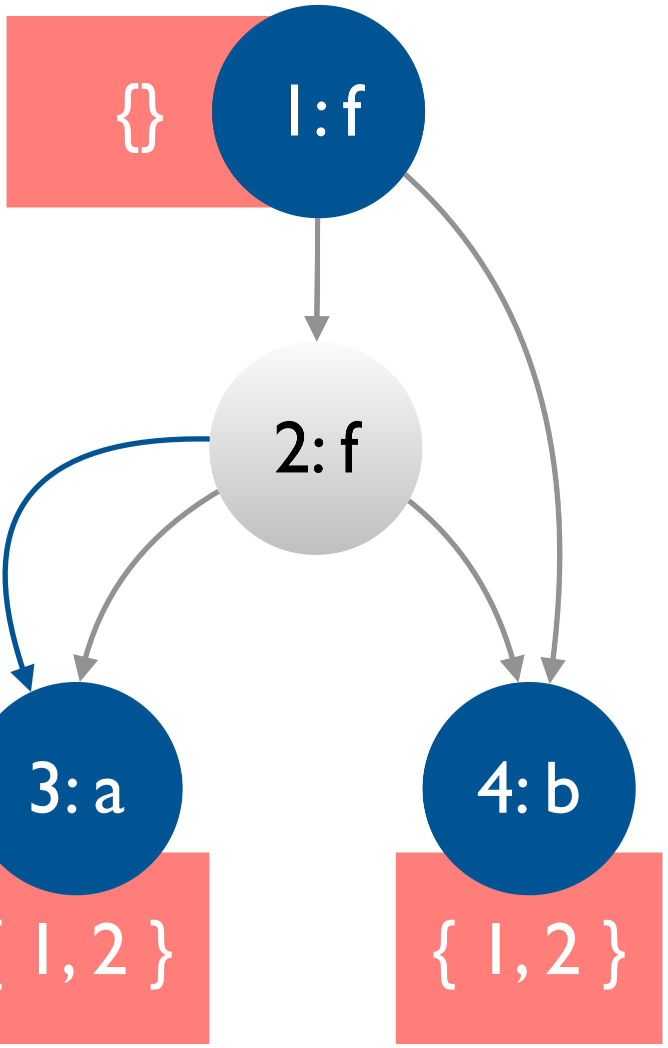
What is UNION(1, 2)?

Congruence closure algorithm: congruent

- CONGRUENT takes as input two nodes and returns true iff their
  - functions are the same
  - corresponding arguments are in the same congruence class

CONGRUENT(1, 2)?

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: merge

MERGE (i_1 , i_2)

$n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)

if $n_1 = n_2$ **then return**

$p_1, p_2 \leftarrow n_1$.ccp, n_2.ccp

UNION(n_1, n_2)

for each $t_1, t_2 \in p_1 \times p_2$

if FIND(t_1) $\neq$ FIND(t_2) $\wedge$ CONGRUENT(t_1, t_2)

then MERGE(t_1, t_2)

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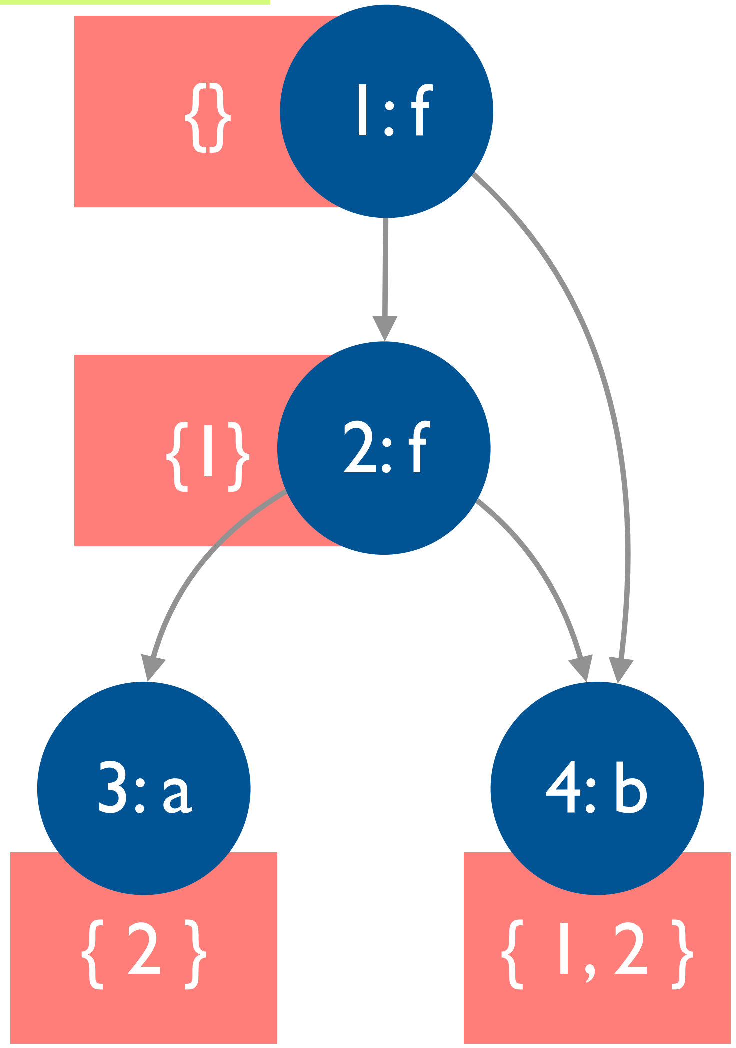

Congruence closure algorithm: merge

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- **if** $n_1 = n_2$ **then return**
- $p_1, p_2 \leftarrow n_1.ccp, n_2.ccp$
- UNION(n_1, n_2)
- **for** each $t_1, t_2 \in p_1 \times p_2$
  - **if** FIND$(t_1) \neq$ FIND$(t_2) \wedge$ CONGRUENT(t_1, t_2)
  - **then** MERGE(t_1, t_2)

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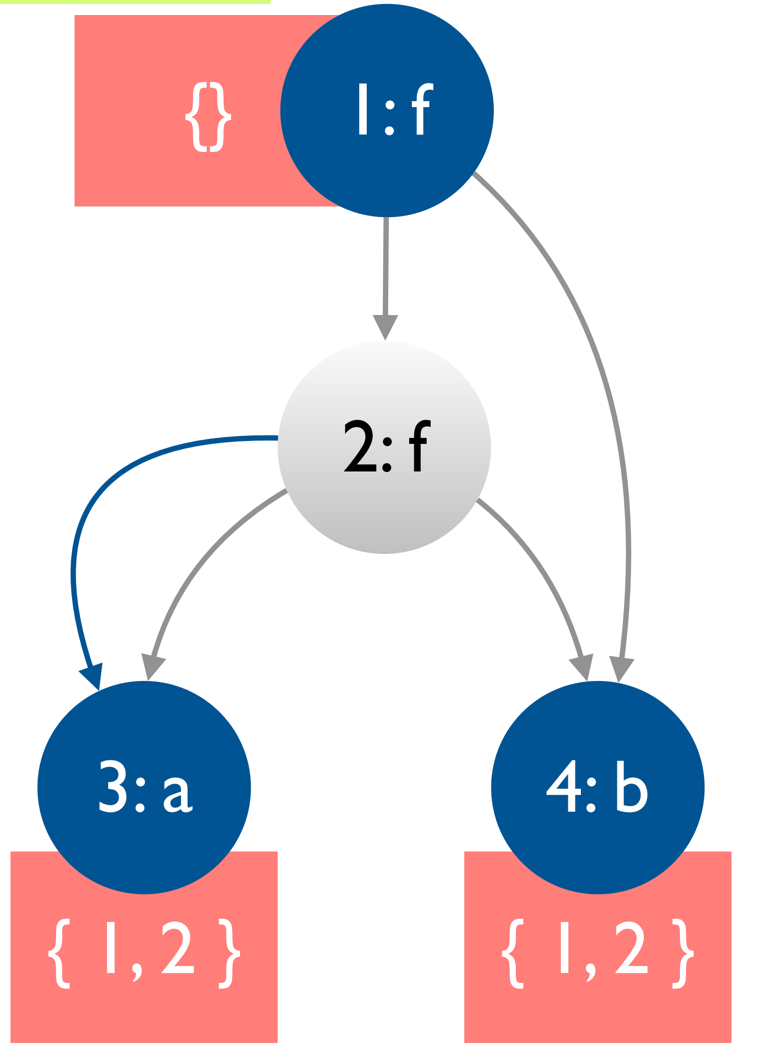
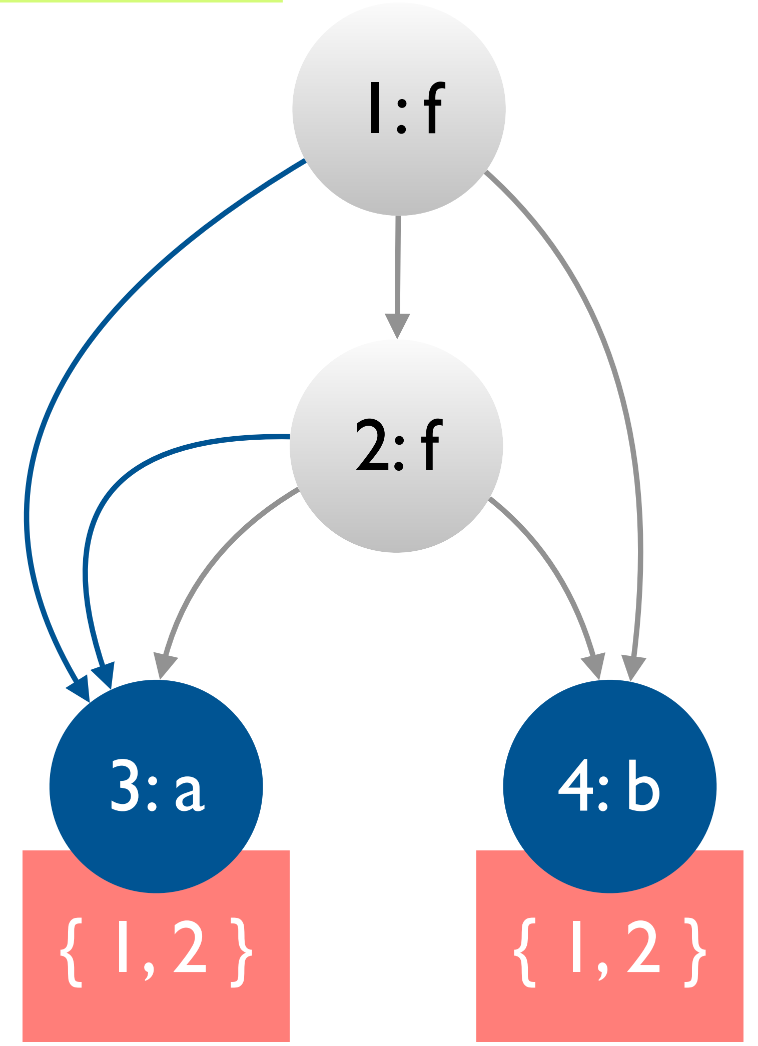


Congruence closure algorithm: merge

$f(a, b) = a \wedge f(f(a, b), b) \neq a$

```
MERGE (i1 , i2)
    n1, n2 ← FIND(i1), FIND(i2)
    if n1 = n2 then return
    p1, p2 ← n1.ccp, n2.ccp
    UNION(n1, n2)
    for each t1, t2 ∈ p1 × p2
        if FIND(t1) ≠ FIND(t2) ∧ CONGRUENT(t1, t2)
        then MERGE(t1, t2)
```



Congruence closure algorithm: deciding $T_=$

$f(a, b) = a \wedge f(f(a, b), b) \neq a$

DECIDE (F)

construct the DAG for F's subterms

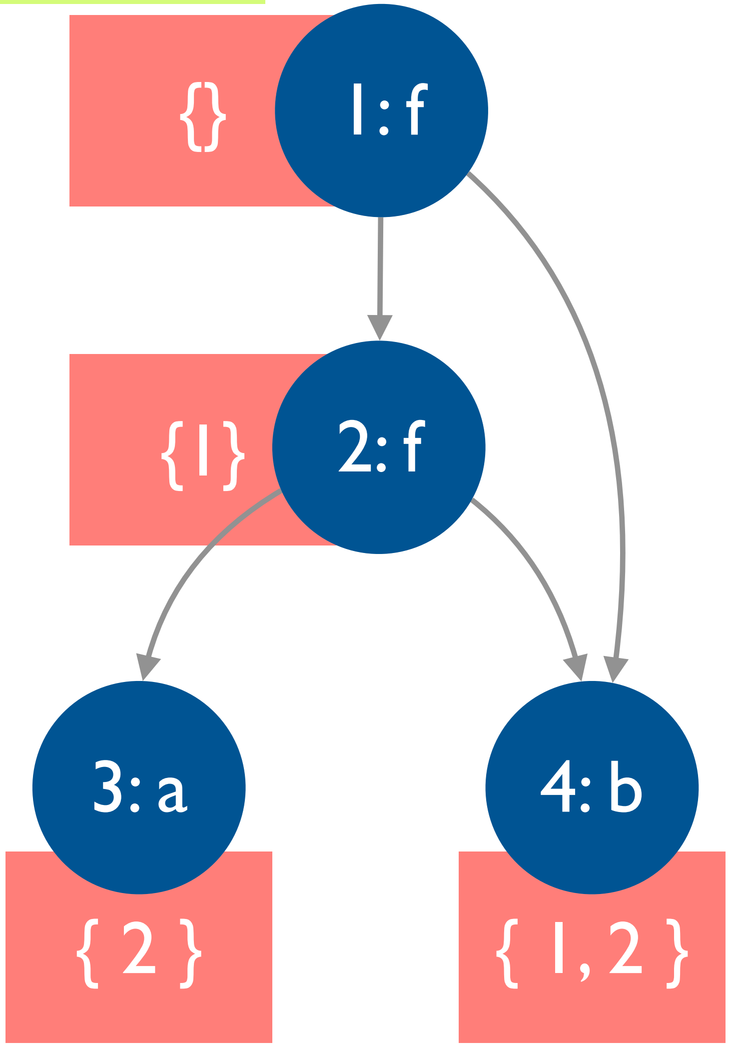
for $s_i = t_i \in F$

MERGE(s_i, t_i)

for $s_i \neq t_i \in F$

if FIND(s_i) = FIND(t_i) **then return** UNSAT

return SAT



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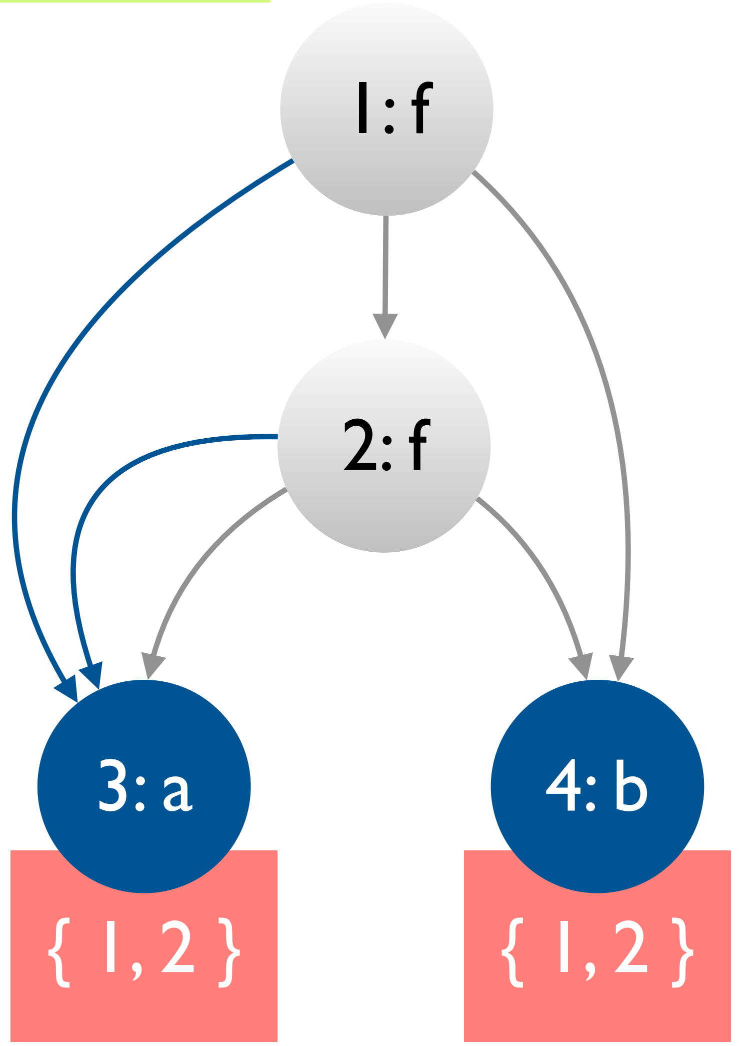

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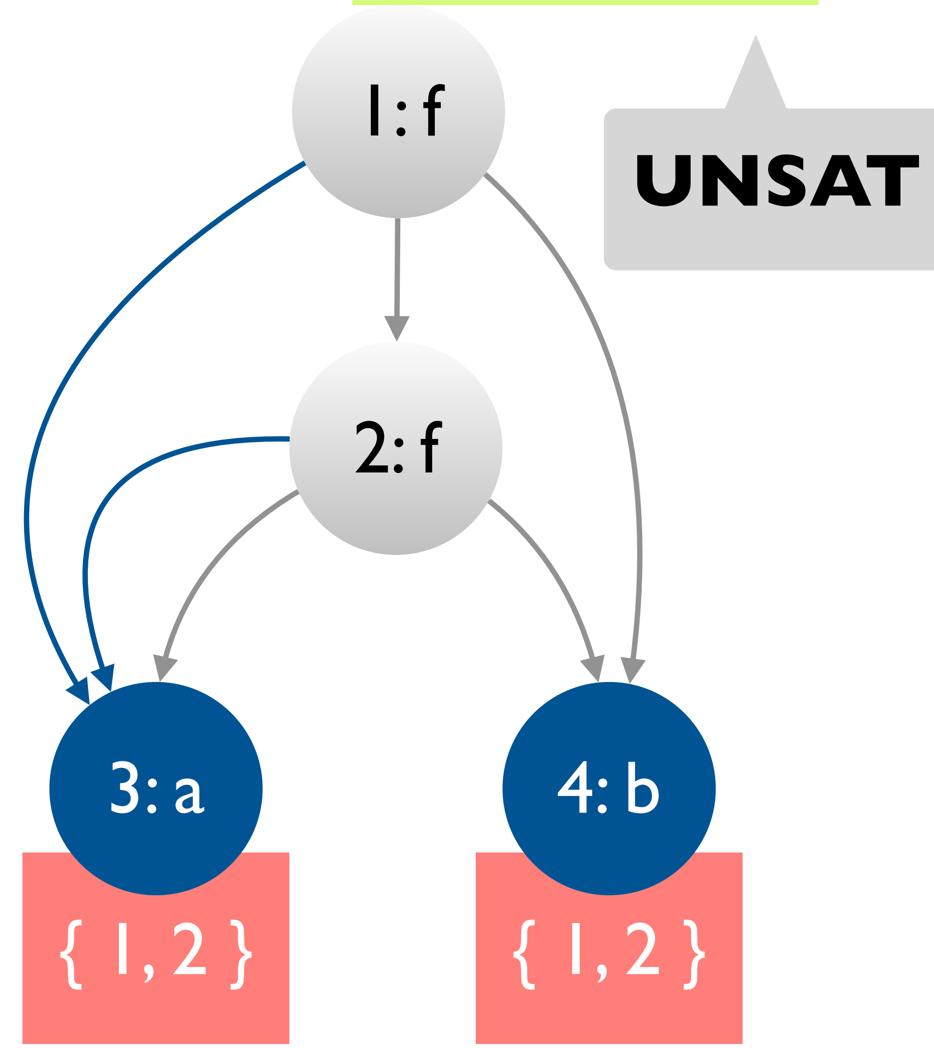
  return SAT

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Summary

Today

- A brief survey of theory solvers
- Congruence closure algorithm for deciding conjunctive $T_=$ formulas

Next lecture

- Combining (decision procedures for different) theories