Practical Applications of SAT

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Today

Past 2 lectures

• The theory and mechanics of SAT solving

Today

• Practical applications of SAT
• Variants of the SAT problem
• Motivating the next lecture on SMT

But first ...

• A brief Q&A session for Homework 1
A brief history of SAT solving and applications

Based on a slide from Vijay Ganesh
Bounded Model Checking.
First presented at FMCAD’98. In an unusual move, the Chairs included an extra talk on BMC. A 1999 paper describes its application at Motorola to verify a PowerPC processor.
A brief history of SAT solving and applications

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**Concolic Testing, Program Analysis, Mercedes Product Configuration**

**Synthesis, Type Systems, Bio, Configuration Management, SMT**

Based on a slide from Vijay Ganesh
Bounded Model Checking (BMC) & Configuration Management
Bounded Model Checking (in general)

Given a system and a property, BMC checks if the property is satisfied by all executions of the system with \( \leq k \) steps, on all inputs of size \( \leq n \).
Bounded Model Checking (in general)

Given a system and a property, BMC checks if the property is satisfied by all executions of the system with \( \leq k \) steps, on all inputs of size \( \leq n \).

We will focus on safety properties (i.e., making sure a bad state, such as an assertion violation, is not reached).
Bounded Model Checking (in general)

Testing: checks a few executions of arbitrary size

BMC: checks all executions of size $\leq k$

Verification: checks all executions of every size

low confidence  |  high confidence
---|---
low human labor  |  high human labor
Bounded Model Checking (in general)

Testing: checks a few executions of arbitrary size

BMC: checks all executions of size $\leq k$

Verification: checks all executions of every size

The **small scope hypothesis** says that many bugs can be triggered with small inputs and executions.

low confidence → low human labor

high confidence → high human labor
BMC by example
int daysToYear(int days) {
    int year = 1980;
    while (days > 365) {
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
    }
    return year;
}

The Zune Bug: on December 31, 2008, all first generation Zune players from Microsoft became unresponsive because of this code. What’s wrong?
BMC by example

```c
int daysToYear(int days) {
    int year = 1980;
    while (days > 365) {
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
    }
    return year;
}
```

Infinite loop triggered on the last day of every leap year.
BMC by example

```c
int daysToYear(int days) {
    int year = 1980;
    while (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
    }
    return year;
}
```

A desired safety property: the value of the `days` variable decreases in every loop iteration.
BMC step 1 of 4: finitize loops & inline calls

```c
int daysToYear(int days) {
    int year = 1980;
    while (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
    }
    return year;
}
```
BMC step 1 of 4: finitize loops & inline calls

```c
int daysToYear(int days) {
    int year = 1980;
    if (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            } else {
                days -= 365;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
    }
    assert days < oldDays;
    assert days <= 365;
}
return year;
```

- Unwind all loops k times (e.g., k=1), and add an unwinding assertion after each.
BMC step 1 of 4: finitize loops & inline calls

```c
int daysToYear(int days) {
    int year = 1980;
    if (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
    }
    assert days <= 365;
    return year;
}
```

- Unwind all loops \( k \) times (e.g., \( k=1 \)), and add an **unwinding assertion** after each.
- If a CEX violates a program assertion, we have found a buggy behavior of length \( \leq k \).
BMC step 1 of 4: finitize loops & inline calls

```c
int daysToYear(int days) {
    int year = 1980;
    if (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
    }
    assert days <= 365;
}
return year;
}
```

- Unwind all loops k times (e.g., k=1), and add an **unwinding assertion** after each.
- If a CEX violates a program assertion, we have found a buggy behavior of length \( \leq k \).
- If a CEX violates an unwinding assertion, the program has no buggy behavior of length \( \leq k \), but it may have a longer one.
int daysToYear(int days) {
    int year = 1980;
    if (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
        assert days <= 365;
    }
    return year;
}
int daysToYear(int days) {
    int year = 1980;
    if (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
    }
    assert days <= 365;
    return year;
}
int daysToYear(int days) {
    int year = 1980;
    if (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
        assert days <= 365;
    }
    return year;
}
BMC step 2 of 4: eliminate side effects

```c
int days;
int year = 1980;
if (days > 365) {
    int oldDays = days;
    if (isLeapYear(year)) {
        if (days > 366) {
            days = days - 366;
            year = year + 1;
        }
    } else {
        days = days - 365;
        year = year + 1;
    }
    assert days < oldDays;
    assert days <= 365;
}
return year;
```
BMC step 2 of 4: eliminate side effects

```c
int days;
int year = 1980;
if (days > 365) {
    int oldDays = days;
    if (isLeapYear(year)) {
        if (days > 366) {
            days = days - 366;
            year = year + 1;
        }
    } else {
        days = days - 365;
        year = year + 1;
    }
    assert days < oldDays;
    assert days <= 365;
}
return year;
```

Convert to Static Single Assignment (SSA) form:

- Replace each assignment to a variable \( v \) with a definition of a fresh variable \( v_i \).
- Change uses of variables so that they refer to the correct definition (version).
- Make conditional dependences explicit with gated \( \varphi \) nodes.
BMC step 2 of 4: eliminate side effects

```c
int days0;
int year0 = 1980;
if (days0 > 365) {
    int oldDays0 = days0;
    if (isLeapYear(year0)) {
        if (days0 > 366) {
            days1 = days0 - 366;
            year1 = year0 + 1;
        }
    } else {
        days3 = days0 - 365;
        year3 = year0 + 1;
    }
    assert days4 < oldDays0;
    assert days4 <= 365;
}
return year5;
```

Convert to **Static Single Assignment** (SSA) form:

- Replace each assignment to a variable v with a definition of a fresh variable $v_i$.
- Change uses of variables so that they refer to the correct definition (version).
- Make conditional dependences explicit with gated $\phi$ nodes.
BMC step 2 of 4: eliminate side effects

```java
int days0;
int year0 = 1980;
boolean g0 = (days0 > 365);
int oldDays0 = days0;
boolean g1 = isLeapYear(year0);
boolean g2 = days0 > 366;
days1 = days0 - 366;
year1 = year0 + 1;
days2 = φ(g1 && g2, days1, days0);
year2 = φ(g1 && g2, year1, year0);
days3 = days0 - 365;
year3 = year0 + 1;
days4 = φ(g1, days2, days3);
year4 = φ(g1, year2, year3);
assert days4 < oldDays0;
assert days4 <= 365;
year5 = φ(g0, year4, year0);
return year5;
```

Convert to **Static Single Assignment** (SSA) form:

- Replace each assignment to a variable v with a definition of a fresh variable v_i.
- Change uses of variables so that they refer to the correct definition (version).
- Make conditional dependences explicit with gated φ nodes.
BMC step 2 of 4: eliminate side effects

```c
int days0;
int year0 = 1980;
if (days0 > 365) {
    int oldDays0 = days0;
    if (isLeapYear(year0)) {
        if (days0 > 366) {
            days1 = days0 - 366;
            year1 = year0 + 1;
        }
    } else {
        days3 = days0 - 365;
        year3 = year0 + 1;
    }
    assert days4 < oldDays0;
    assert days4 <= 365;
}
return year4;
```

```c
int days0;
int year0 = 1980;
boolean g0 = (days0 > 365);
int oldDays0 = days0;
boolean g1 = isLeapYear(year0);
boolean g2 = days0 > 366;
days1 = days0 - 366;
year1 = year0 + 1;
days2 = φ(g1 && g2, days1, days0);
year2 = φ(g1 && g2, year1, year0);
days3 = days0 - 365;
year3 = year0 + 1;
days4 = φ(g1, days2, days3);
year4 = φ(g1, year2, year3);
assert days4 < oldDays0;
assert days4 <= 365;
year5 = φ(g0, year4, year0);
return year5;
```
int days₀;
int year₀ = 1980;
boolean g₀ = (days₀ > 365);
int oldDays₀ = days₀;
boolean g₁ = isLeapYear(year₀);
boolean g₂ = days₀ > 366;
days₁ = days₀ - 366;
year₁ = year₀ + 1;
days₂ = φ(g₁ && g₂, days₁, days₀);
year₂ = φ(g₁ && g₂, year₁, year₀);
days₃ = days₀ - 365;
year₃ = year₀ + 1;
days₄ = φ(g₁, days₂, days₃);
year₄ = φ(g₁, year₂, year₃);
assert days₄ < oldDays₀;
assert days₄ <= 365;
year₅ = φ(g₀, year₄, year₀);
return year₅;
BMC step 3 of 4: convert into equations

```plaintext
year_0 = 1980 \land 
g_0 = (days_0 > 365) \land 
oldDays_0 = days_0 \land 
g_1 = isLeapYear(year_0) \land 
g_2 = days_0 > 366 \land 
days_1 = days_0 - 366 \land 
year_1 = year_0 + 1 \land 
days_2 = \text{ite}(g_1 \land g_2, days_1, days_0) \land 
year_2 = \text{ite}(g_1 \land g_2, year_1, year_0) \land 
days_3 = days_0 - 365 \land 
year_3 = year_0 + 1 \land 
days_4 = \text{ite}(g_1, days_2, days_3) \land 
year_4 = \text{ite}(g_1, year_2, year_3) \land 
year_5 = \text{ite}(g_0, year_4, year_0) \land 
\neg(days_4 < oldDays_0) \lor 
\neg(days_4 <= 365))
```

A solution to these equations is a sound **counterexample**: an interpretation for all logical variables that satisfies the program semantics (for up to \( k \) unwindings) but violates at least one of the assertions.
BMC step 4 of 4: convert into CNF

\[ \text{year}_1 = \text{year}_0 + 1 \]
BMC step 4 of 4: convert into CNF

$\text{year}_1 = \text{year}_0 + 1$

$\text{year}_0 = 000 \ldots 000$

Represent numbers as arrays of bits …
BMC step 4 of 4: convert into CNF

Represent numbers as arrays of bits, and create one propositional variable per bit for each number.

\[ \text{year}_1 = \text{year}_0 + 1 \]

\[ \text{year}_0 = \underbrace{000 \ldots 000}_{31 \ 30 \ 29 \ 2 \ 1 \ 0} \]
BMC step 4 of 4: convert into CNF

Represent numbers as arrays of bits, and create one propositional variable per bit for each number.

\[ \text{year}_1 = \text{year}_0 + 1 \]

\[ \text{year}_0 = 000 \ldots 000 \]

Construct an adder circuit for \( \text{year}_0 + 1 \).
Represent numbers as arrays of bits, and create one propositional variable per bit for each number.

Introduce new clauses to constrain bits in \( \text{year}_1 \) to match bits in the sum.

Construct an adder circuit for \( \text{year}_0 + 1 \).

\[ \text{year}_1 = \text{year}_0 + 1 \]

\[ \text{year}_0 = 000 \ldots 000 \]

\[ 31 \ 30 \ 29 \ 2 \ 1 \ 0 \]

BMC step 4 of 4: convert into CNF
BMC counterexample for k=1

```c
int daysToYear(int days) {
    int year = 1980;
    while (days > 365) {
        int oldDays = days;
        if (isLeapYear(year)) {
            if (days > 366) {
                days -= 366;
                year += 1;
            }
        } else {
            days -= 365;
            year += 1;
        }
        assert days < oldDays;
    }
    return year;
}
```

`days = 366`
Bounded Model Checking (BMC) & Configuration Management
Configuration Management

Given a configuration, consisting of a set of components, their dependencies, and conflicts:

• Decide if a new component can be added to the configuration.

• Add the component while optimizing some linear function.

• If the component cannot be added, find a way to add it by removing as few conflicting components from the current configuration as possible.
Given a configuration, consisting of a set of components, their dependencies, and conflicts:

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Configuration Management

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SAT
Pseudo-Boolean Constraints
Configuration Management

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• Decide if a new component can be added to the configuration.

• Add the component while optimizing some linear function.

• If the component cannot be added, find a way to add it by removing as few conflicting components from the current configuration as possible.

SAT
Pseudo-Boolean Constraints
Partial (Weighted) MaxSAT
Deciding if a component can be installed
Deciding if a component can be installed

z already installed.
Deciding if a component can be installed

- Component a depends on b, c, and z.
- Component z is already installed.
Deciding if a component can be installed

- a depends on b, c, z.
- z already installed.
- c needs f or g.
Deciding if a component can be installed

Conflict: d and e cannot both be installed.
Deciding if a component can be installed

Conflict: d and e cannot both be installed.

To install a, CNF constraints are:

a depends on b, c, z.

z already installed.

c needs f or g.

a depends on b, c, z.
Deciding if a component can be installed

To install a, CNF constraints are:

\((\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land \)

Conflict: d and e cannot both be installed.
Deciding if a component can be installed

To install a, CNF constraints are:

\[(\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land (\neg b \lor d) \land \]

Conflict: d and e cannot both be installed.
Deciding if a component can be installed

Conflict: d and e cannot both be installed.

To install a, CNF constraints are:

\((\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land (\neg b \lor d) \land (\neg c \lor d \lor e) \land (\neg c \lor f \lor g) \land z \text{ already installed.}\)
Deciding if a component can be installed

Conflict: d and e cannot both be installed.

To install a, CNF constraints are:

\((\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land (\neg b \lor d) \land (\neg c \lor d \lor e) \land (\neg c \lor f \lor g) \land (\neg d \lor \neg e) \land \)
Deciding if a component can be installed

To install a, CNF constraints are:

\[(\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land \]
\[(\neg b \lor d) \land \]
\[(\neg c \lor d \lor e) \land (\neg c \lor f \lor g) \land \]
\[(\neg d \lor \neg e) \land \]
\[(\neg y \lor z) \land \]

Conflict: d and e cannot both be installed.
Deciding if a component can be installed

To install a, CNF constraints are:

\[(\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land (\neg b \lor d) \land (\neg c \lor d \lor e) \land (\neg c \lor f \lor g) \land (\neg d \lor \neg e) \land (\neg y \lor z) \land a \land z\]
Optimal installation
Assume f and g are 5MB and 2MB each, and all other components are 1MB. To install a, while minimizing total size, pseudo-boolean constraints are:

\[
\begin{align*}
\text{min } & a + b + c + d + e + 5f + 2g + y + 0z \\
\text{subject to } & -a + b \geq 0 \\
& -a + c \geq 0 \\
& -a + z \geq 0 \\
& -b + d \geq 0 \\
& -c + d + e \geq 0 \\
& -c + f + g \geq 0 \\
& -d + -e \geq -1 \\
& -y + z \geq 0 \\
& \text{and } a \geq 1 \\
& \text{and } z \geq 1
\end{align*}
\]

Optimal installation

Pseudo-boolean solvers accept a linear function to minimize, in addition to a (weighted) CNF.
Assume $f$ and $g$ are 5MB and 2MB each, and all other components are 1MB. To install $a$, while minimizing total size, pseudo-boolean constraints are:

$$\text{min } c_1x_1 + \ldots + c_nx_n$$

$$a_1x_1 + \ldots + a_nx_n \geq b_1$$

$$\ldots$$

$$a_kx_1 + \ldots + a_kx_n \geq b_k$$
Assume $f$ and $g$ are 5MB and 2MB each, and all other components are 1MB. To install $a$, while minimizing total size, pseudo-boolean constraints are:

$$\text{min } a + b + c + d + e + 5f + 2g + y + 0z$$

$$(-a + b \geq 0) \land (-a + c \geq 0) \land (-a + z \geq 0) \land (-b + d \geq 0) \land (-c + d + e \geq 0) \land (-c + f + g \geq 0) \land (-d + -e \geq -1) \land (-y + z \geq 0) \land (a \geq 1) \land (z \geq 1)$$

Optimal installation:

\[ \begin{align*}
\min & \quad c_1 x_1 + \ldots + c_n x_n \\
\text{subject to} & \quad a_{11} x_1 + \ldots + a_{1n} x_n \geq b_1 \\
& \quad \ldots \\
& \quad a_{k1} x_1 + \ldots + a_{kn} x_n \geq b_k \\
& \quad a \geq 1 \land z \geq 1
\end{align*} \]
Installation in the presence of conflicts
Installation in the presence of conflicts

a cannot be installed because it requires b, which requires d, which conflicts with e.
Installation in the presence of conflicts

To install a, while minimizing the number of removed components, Partial MaxSAT constraints are:

**hard:** \((\neg a \lor b) \land (\neg a \lor c) \land (\neg a \lor z) \land (\neg b \lor d) \land (\neg c \lor d \lor e) \land (\neg c \lor f \lor g) \land (\neg d \lor \neg e) \land (\neg y \lor z) \land a\)

**soft:** \(e \land z\)

Partial MaxSAT solver takes as input a set of **hard** clauses and a set of **soft** clauses, and it produces an assignment that satisfies all hard clauses and the greatest number of soft clauses.
Summary

Today

• SAT solvers have been used successfully in many applications & domains
• But reducing problems to SAT is a lot like programming in assembly …
• We need higher-level logics!

Next lecture

• On to richer logics: introduction to Satisfiability Modulo Theories (SMT)