Computer-Aided Reasoning for Software

CSE507

# Satisfiability Modulo Theories

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# Today

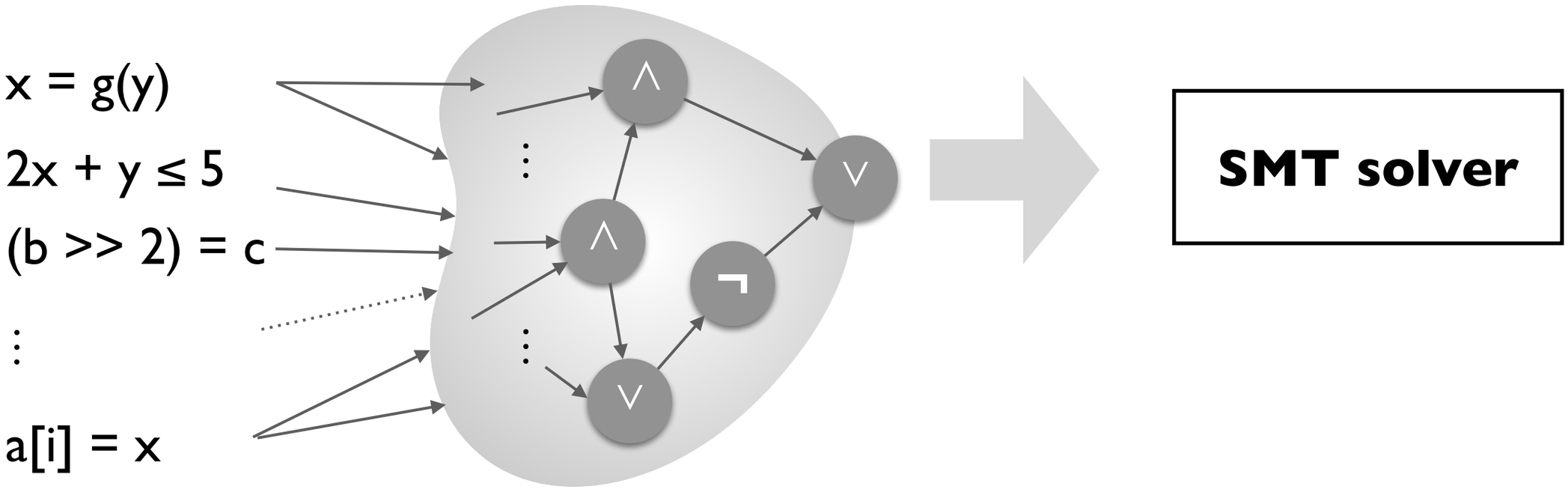
## Last lecture

- Practical applications of SAT and the need for a richer logic

## Today

- Introduction to Satisfiability Modulo Theories (SMT)
- Syntax and semantics of (quantifier-free) first-order logic
- Overview of key theories

# Satisfiability Modulo Theories (SMT)



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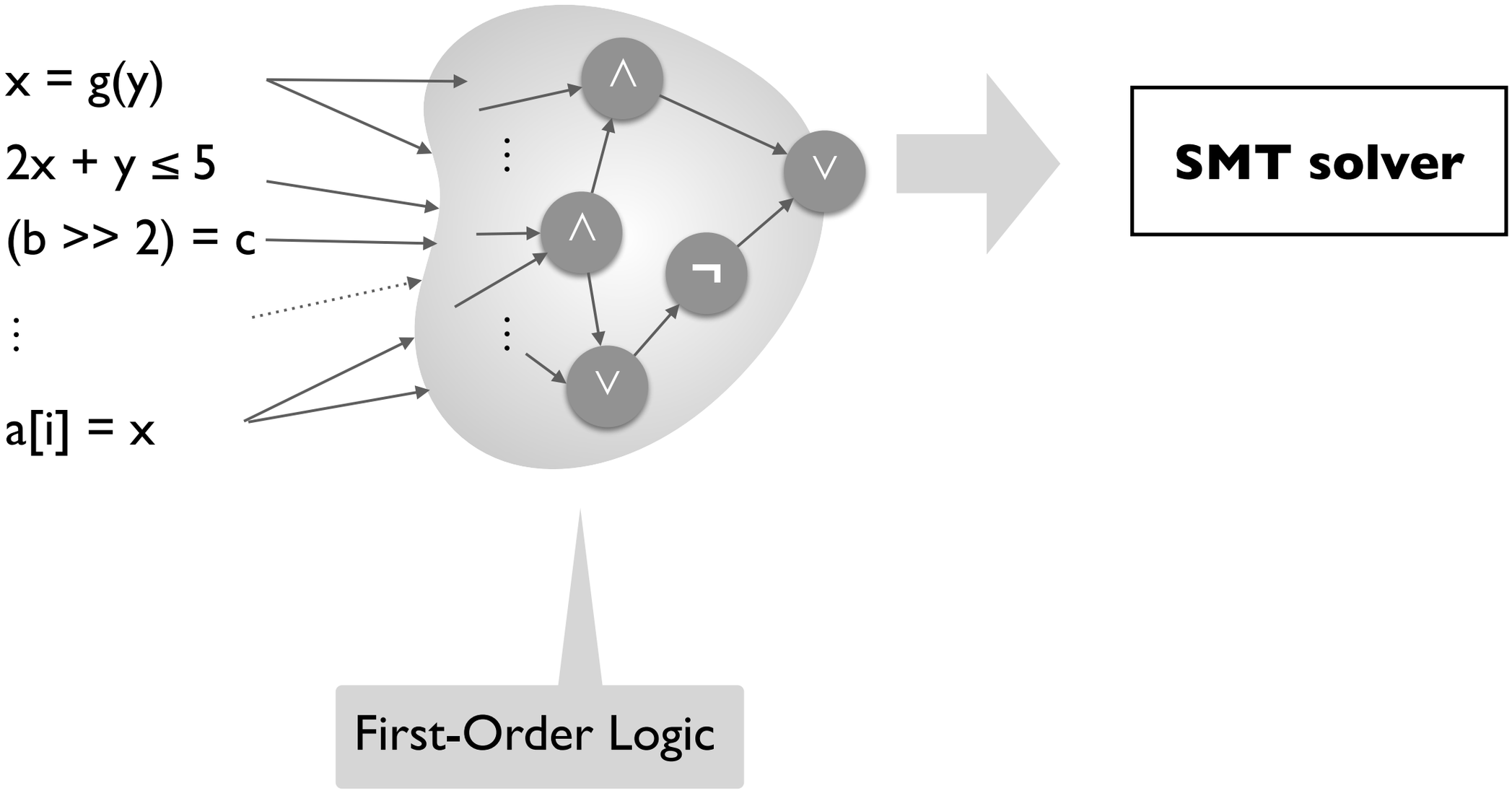

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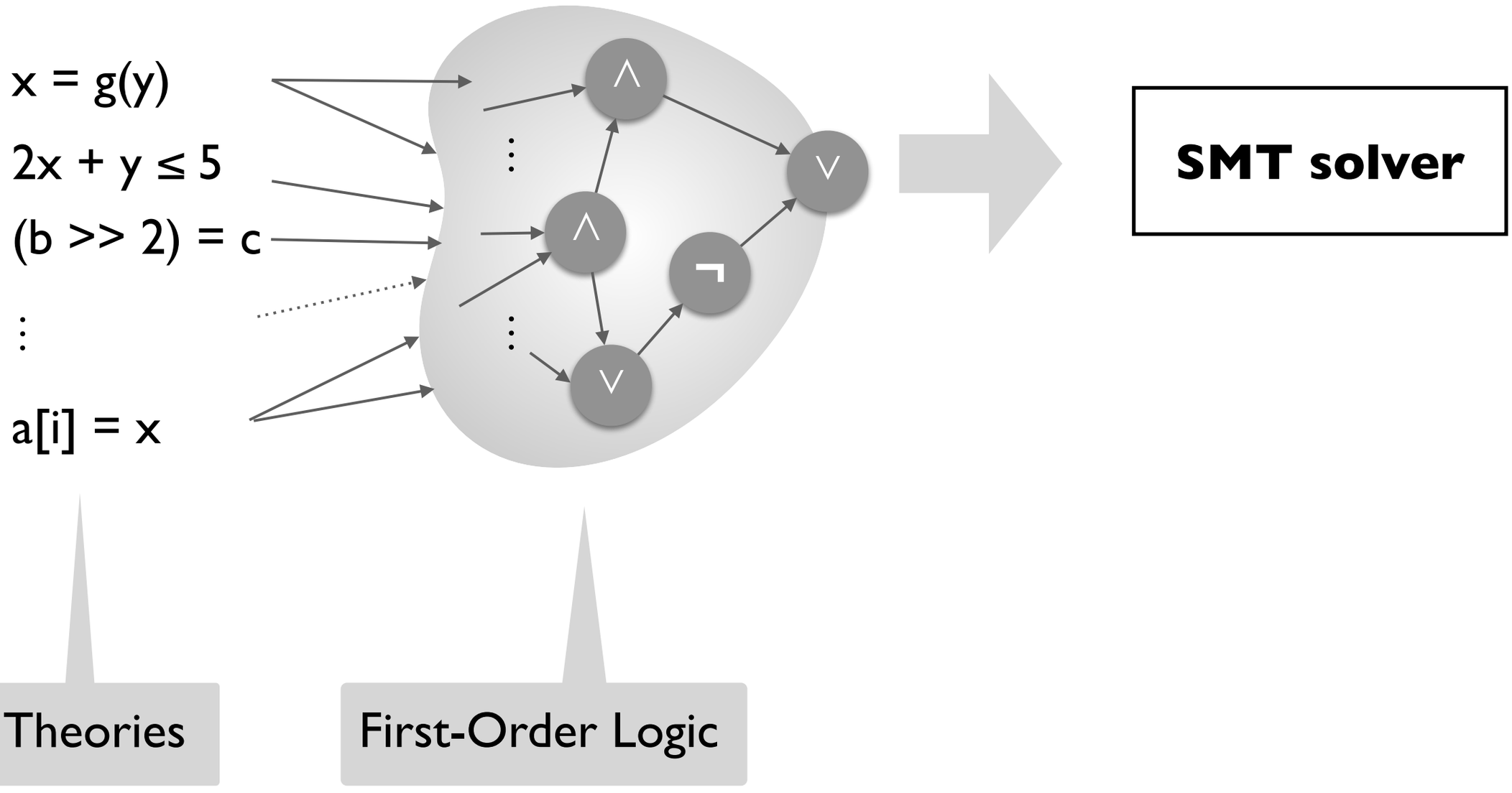

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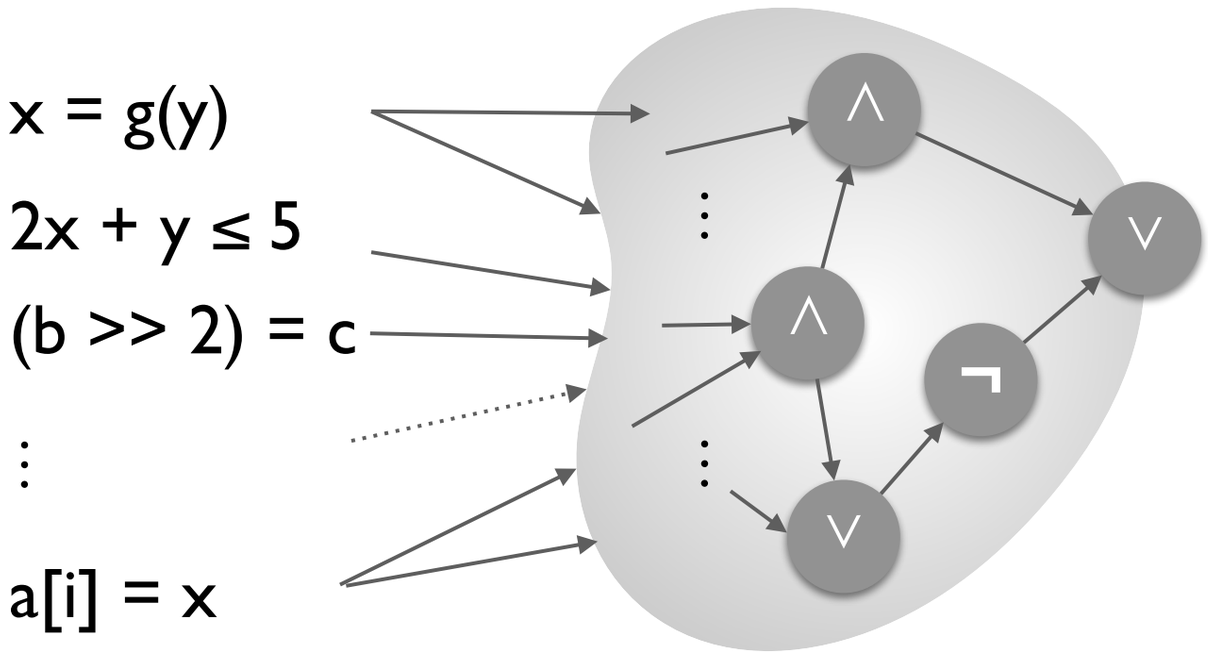


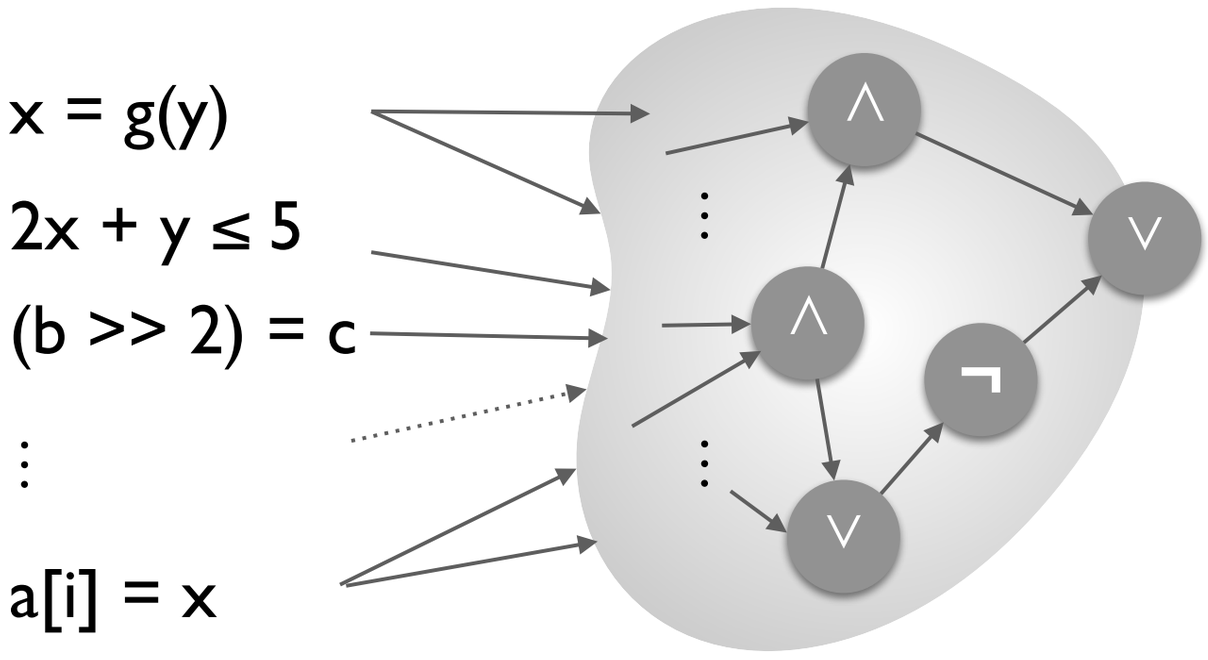

(un)satisfiable

**SMT solver**

Theories

First-Order Logic

# Satisfiability Modulo Theories (SMT)




Theories

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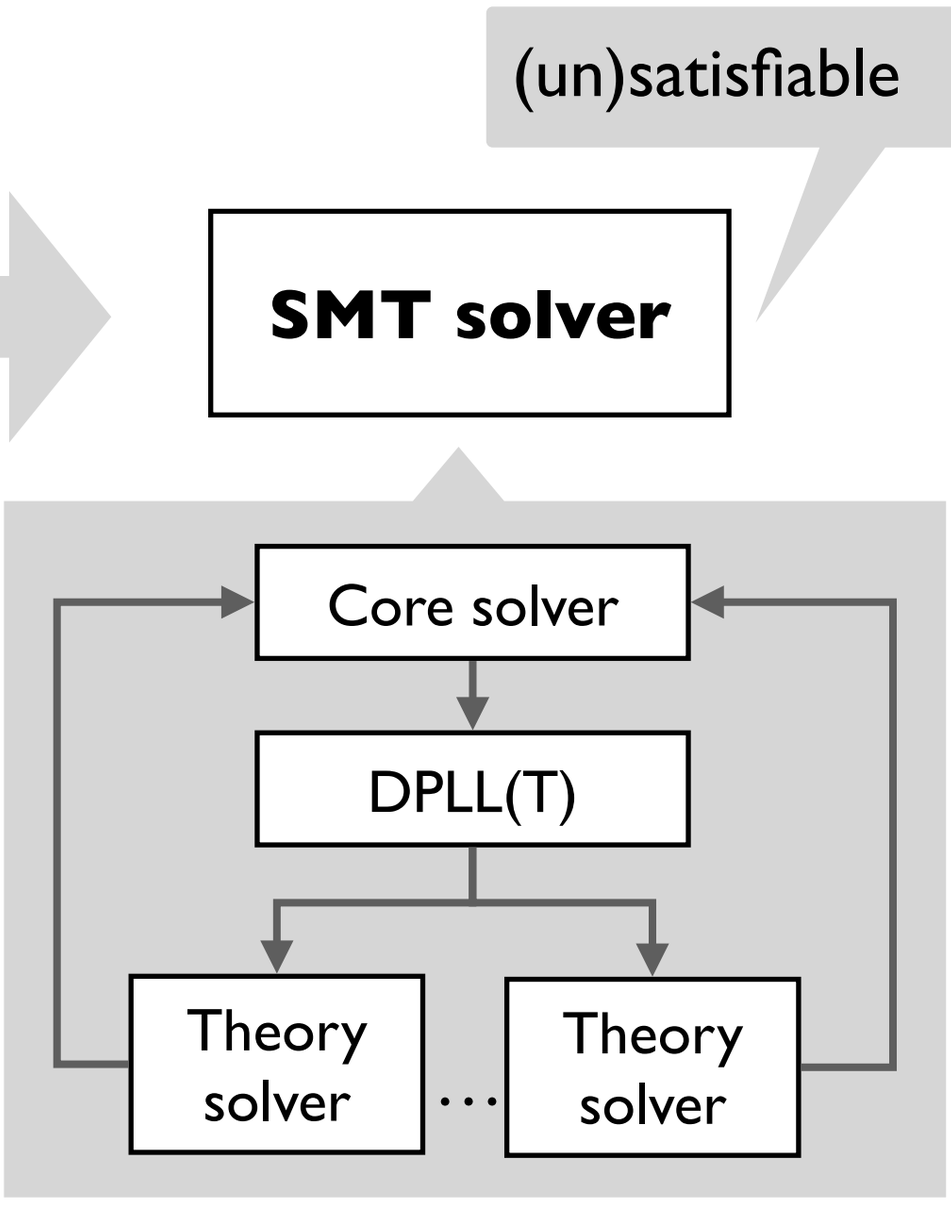

# Syntax of First-Order Logic (FOL)

## Logical symbols

- Connectives: ¬, ∧, ∨, →, ↔
- Parentheses: ()
- Quantifiers: ∀, ∃

## Non-logical symbols

- Constants: x, y, z
- N-ary functions: f, g
- N-ary predicates: p, q
- Variables: u, v, w

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We will only consider the **quantifier-free** fragment of FOL.

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In particular, we will consider quantifier-free **ground** formulas.

# Syntax of quantifier-free ground FOL formulas

**Logical symbols**

- Connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$
- Parentheses: ()

**Non-logical symbols**

- Constants: x, y, z
- N-ary functions: f, g
- N-ary predicates: p, q

- A **term** is a constant, or an n-ary function applied to n terms.
- An **atom** is $\top$, $\bot$, or an n-ary predicate applied to n terms.
- A **literal** is an atom or its negation.
- A (quantifier-free ground) **formula** is a literal or the application of logical connectives to formulas.

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isPrime(x) → ¬ isInteger(sqrt(x))

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# Semantics of FOL:  first-order structures ⟨U, I⟩

**U**niverse

**I**nterpretation

# Semantics of FOL:  universe

## Universe

- A non-empty set of values
- Finite or (un)countably infinite

## Interpretation

# Semantics of FOL: interpretation

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## Interpretation

- Maps a constant symbol c to an element of U: $I[c] \in U$
- Maps an n-ary function symbol f to a function $f_I : U^n \rightarrow U$
- Maps an n-ary predicate symbol p to an n-ary relation $p_I \subseteq U^n$

# Semantics of FOL: inductive definition

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$I[f(t_1, \ldots, t_n)] = I[f](I[t_1], \ldots, I[t_n])$

$I[p(t_1, \ldots, t_n)] = (\langle I[t_1], \ldots, I[t_n] \rangle \in I[p])$

$\langle U, I \rangle \models \top$

$\langle U, I \rangle \not\models \bot$

$\langle U, I \rangle \models p(t_1, \ldots, t_n)$ **iff** $I[p(t_1, \ldots, t_n)] = \text{true}$

$\langle U, I \rangle \models \neg F$ **iff** $\langle U, I \rangle \not\models F$

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This is the semantics of **unsorted FOL**. SMT solvers work on **many-sorted FOL**, which partitions the universe into different types or sorts, and assigns types to non-logical symbols. SMT interpretations respect these types.

# Semantics of FOL: example

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- Finite or (un)countably infinite

## Interpretation

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- Maps an n-ary function symbol f to a function $f_I : U^n \to U$
- Maps an n-ary predicate symbol p to an n-ary relation $p_I \subseteq U^n$

$U = \{☀, ☁\}$

$I[x] = ☀$

$I[y] = ☁$

$I[f] = \{☀ \mapsto ☁, ☁ \mapsto ☀\}$

$I[p] = \{\langle ☀, ☀\rangle, \langle ☀, ☁\rangle\}$

$\langle U, I\rangle \models p(f(y), f(f(x)))$ ?

# Satisfiability and validity of FOL

F is **satisfiable** iff $M \models F$ for some structure $M = \langle U, I \rangle$.

F is **valid** iff $M \models F$ for all structures $M = \langle U, I \rangle$.

**Duality** of satisfiability and validity:

*F* is valid iff ¬*F* is unsatisfiable.

# First-order theories

## Signature $\Sigma_T$

## Set of T-models

# First-order theories

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- Set of constant, predicate, and function symbols

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- Can also view a theory as a set of axioms over $\Sigma_T$ (and T-models are the models of the theory axioms)

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A formula F is **satisfiable modulo T** iff $M \models F$ for some T-model M.

A formula F is **valid modulo T** iff $M \models F$ for all T-models M.

# First-order theories: expansion

**Signature $\Sigma_T$**

- Set of constant, predicate, and function symbols

**Set of T-models**

- One or more (possibly infinitely many) models that fix the interpretation of the symbols in $\Sigma_T$
- Can also view a theory as a set of axioms over $\Sigma_T$ (and T-models are the models of the theory axioms)

We can **expand** a theory's signature to include additional *uninterpreted* symbols (e.g., constants).

If $E_T$ is an expansion of $\Sigma_T$, then the T-models of $E_T$ are the set of all possible expansions of the T-models of $\Sigma_T$ to include interpretations for the symbols in $E_T \setminus \Sigma_T$.

# Common theories

**Equality (and uninterpreted functions)**

- $x = g(y)$

**Fixed-width bitvectors**

- $(b >> 1) = c$

**Linear arithmetic (over R and Z)**

- $2x + y \leq 5$

**Arrays**

- $a[i] = x$

# Theory of equality with uninterpreted functions

**Signature: {=, x, y, z, …, f, g, …, p, q, …}**

- The binary predicate = is *interpreted*.
- All constant, function, and predicate symbols are *uninterpreted*.

**Axioms**

- $\forall x.\ x = x$
- $\forall x, y.\ x = y \rightarrow y = x$
- $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$
- $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$
- $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \rightarrow (p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n))$

**Deciding $T_=$**

- Conjunctions of literals modulo $T_=$ is decidable in polynomial time.

# $T_=$ example: checking program equivalence

```
int abs(int y) {
  return y<0 ? -y : y;
}

int sq(int y) {
  return y*y;
}
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```
int sqabs(int y) {
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Yes, but the solver takes a while to return an answer because reasoning about multiplication is expensive.

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What happens if we replace the multiplication with an uninterpreted function?

# Theory of fixed-width bitvectors

**Signature**

- Fixed-width words modeling machine ints, longs, …
- Arithmetic operations: bvadd, bvsub, bvmul, …
- Bitwise operations: bvand, bvor, bvnot, …
- Comparison predicates: bvlt, bvgt, …
- Equality: =
- Expanded with all constant symbols: x, y, z, …

**Deciding $T_{BV}$**

- NP-complete.

# Theories of linear integer and real arithmetic

**Signature**

- Integers (or reals)
- Arithmetic operations: multiplication by an integer (or real) number, +, -.
- Predicates: =, ≤.
- Expanded with all constant symbols: x, y, z, …

**Deciding $T_{LIA}$ and $T_{LRA}$**

- NP-complete for linear integer arithmetic (LIA).
- Polynomial time for linear real arithmetic (LRA).
- Polynomial time for difference logic (conjunctions of the form $x - y \leq c$, where c is an integer or real number).

# LIA example: compiler optimization

```
for (i=1; i<=10; i++) {
  a[j+i] = a[j];
}
```

A LIA formula that is unsatisfiable iff this transformation is valid:

```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
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# LIA example: compiler optimization

```
for (i=1; i<=10; i++) {
  a[j+i] = a[j];
}
```

A LIA formula that is unsatisfiable iff this transformation is valid:

$(i \geq 1) \wedge (i \leq 10) \wedge$

$(j + i = j)$

```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
}
```

**Polyhedral model**

# Theory of arrays

**Signature**

- Array operations: read, write
- Equality: =
- Expanded with all constant symbols: x, y, z, …

**Axioms**

- ∀a, i, v. read(write(a, i, v), i) = v
- ∀a, i, j, v. ¬(i = j) → (read(write(a, i, v), j) = read(a, j))
- ∀a, b. (∀i. read(a, i) = read(b, i)) → a = b

**Deciding $T_A$**

- Satisfiability problem: NP-complete.
- Used in many software verification tools to model memory.

# Summary

## Today

- Introduction to SMT
- Quantifier-free FOL (syntax & semantics)
- Overview of common theories

## Next lecture

- Survey of theory solvers