Computer-Aided Reasoning for Software

CSE507

A Survey of Theory Solvers

Emina Torlak
emina@cs.washington.edu

Today

Last lecture

- Introduction to Satisfiability Modulo Theories (SMT)

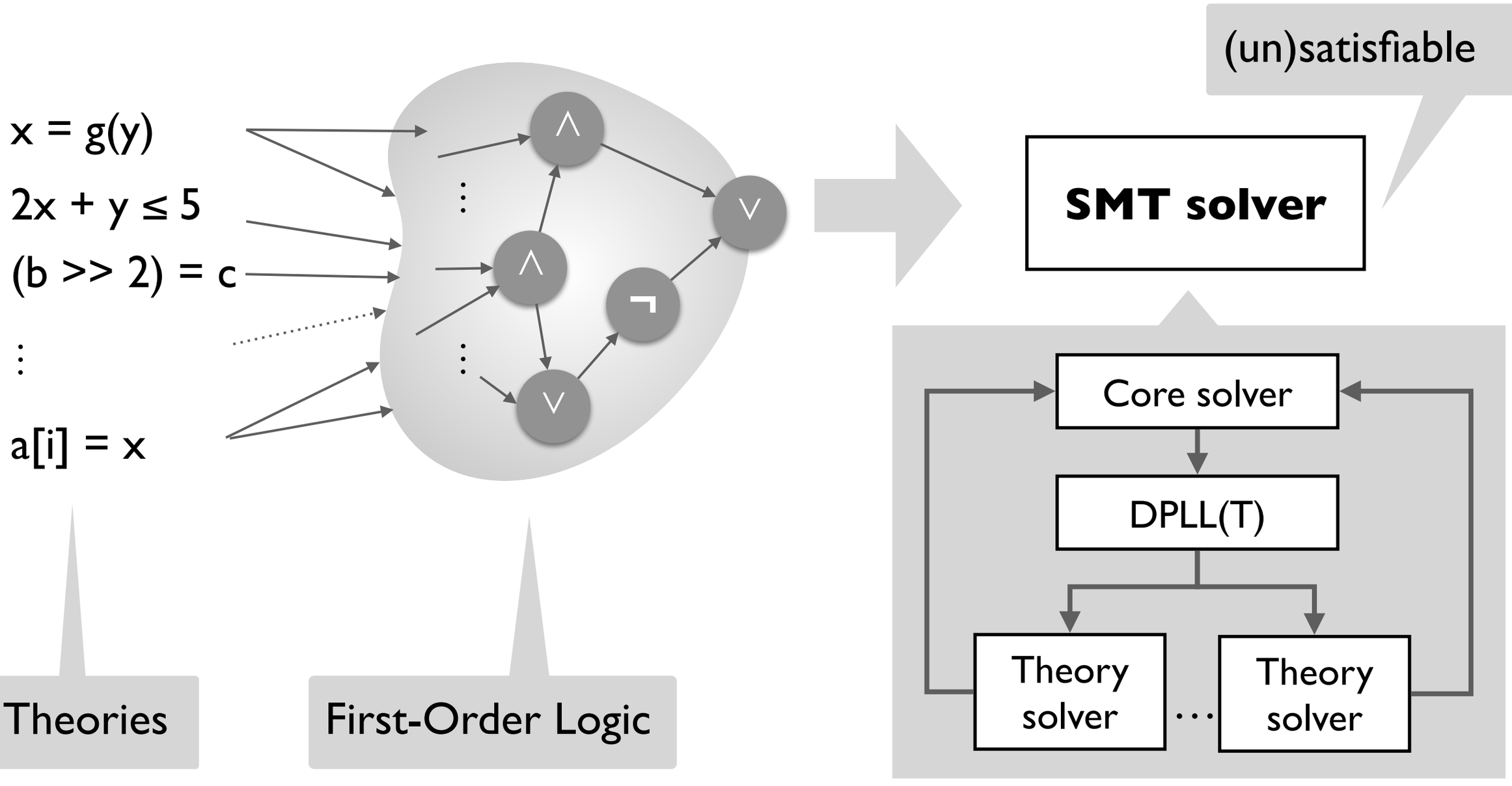
Today

- A quick survey of theory solvers
- An in-depth look at the core theory solver (theory of equality and uninterpreted functions)

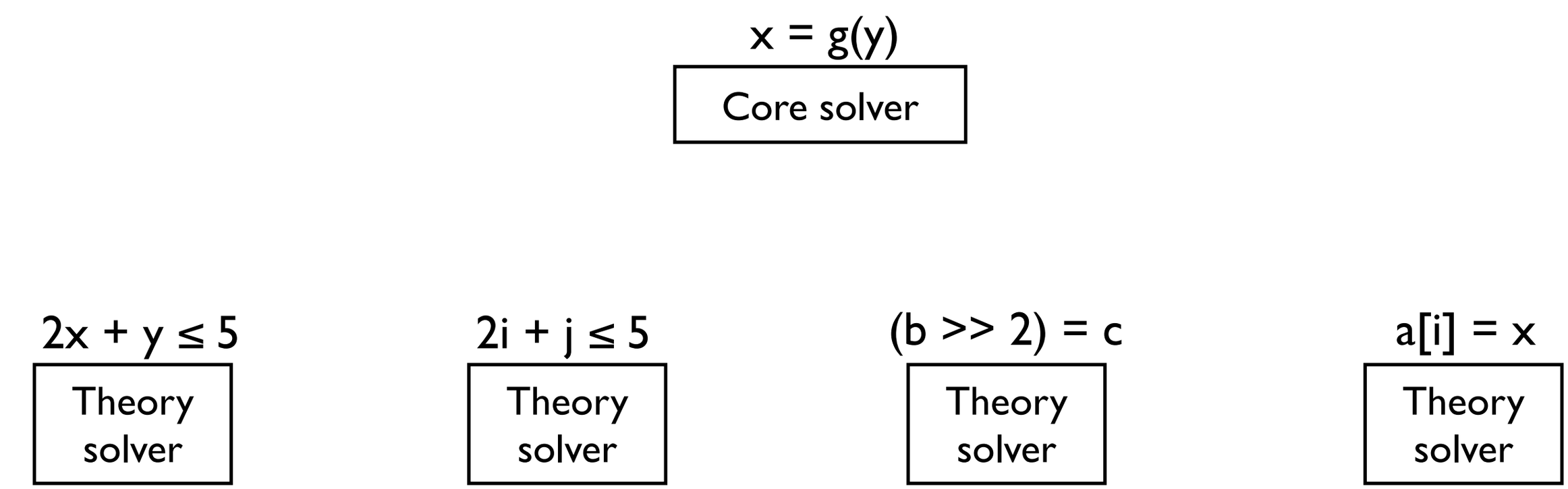
Reminders

- HW1 due tonight.
- Project proposal due next week. Find a partner and start brainstorming if you haven't already!

Recall: Satisfiability Modulo Theories (SMT)



A brief survey of common theory solvers



A brief survey of common theory solvers

x = g(y)

Equality and UF

2x + y ≤ 5

Linear Real Arithmetic

2i + j ≤ 5

Linear Integer Arithmetic

(b >> 2) = c

Bitvectors

a[i] = x

Arrays

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Bitvectors

$a[i] = x$

Arrays

- **Conjunctions** of linear constraints over R
  - Can be decided in polynomial time, but in practice solved with the **General Simplex** method (worst case exponential)
  - Can also be decided with **Fourier-Motzkin** elimination (exponential)

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

- **Conjunctions** of linear constraints over Z
  - **Branch-and-cut** (based on Simplex)
  - **Omega Test** (extension of Fourier-Motzkin)
- **Small-Domain Encoding** used for arbitrary combinations of linear constraints over Z
- NP-complete

$(b >> 2) = c$

Bitvectors

$a[i] = x$

Arrays

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Bitvectors

- **Arbitrary combination** of constraints over bitvectors
  - **Bit blasting** (reduction to SAT)
  - NP-complete

$a[i] = x$

Arrays

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Bitvectors

$a[i] = x$

Arrays

- **Conjunctions** of constraints over read/write terms in the theory of arrays
  - Reduce to $T_=$ satisfiability
  - NP-complete (because the reduction introduces disjunctions)

A brief survey of common theory solvers

x = g(y)

Equality and UF

- **Conjunctions** of equality constraints over uninterpreted functions
  - **Congruence closure**
  - Polynomial time

2x + y ≤ 5

Linear Real Arithmetic

2i + j ≤ 5

Linear Integer Arithmetic

(b >> 2) = c

Bitvectors

a[i] = x

Arrays

Theory of equality and UF ($T_=$)

Signature (all symbols)

- {=, a, b, c, …, f, g, …, p, q, …}

Axioms

- reflexivity: $\forall x.\ x = x$
- symmetry: $\forall x, y.\ x = y \rightarrow y = x$
- transitivity: $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \le i \le n} x_i = y_i) \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \le i \le n} x_i = y_i) \rightarrow p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)$

Theory of equality and UF ($T_=$)

Signature (all symbols)

- {=, a, b, c, …, f, g, …, p, q, …}

Axioms

- reflexivity: $\forall x.\ x = x$
- symmetry: $\forall x, y.\ x = y \rightarrow y = x$
- transitivity: $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \le i \le n} x_i = y_i) \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
- ✗ congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \le i \le n} x_i = y_i) \rightarrow p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)$

Replace predicates with equality constraints over functions:

- introduce a fresh constant T
- for each predicate p, introduce a fresh function f_p
- $p(x_1, \ldots, x_n) \rightsquigarrow f_p(x_1, \ldots, x_n) = T$

Is a conjunction of $T_=$ literals satisfiable?

$$f(f(f(a))) = a \wedge f(f(f(f(f(a))))) = a \wedge f(a) \neq a$$

Is a conjunction of $T_=$ literals satisfiable?

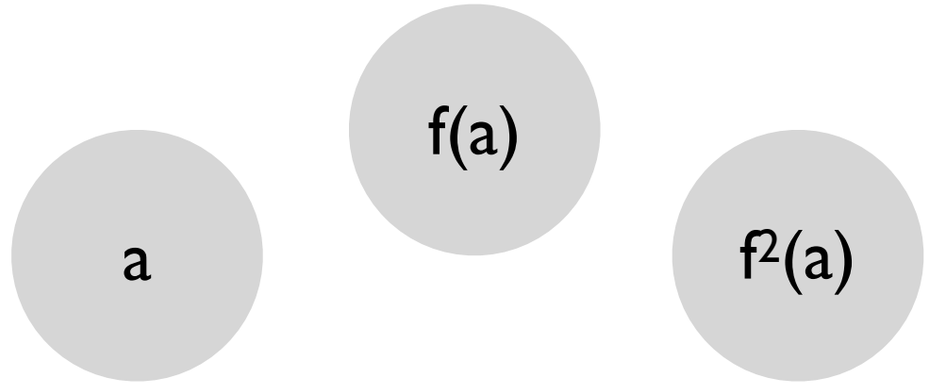
$$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

Congruence closure algorithm: example

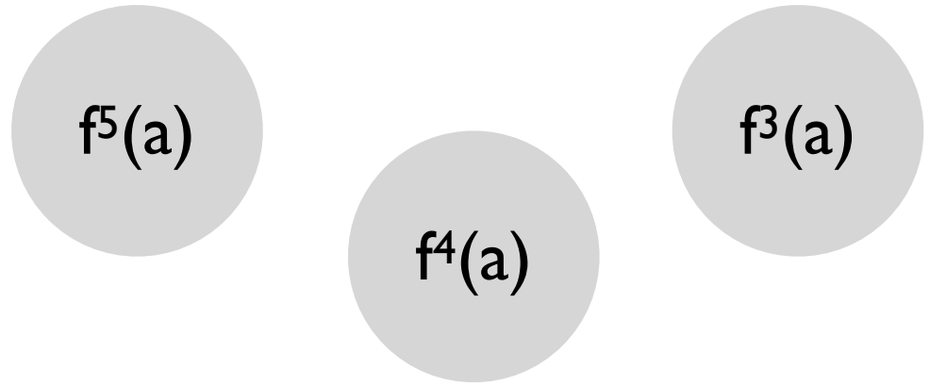
$$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**

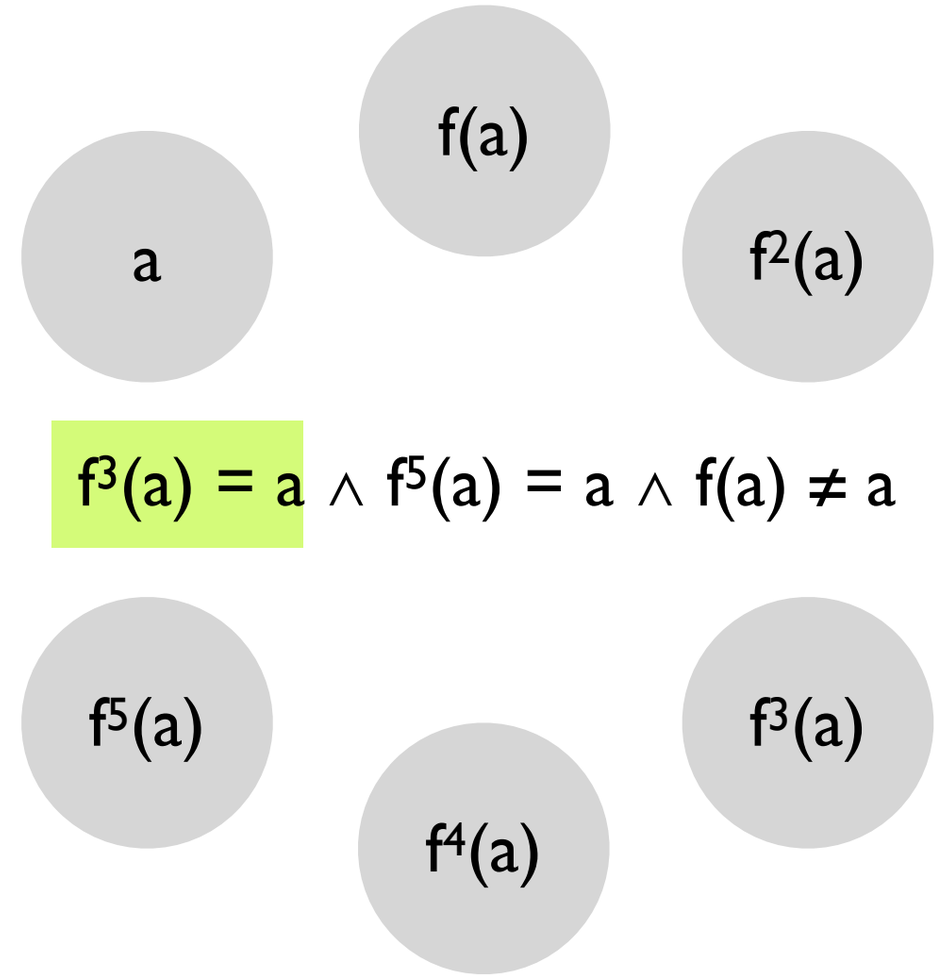


$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$



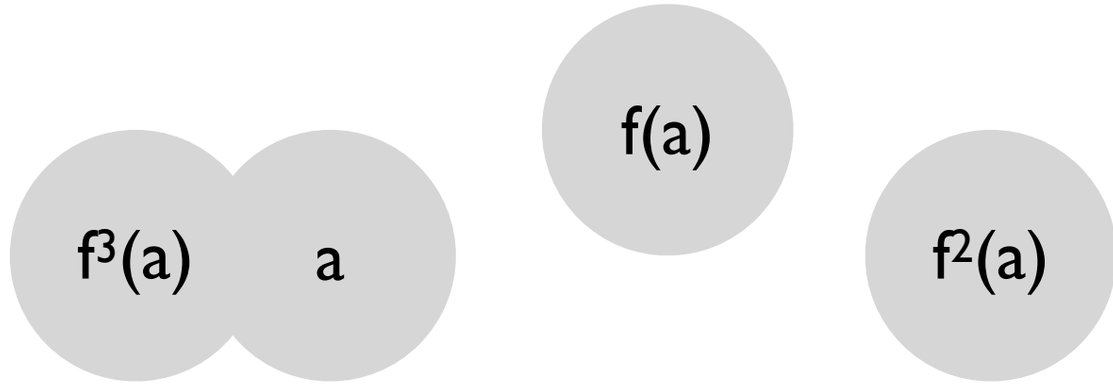
Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2

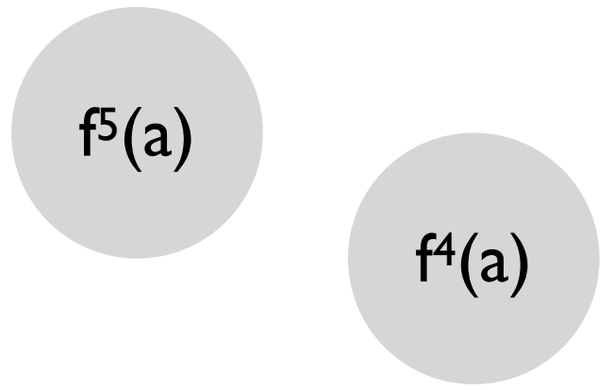


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2

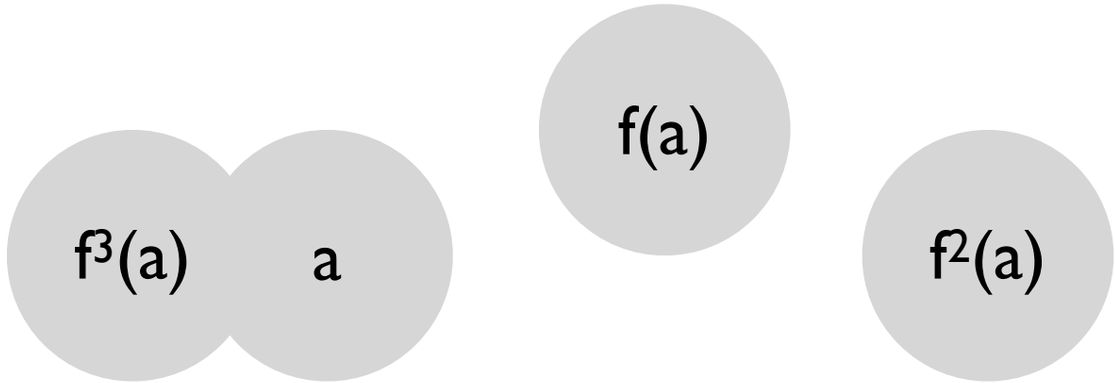


$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$

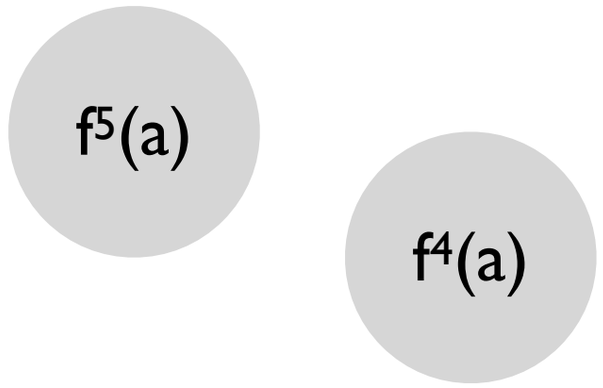


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences

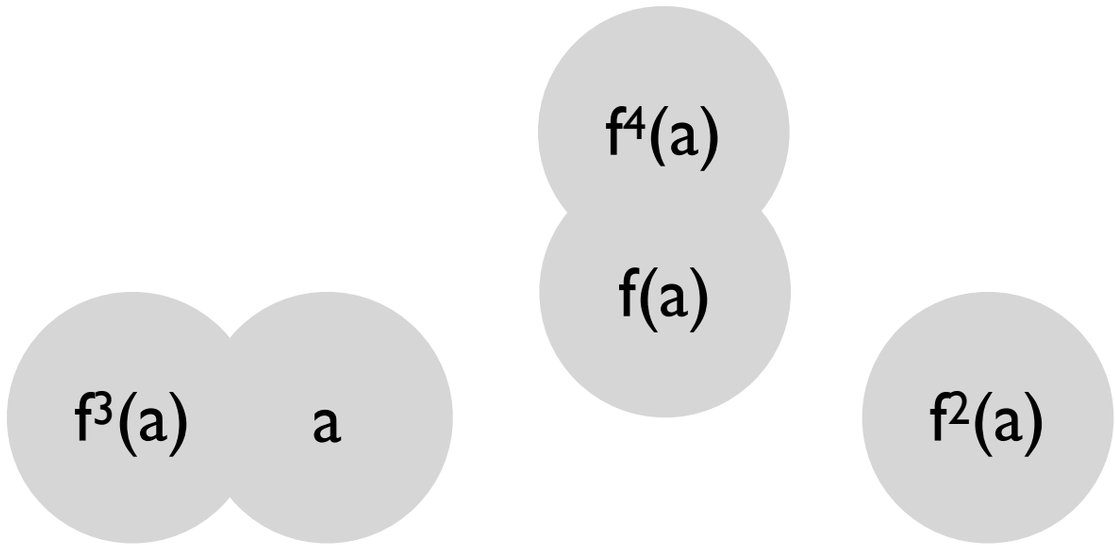


$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$

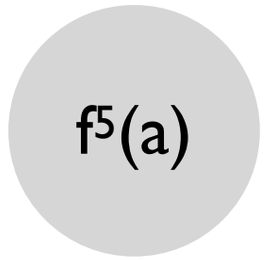


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences

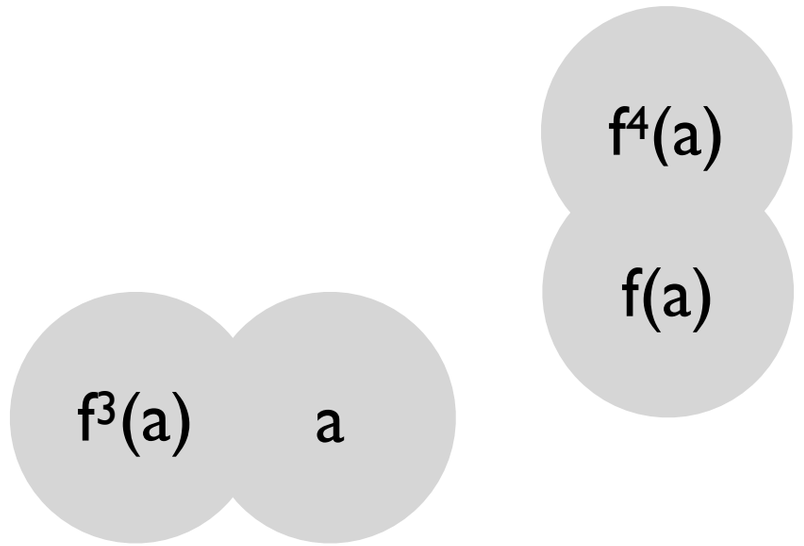


$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$

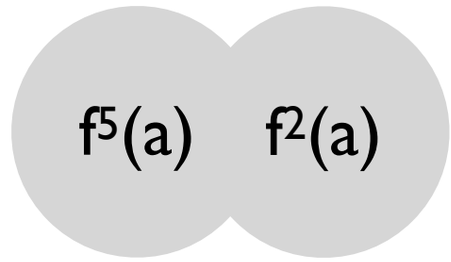


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences

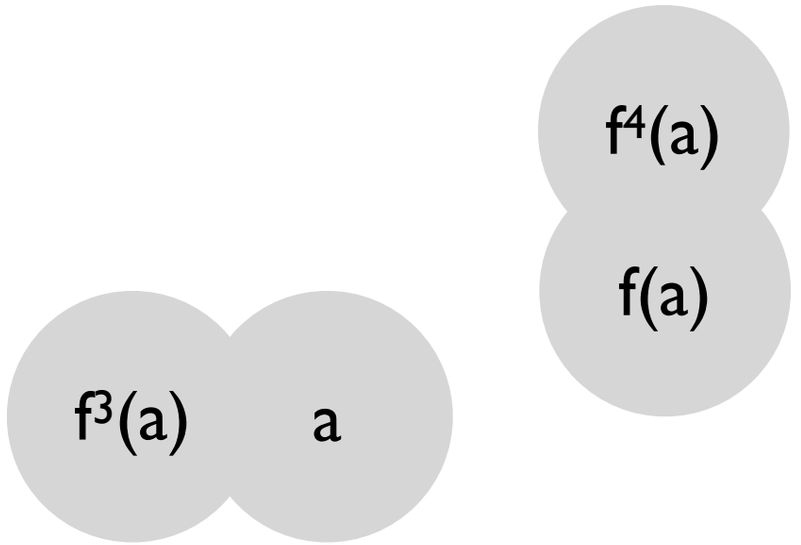


$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$

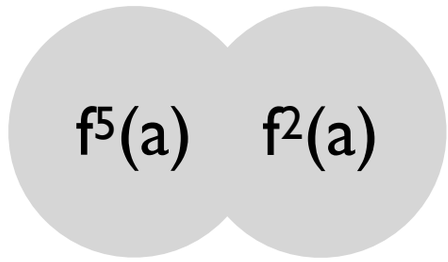


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences

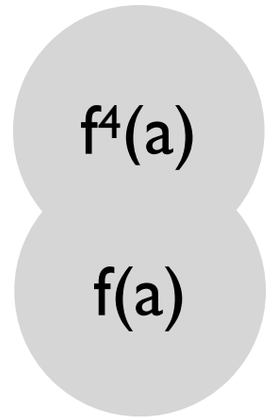


$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$

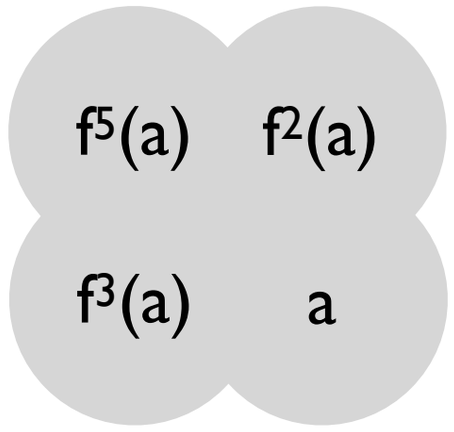


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences



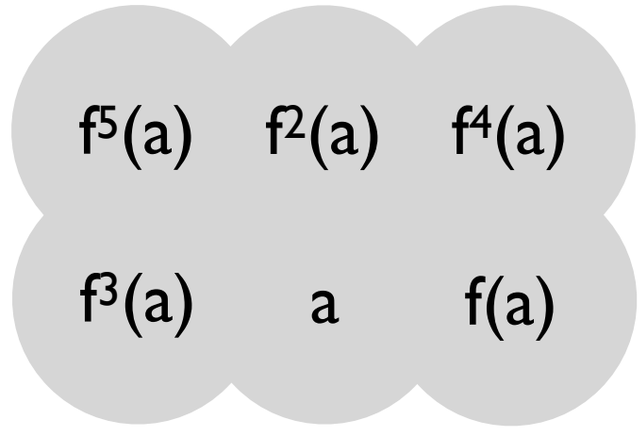
$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$



Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences

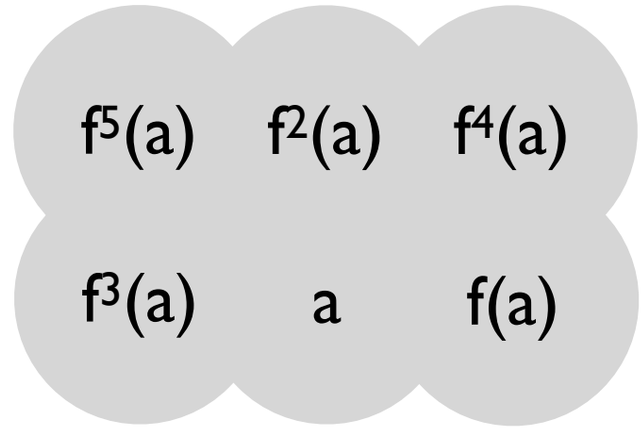
$$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$



Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT

$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$

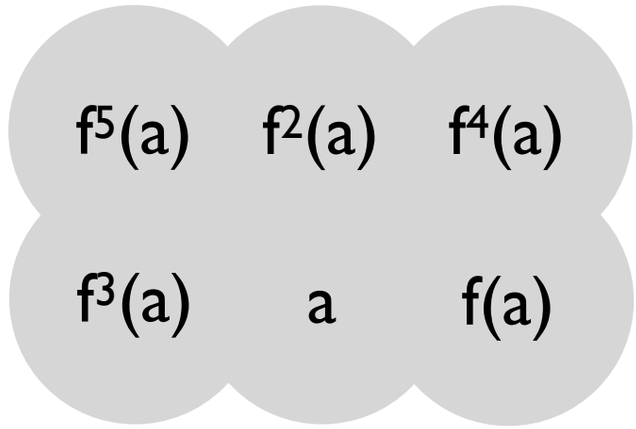


Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT

UNSAT

$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$



Congruence closure algorithm: another example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT

$$f(x) = f(y) \wedge x \neq y$$

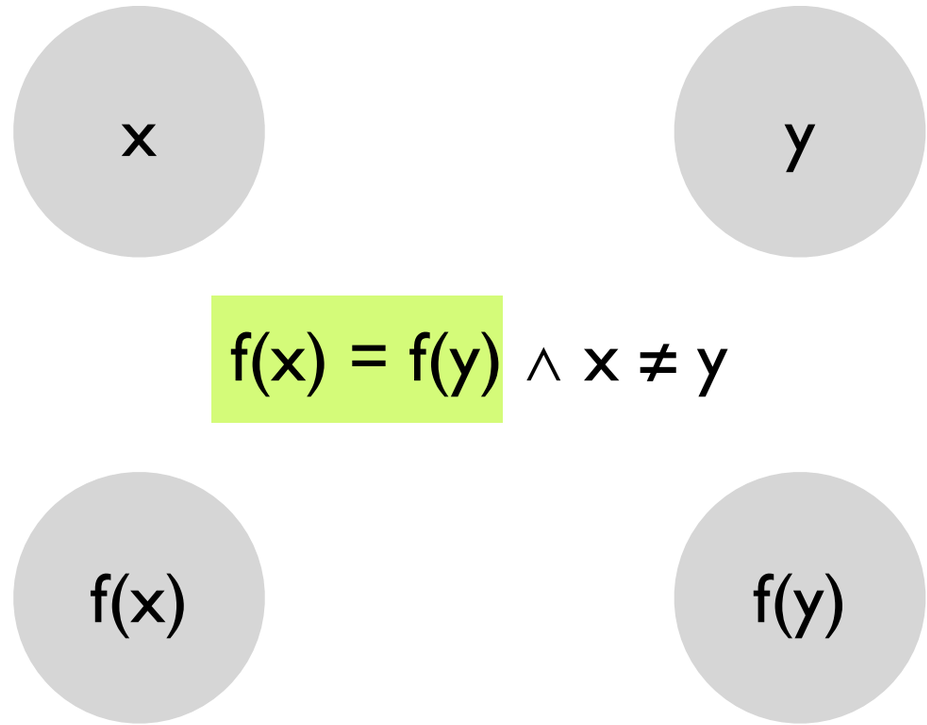
Congruence closure algorithm: another example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT

$f(x) = f(y) \wedge x \neq y$

Congruence closure algorithm: another example

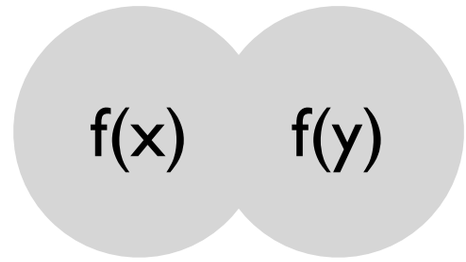
- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT



Congruence closure algorithm: another example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT

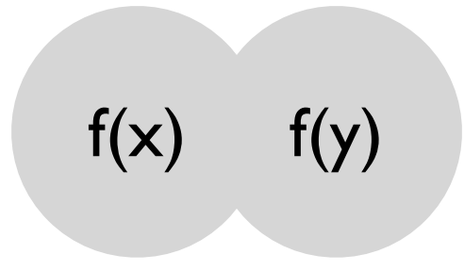
$f(x) = f(y) \wedge x \neq y$



Congruence closure algorithm: another example

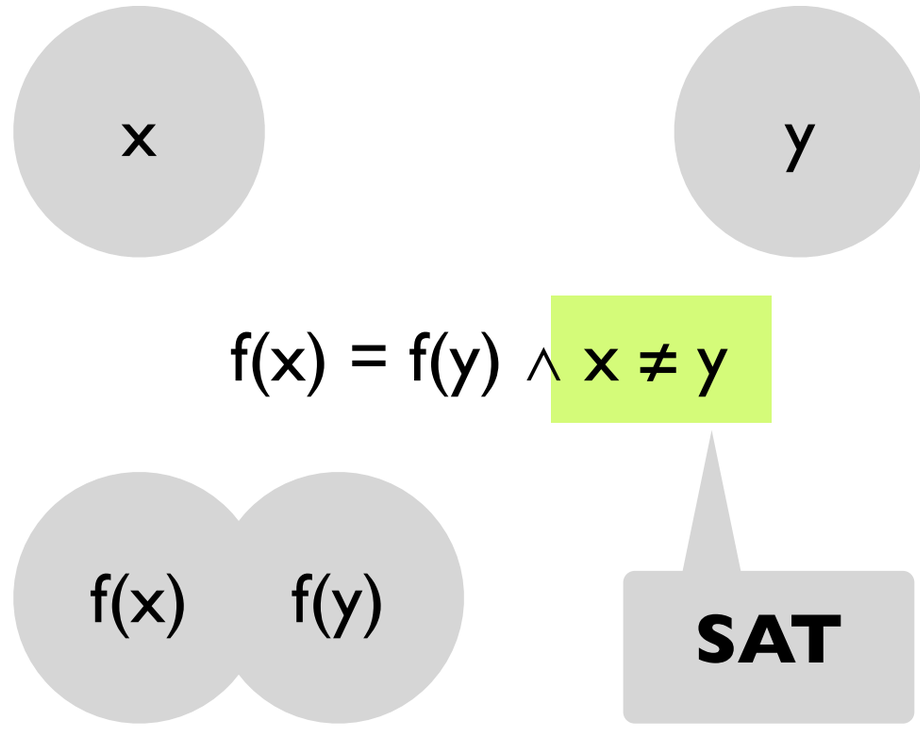
- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT

$f(x) = f(y) \wedge x \neq y$



Congruence closure algorithm: another example

- Place each subterm of F into its own **congruence class**
- For each positive literal $t_1 = t_2$ in F
  - Merge the classes for t_1 and t_2
  - Propagate the resulting congruences
- If F has a negative literal $t_1 \neq t_2$ with both terms in the same congruence class, output UNSAT
- Otherwise, output SAT



Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

An equivalence relation R is a **congruence relation** if for every n-ary function f

$$\forall \overline{x}, \overline{y}.\ \bigwedge R(x_i, y_i) \rightarrow R(f(\overline{x}), f(\overline{y}))$$

Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

An equivalence relation R is a **congruence relation** if for every n-ary function f

$$\forall \overline{x}, \overline{y}. \bigwedge R(x_i, y_i) \rightarrow R(f(\overline{x}), f(\overline{y}))$$

The **equivalence class** of an element $s \in S$ under an equivalence relation R:

$$\{ s' \in S \mid R(s, s') \}$$

Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

An equivalence relation R is a **congruence relation** if for every n-ary function f

$$\forall \overline{x}, \overline{y}.\ \bigwedge R(x_i, y_i) \rightarrow R(f(\overline{x}), f(\overline{y}))$$

The **equivalence class** of an element $s \in S$ under an equivalence relation R:

$$\{\ s' \in S \mid R(s, s')\ \}$$

What is the equivalence class of 9 under $\equiv_3$?

Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

An equivalence relation R is a **congruence relation** if for every n-ary function f

$$\forall \overline{x}, \overline{y}. \bigwedge R(x_i, y_i) \rightarrow R(f(\overline{x}), f(\overline{y}))$$

The **equivalence class** of an element $s \in S$ under an equivalence relation R:

$$\{ s' \in S \mid R(s, s') \}$$

An equivalence class is called a **congruence class** if R is a congruence relation.

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

What is the equivalence closure of R = {⟨a, b⟩, ⟨b, c⟩, ⟨d, d⟩}?

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

What is the equivalence closure of R = {⟨a, b⟩, ⟨b, c⟩, ⟨d, d⟩}?

R^E = {⟨a, a⟩, ⟨b, b⟩, ⟨c, c⟩,⟨d, d⟩
⟨a, b⟩, ⟨b, a⟩, ⟨b, c⟩, ⟨c, b⟩,
⟨a, c⟩, ⟨c, a⟩}

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

The **congruence closure** R^C of a binary relation R is the smallest congruence relation that contains R.

The congruence closure algorithm computes the congruence closure of the equality relation over terms asserted by a conjunctive quantifier-free formula in $T_=$.

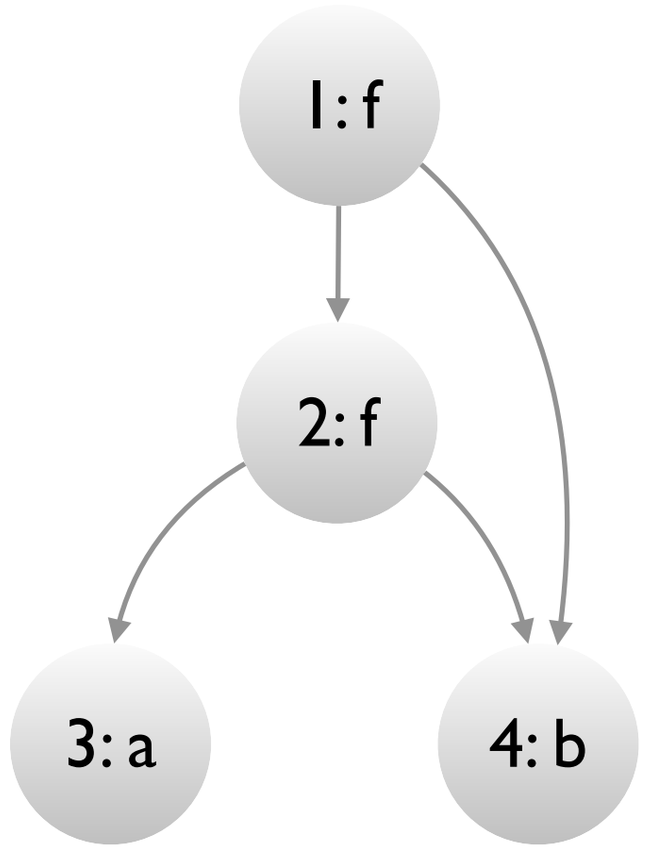
Congruence closure algorithm: data structure

$$f(a, b) = a \wedge f(f(a, b), b) \neq a$$

Congruence closure algorithm: data structure

- Represent subterms with a DAG

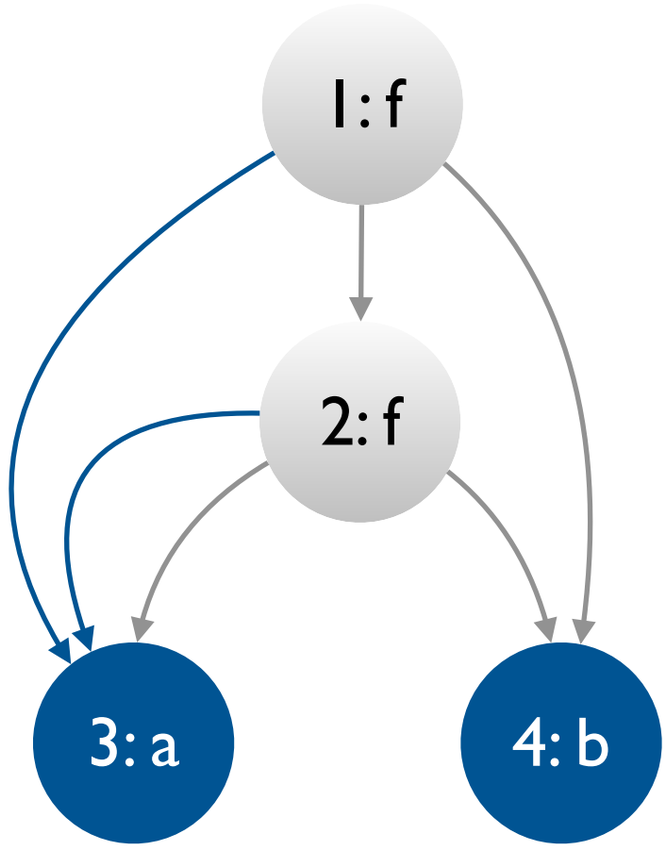
$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: data structure

- Represent subterms with a DAG
- Each node has a **find** pointer to another node in its congruence class (or to itself if it is the **representative**)

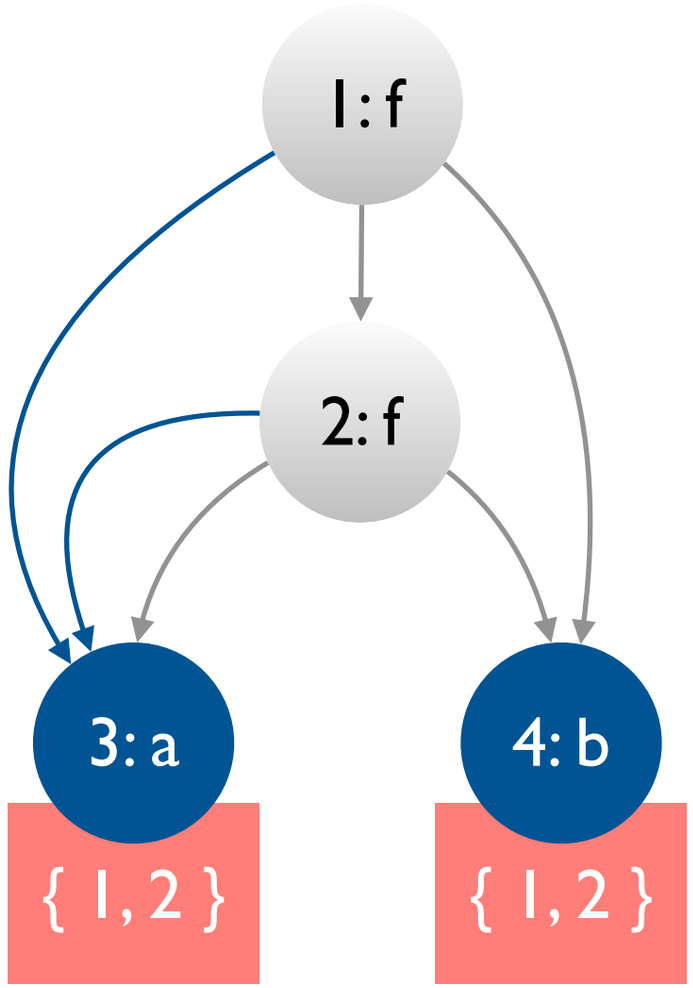
$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: data structure

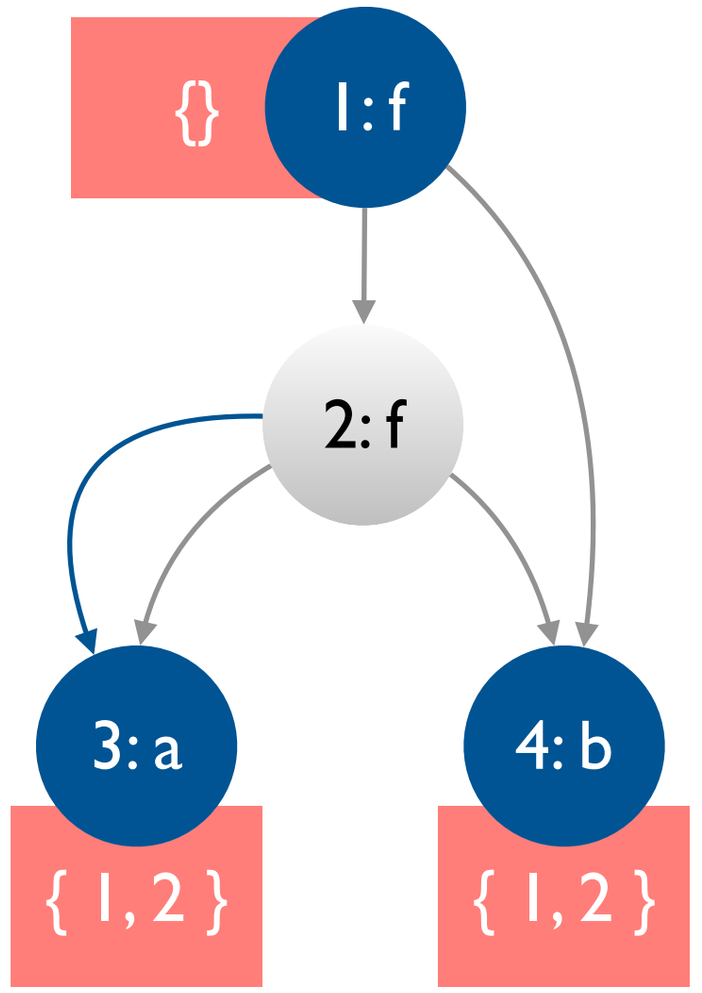
- Represent subterms with a DAG
- Each node has a **find** pointer to another node in its congruence class (or to itself if it is the **representative**)
- Each representative has a **ccp** field that stores all parents of all nodes in its congruence class.

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



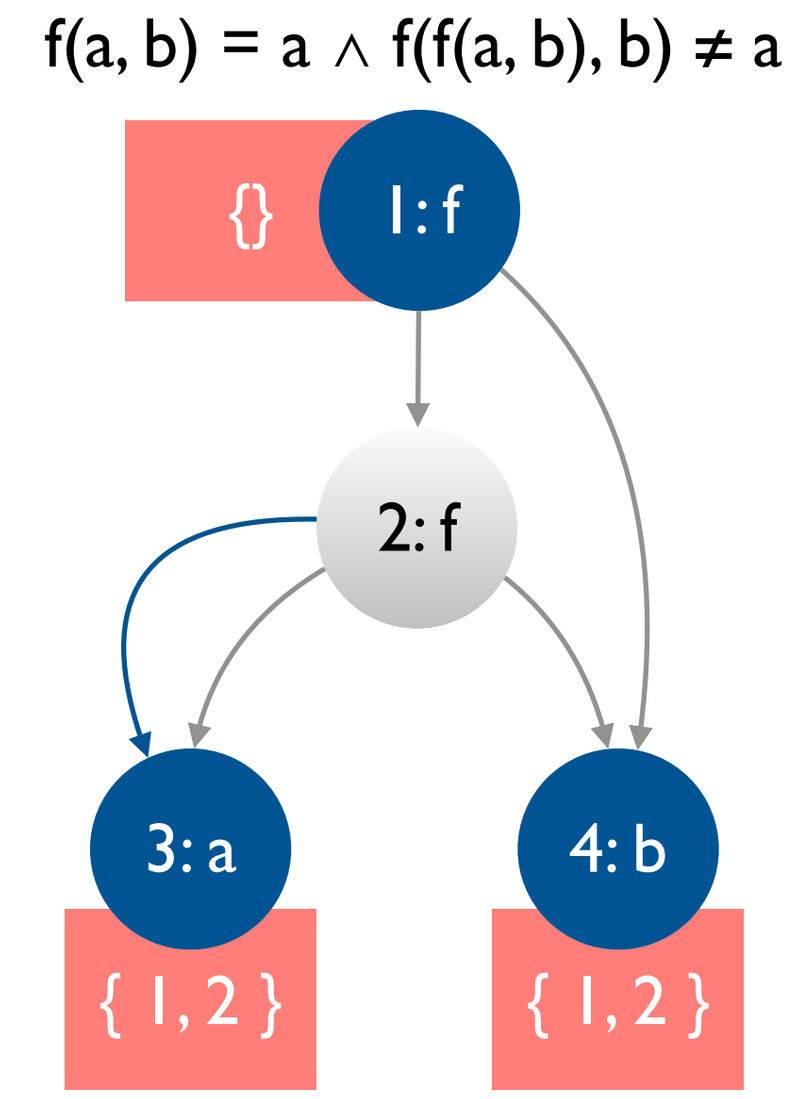
Congruence closure algorithm: union-find

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: union-find

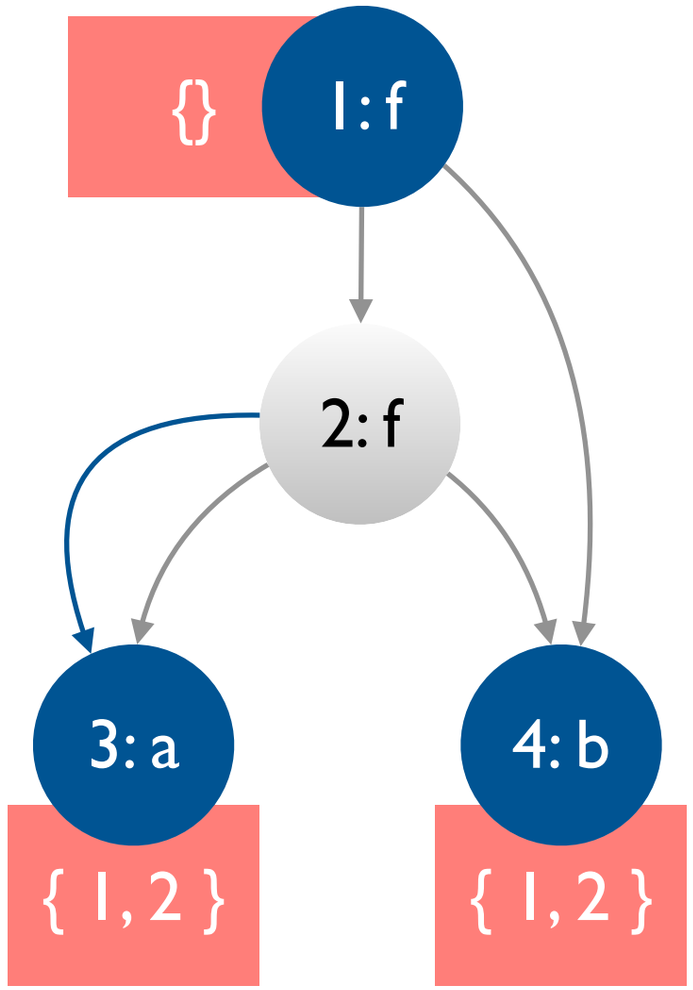
- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.



Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.
- UNION combines equivalence classes for nodes i_1 and i_2:
  - $n_1, n_2 \leftarrow \text{FIND}(i_1), \text{FIND}(i_2)$
  - $n_1.\text{find} \leftarrow n_2$
  - $n_2.\text{ccp} \leftarrow n_1.\text{ccp} \cup n_2.\text{ccp}$
  - $n_1.\text{ccp} \leftarrow \varnothing$

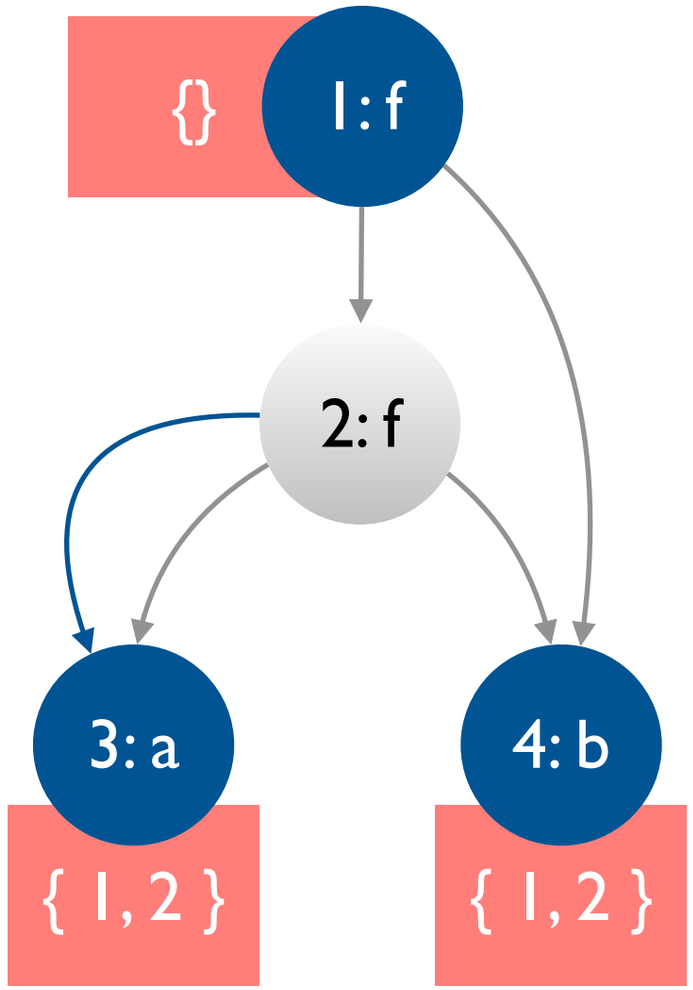
$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.
- UNION combines equivalence classes for nodes i_1 and i_2:
  - $n_1, n_2 \leftarrow \text{FIND}(i_1), \text{FIND}(i_2)$
  - $n_1.\text{find} \leftarrow n_2$
  - $n_2.\text{ccp} \leftarrow n_1.\text{ccp} \cup n_2.\text{ccp}$
  - $n_1.\text{ccp} \leftarrow \varnothing$

$f(a, b) = a \wedge f(f(a, b), b) \neq a$

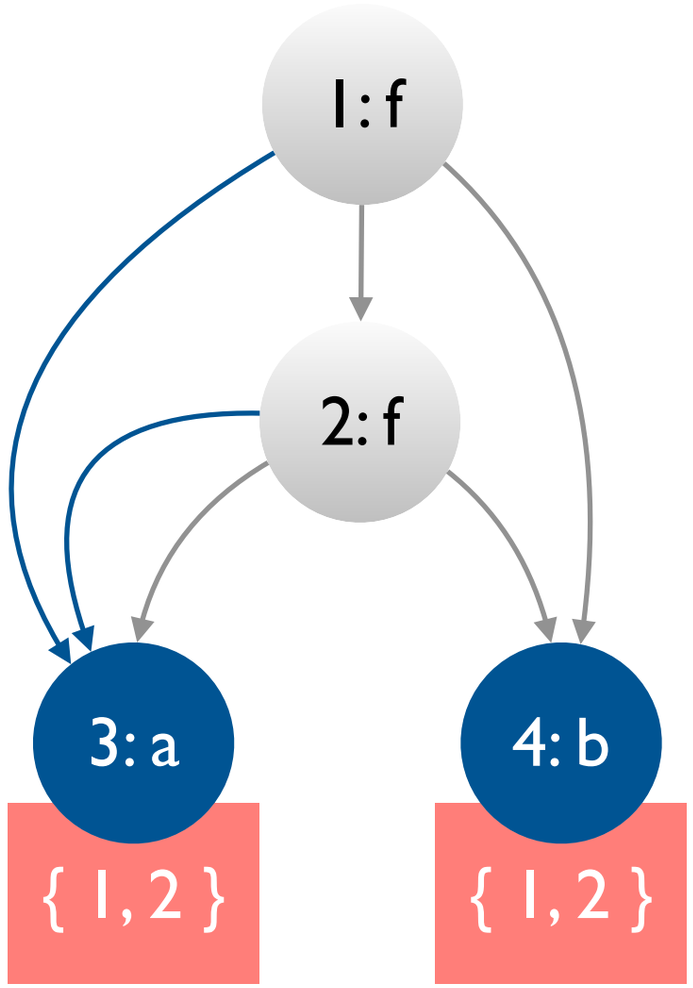


What is UNION(1, 2)?

Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.
- UNION combines equivalence classes for nodes i_1 and i_2:
  - $n_1, n_2 \leftarrow \text{FIND}(i_1), \text{FIND}(i_2)$
  - $n_1.\text{find} \leftarrow n_2$
  - $n_2.\text{ccp} \leftarrow n_1.\text{ccp} \cup n_2.\text{ccp}$
  - $n_1.\text{ccp} \leftarrow \varnothing$

$f(a, b) = a \wedge f(f(a, b), b) \neq a$

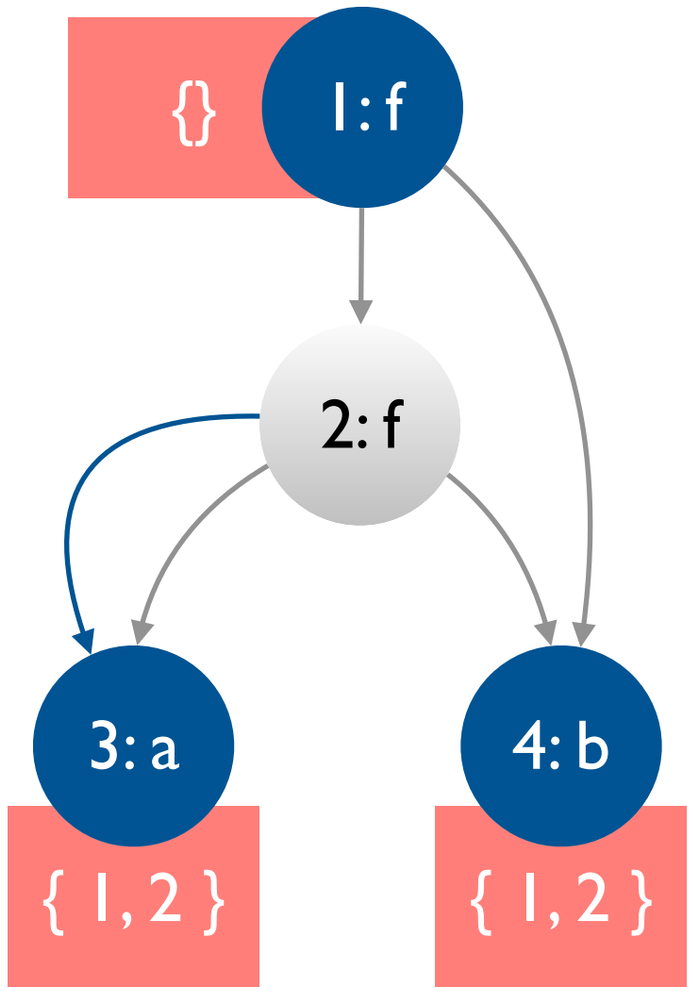


What is UNION(1, 2)?

Congruence closure algorithm: congruent

- CONGRUENT takes as input two nodes and returns true iff their
  - functions are the same
  - corresponding arguments are in the same congruence class

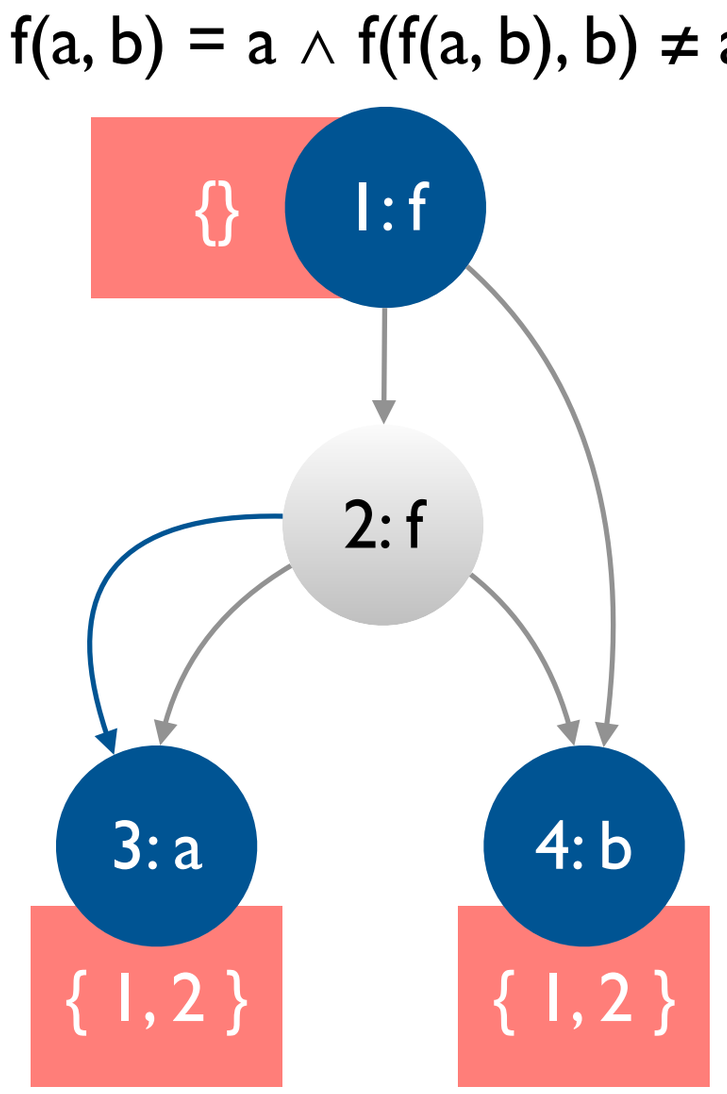
$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: congruent

- CONGRUENT takes as input two nodes and returns true iff their
  - functions are the same
  - corresponding arguments are in the same congruence class

CONGRUENT(1, 2)?



Congruence closure algorithm: merge

MERGE (i_1 , i_2)

$n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)

if $n_1 = n_2$ **then return**

$p_1, p_2 \leftarrow n_1.ccp, n_2.ccp$

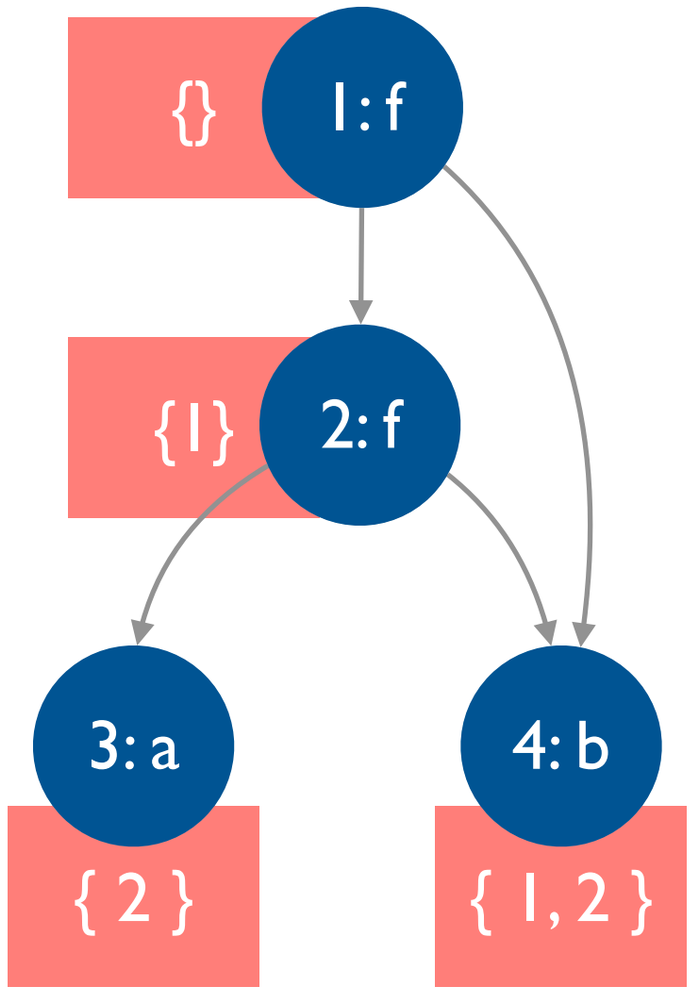
UNION(n_1, n_2)

for each $t_1, t_2 \in p_1 \times p_2$

if FIND(t_1) $\neq$ FIND(t_2) $\wedge$ CONGRUENT(t_1, t_2)

then MERGE(t_1, t_2)

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: merge

MERGE (i_1 , i_2)

  $n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)

  if $n_1 = n_2$ **then return**

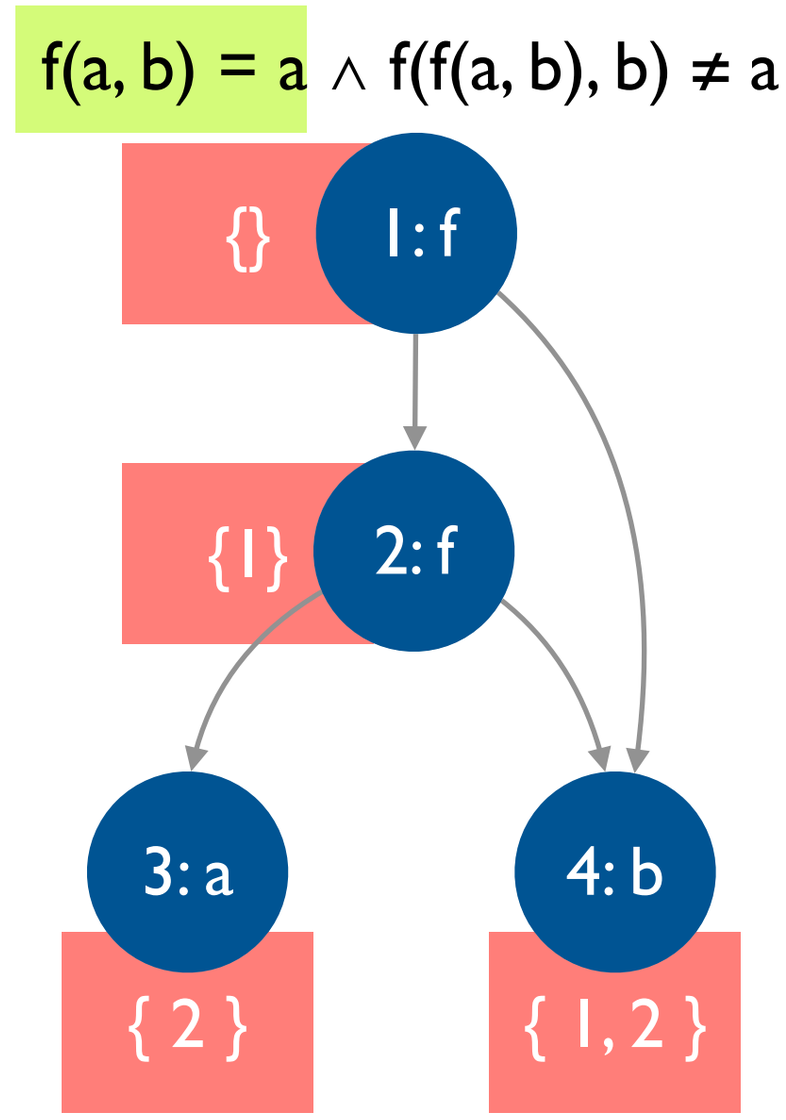
  $p_1, p_2 \leftarrow n_1.ccp, n_2.ccp$

  UNION(n_1, n_2)

  for each $t_1, t_2 \in p_1 \times p_2$

    if FIND(t_1) ≠ FIND(t_2) ∧ CONGRUENT(t_1, t_2)

    then MERGE(t_1, t_2)



Congruence closure algorithm: merge

MERGE (i_1 , i_2)

$n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)

if $n_1 = n_2$ **then return**

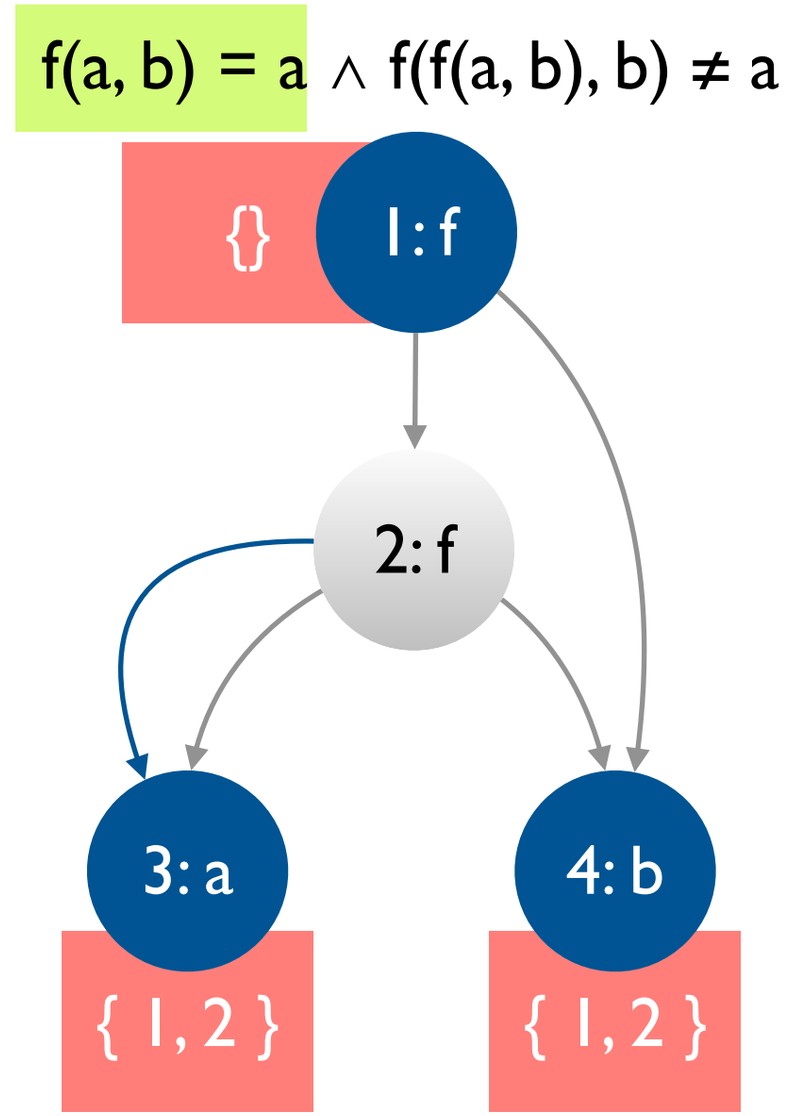
$p_1, p_2 \leftarrow n_1.ccp, n_2.ccp$

UNION(n_1, n_2)

for each $t_1, t_2 \in p_1 \times p_2$

if FIND(t_1) ≠ FIND(t_2) ∧ CONGRUENT(t_1, t_2)

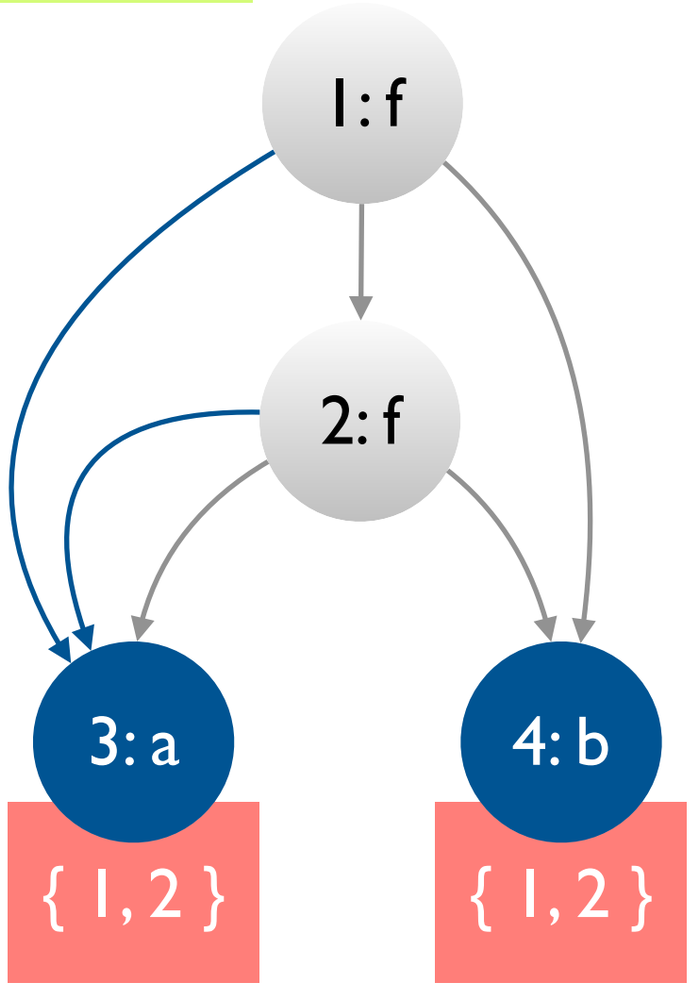
then MERGE(t_1, t_2)



Congruence closure algorithm: merge

```
MERGE (i1 , i2)
    n1, n2 ← FIND(i1), FIND(i2)
    if n1 = n2 then return
    p1, p2 ← n1.ccp, n2.ccp
    UNION(n1, n2)
    for each t1, t2 ∈ p1 × p2
        if FIND(t1) ≠ FIND(t2) ∧ CONGRUENT(t1, t2)
        then MERGE(t1, t2)
```

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: deciding $T_=$

DECIDE (F)

construct the DAG for F's subterms

for $s_i = t_i \in F$

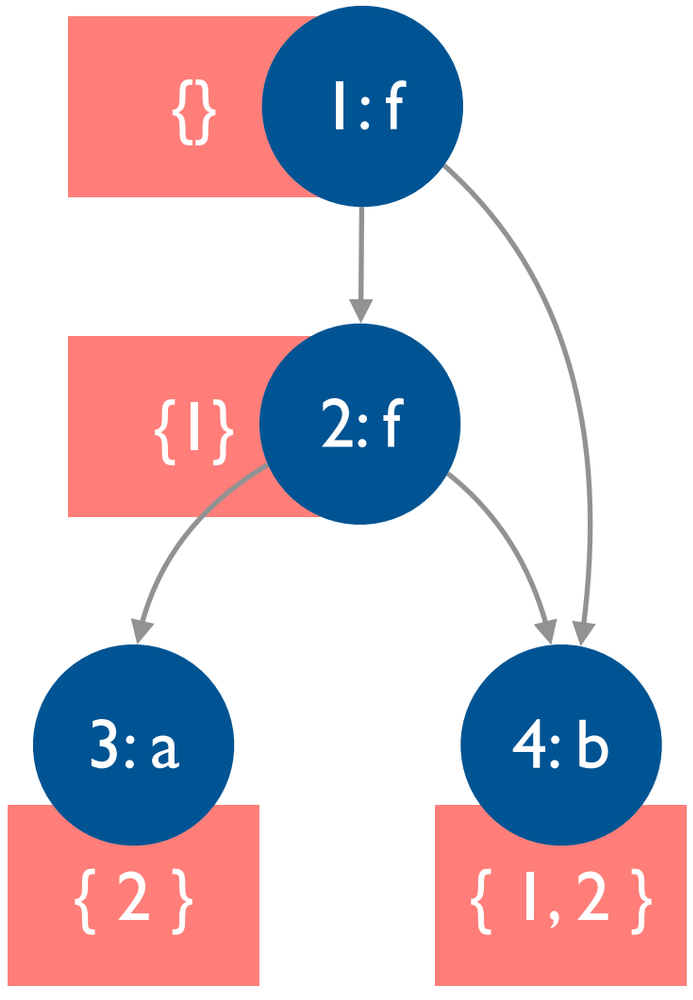
MERGE(s_i, t_i)

for $s_i \neq t_i \in F$

if FIND(s_i) = FIND(t_i) **then return** UNSAT

return SAT

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: deciding $T_=$

DECIDE (F)

construct the DAG for F's subterms

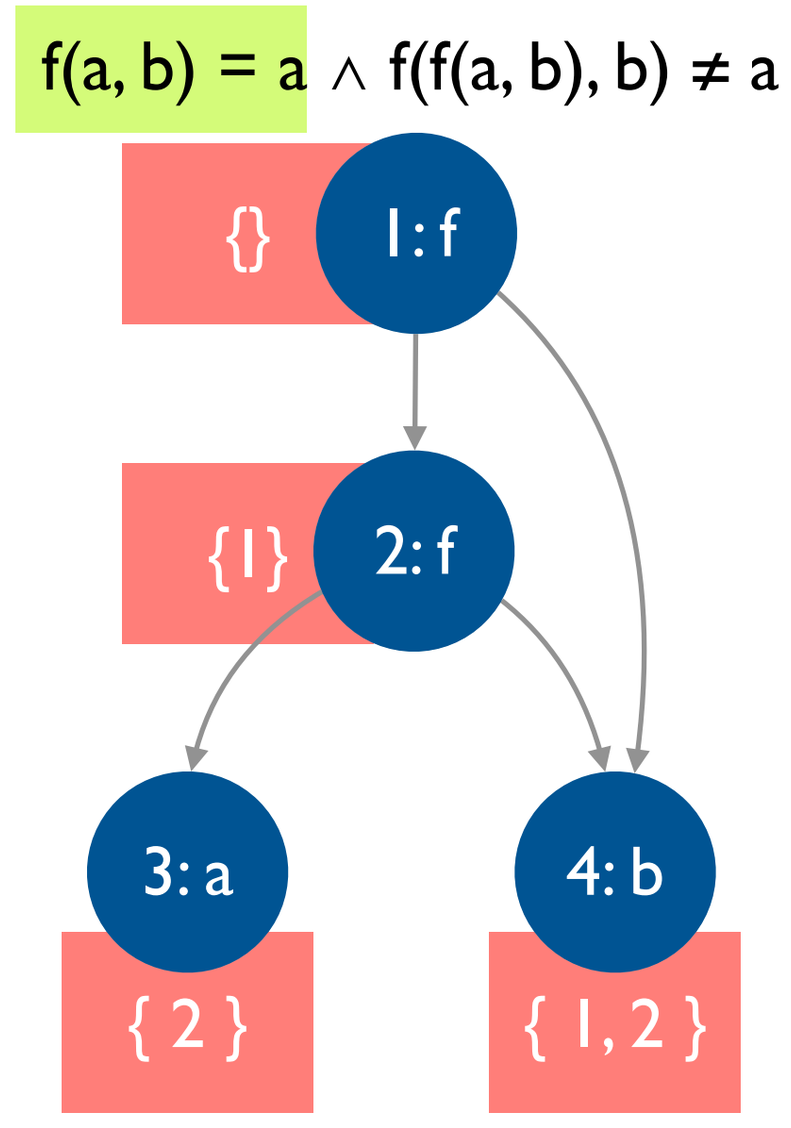
for $s_i = t_i \in F$

MERGE(s_i, t_i)

for $s_i \neq t_i \in F$

if FIND(s_i) = FIND(t_i) **then return** UNSAT

return SAT



Congruence closure algorithm: deciding $T_=$

DECIDE (F)

construct the DAG for F's subterms

for $s_i = t_i \in F$

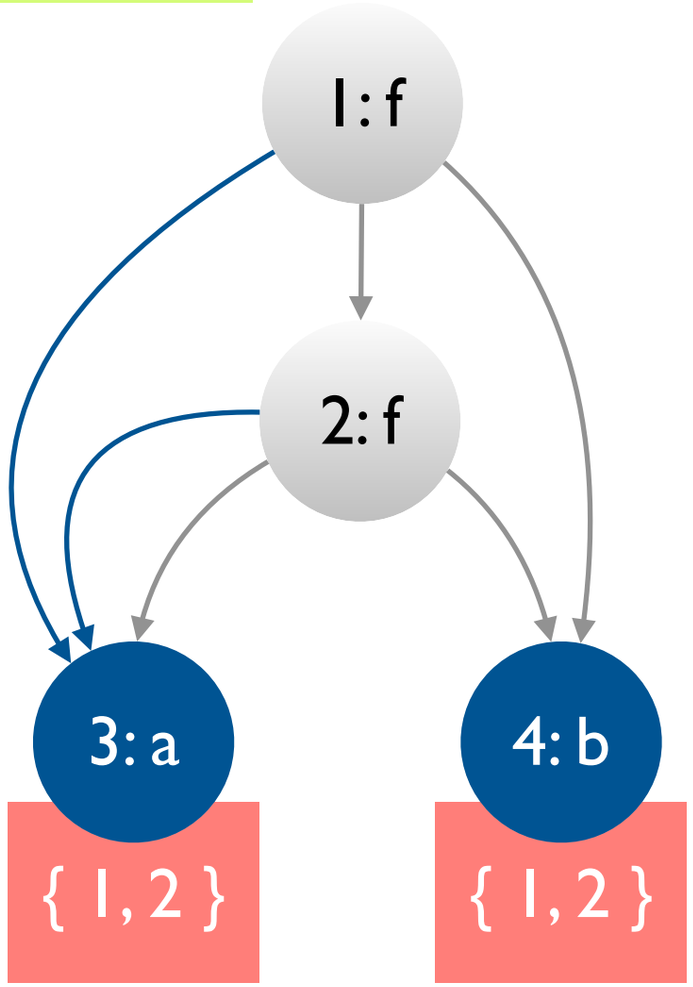
MERGE(s_i, t_i)

for $s_i \neq t_i \in F$

if FIND(s_i) = FIND(t_i) **then return** UNSAT

return SAT

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: deciding $T_=$

DECIDE (F)

construct the DAG for F's subterms

for $s_i = t_i \in F$

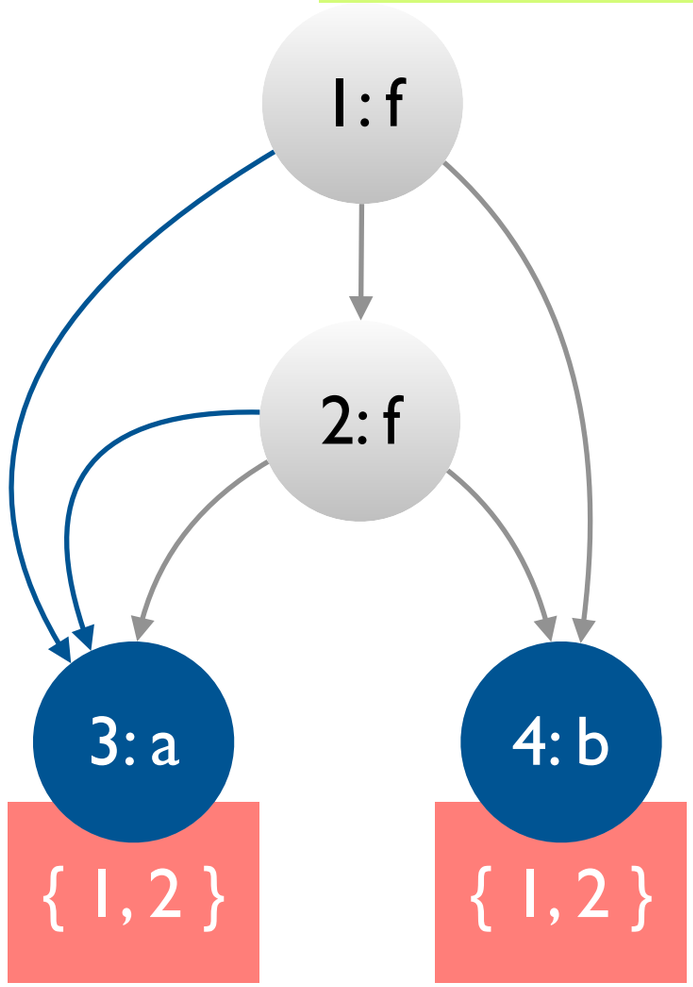
MERGE(s_i, t_i)

for $s_i \neq t_i \in F$

if FIND(s_i) = FIND(t_i) **then return** UNSAT

return SAT

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: deciding $T_=$

DECIDE (F)

construct the DAG for F's subterms

for $s_i = t_i \in F$

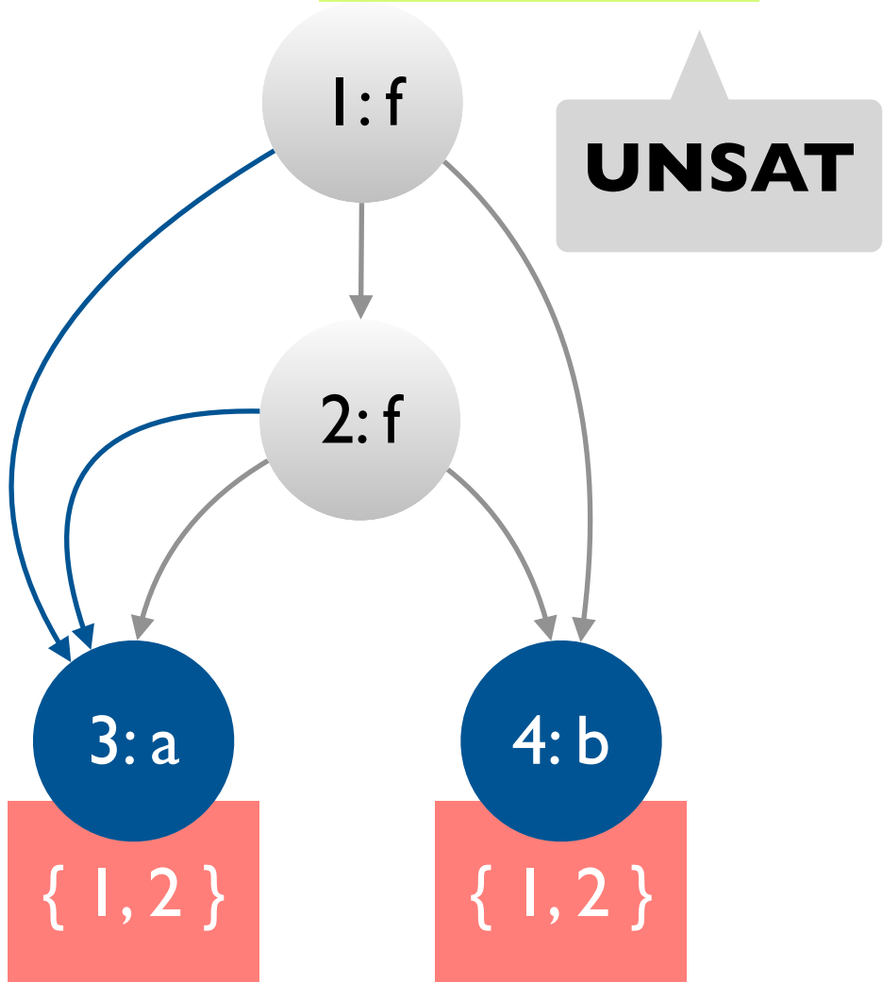
MERGE(s_i, t_i)

for $s_i \neq t_i \in F$

if FIND(s_i) = FIND(t_i) **then return** UNSAT

return SAT

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Summary

Today

- A brief survey of theory solvers
- Congruence closure algorithm for deciding conjunctive $T_=$ formulas

Next lecture

- Combining (decision procedures for different) theories