Computer-Aided Reasoning for Software

CSE507

A Survey of Theory Solvers

courses.cs.washington.edu/courses/cse507/17wi/

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Today

Last lecture

- Introduction to Satisfiability Modulo Theories (SMT)

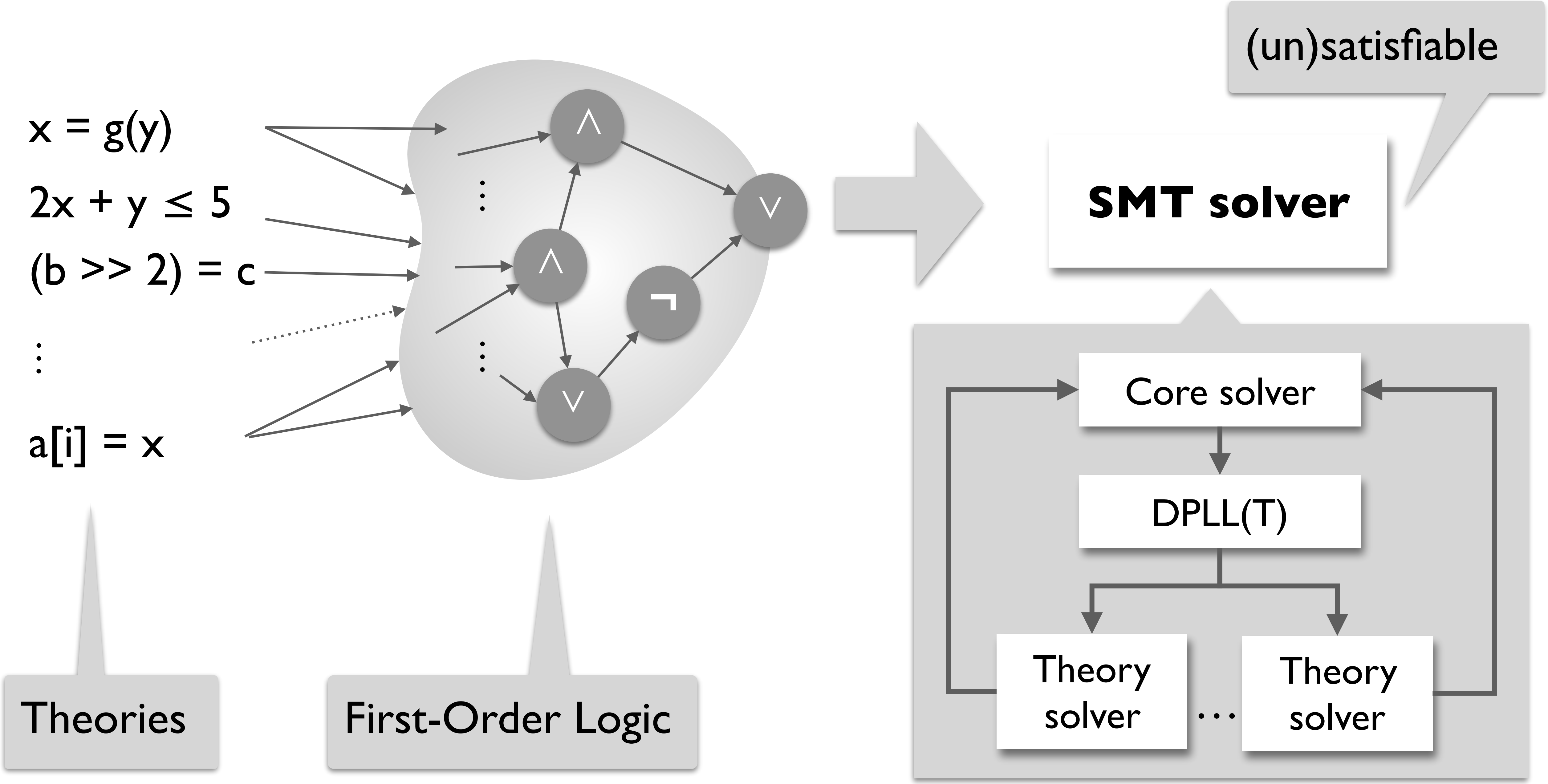
Today

- A quick survey of theory solvers
- An in-depth look at the core theory solver (theory of equality and uninterpreted functions)

Reminder

- Start thinking about your project & find a partner

Recall: Satisfiability Modulo Theories (SMT)



$x = g(y)$

$2x + y \leq 5$

$(b >> 2) = c$

⋮

$a[i] = x$

Theories

First-Order Logic

SMT solver

(un)satisfiable

Core solver

DPLL(T)

Theory solver

…

Theory solver

A brief survey of common theory solvers

$x = g(y)$

Core solver

$2x + y \leq 5$

Theory solver

$2i + j \leq 5$

Theory solver

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Theory solver

$a[i] = x$

Theory solver

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

$(b >> 2) = c$

Fixed-Width Bitvectors

$a[i] = x$

Arrays

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Arrays

- **Conjunctions** of linear constraints over R
  - Can be decided in polynomial time, but in practice solved with the **General Simplex** method (worst case exponential)
  - Can also be decided with **Fourier-Motzkin** elimination (exponential)

A brief survey of common theory solvers

$x = g(y)$

Equality and UF

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

- **Conjunctions** of linear constraints over Z
    - **Branch-and-cut** (based on Simplex)
    - **Omega Test** (extension of Fourier-Motzkin)
- **Small-Domain Encoding** used for arbitrary combinations of linear constraints over Z
- NP-complete

$(b >> 2) = c$

Fixed-Width Bitvectors

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Arrays

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x = g(y)

Equality and UF

2x + y ≤ 5

Linear Real Arithmetic

2i + j ≤ 5

Linear Integer Arithmetic

(b >> 2) = c

Fixed-Width Bitvectors

- **Arbitrary combination** of constraints over bitvectors
  - **Bit blasting** (reduction to SAT)
  - NP-complete

a[i] = x

Arrays

A brief survey of common theory solvers

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Linear Integer Arithmetic

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Fixed-Width Bitvectors

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Arrays

- **Conjunctions** of constraints over read/write terms in the theory of arrays
  - Reduce to $T_=$ satisfiability
  - NP-complete (because the reduction introduces disjunctions)

A brief survey of common theory solvers

x = g(y)

Equality and UF

- **Conjunctions** of equality constraints over uninterpreted functions
  - **Congruence closure**
  - Polynomial time

$2x + y \leq 5$

Linear Real Arithmetic

$2i + j \leq 5$

Linear Integer Arithmetic

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Fixed-Width Bitvectors

a[i] = x

Arrays

Theory of equality and UF ($T_=$)

Signature (all symbols)

- $\{=, a, b, c, \ldots, f, g, \ldots, p, q, \ldots\}$

Axioms

- reflexivity: $\forall x.\ x = x$
- symmetry: $\forall x, y.\ x = y \rightarrow y = x$
- transitivity: $\forall x, y, z.\ x = y \wedge y = z \rightarrow x = z$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \leq i \leq n}\, x_i = y_i) \rightarrow f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$
- congruence: $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (\wedge_{1 \leq i \leq n}\, x_i = y_i) \rightarrow p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)$

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Replace predicates with equality constraints over functions:

- introduce a fresh constant T
- for each predicate p, introduce a fresh function f_p
- $p(x_1, \ldots, x_n) \rightsquigarrow f_p(x_1, \ldots, x_n) = T$

Theory of equality and UF ($T_=$)

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$T_=$ models

- all structures $\langle U, I \rangle$ that satisfy the axioms of $T_=$

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$T_=$ models

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$T_=$ models?

U = {☀, ☁}

I_1[=] : {⟨☀,☁⟩, ⟨☁,☀⟩}

I_2[=] : {⟨☀,☀⟩, ⟨☁,☁⟩}

I_3[=] : {⟨☀,☀⟩, ⟨☁,☁⟩,
⟨☀,☁⟩, ⟨☁,☀⟩}

Is a conjunction of $T_=$ literals satisfiable?

$$f(f(f(a))) = a \land f(f(f(f(f(a))))) = a \land f(a) \neq a$$

Is a conjunction of $T_=$ literals satisfiable?

$$f^3(a) = a \wedge f^5(a) = a \wedge f(a) \neq a$$

Congruence closure algorithm: example

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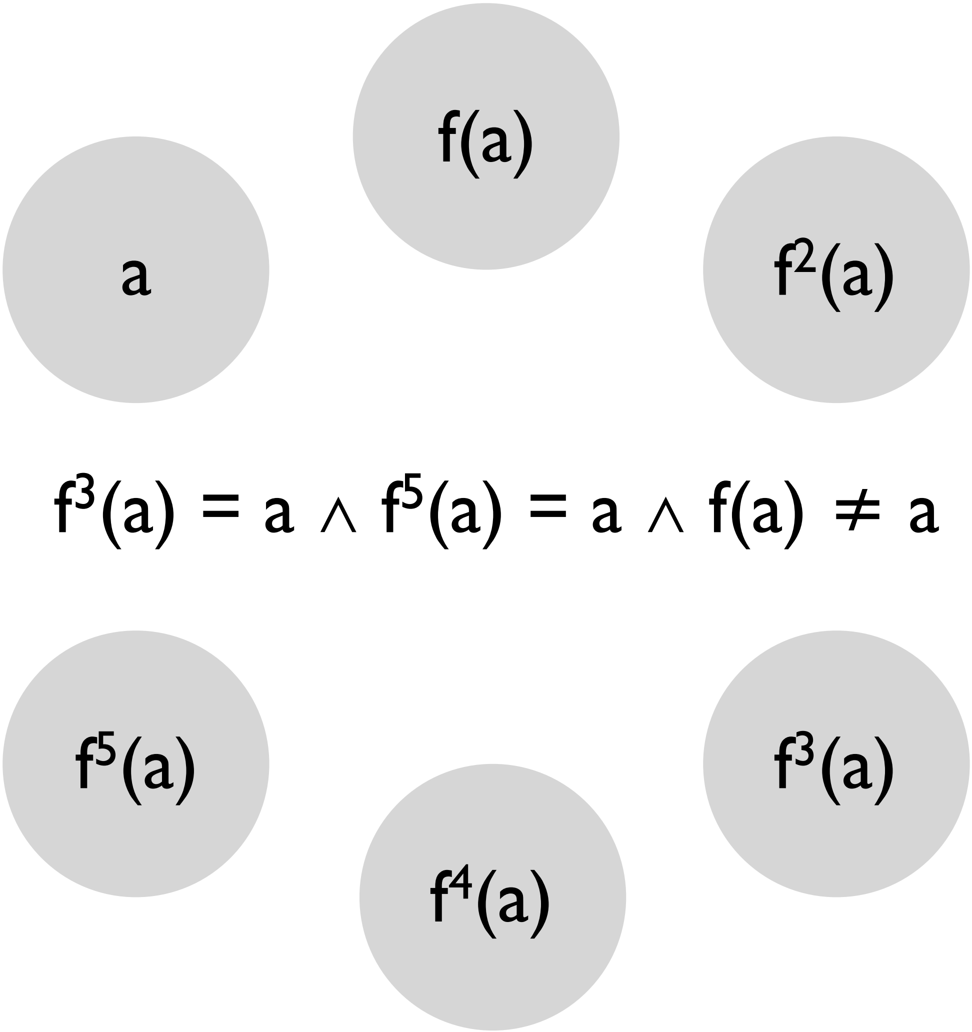
Congruence closure algorithm: example

- Place each subterm of F into its own **congruence class**



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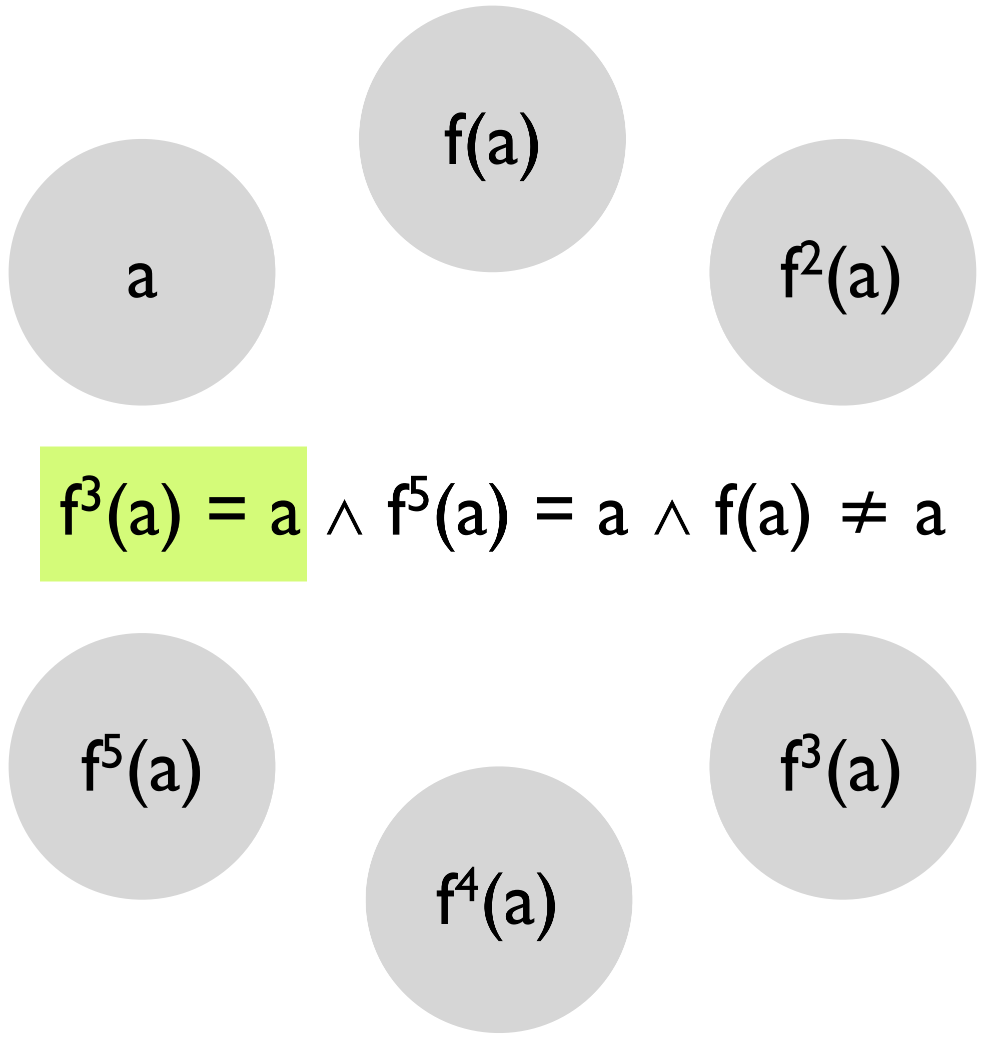

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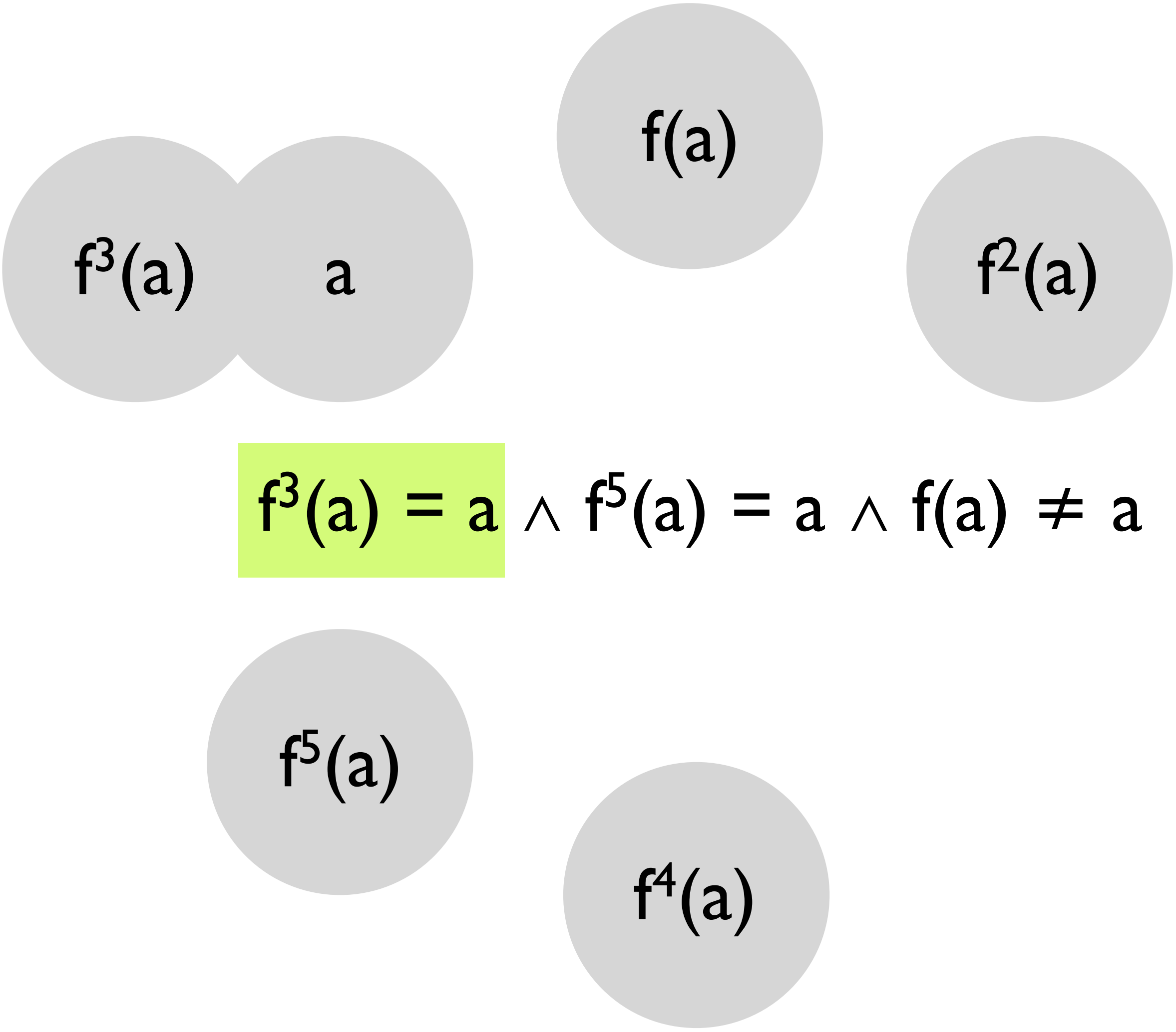

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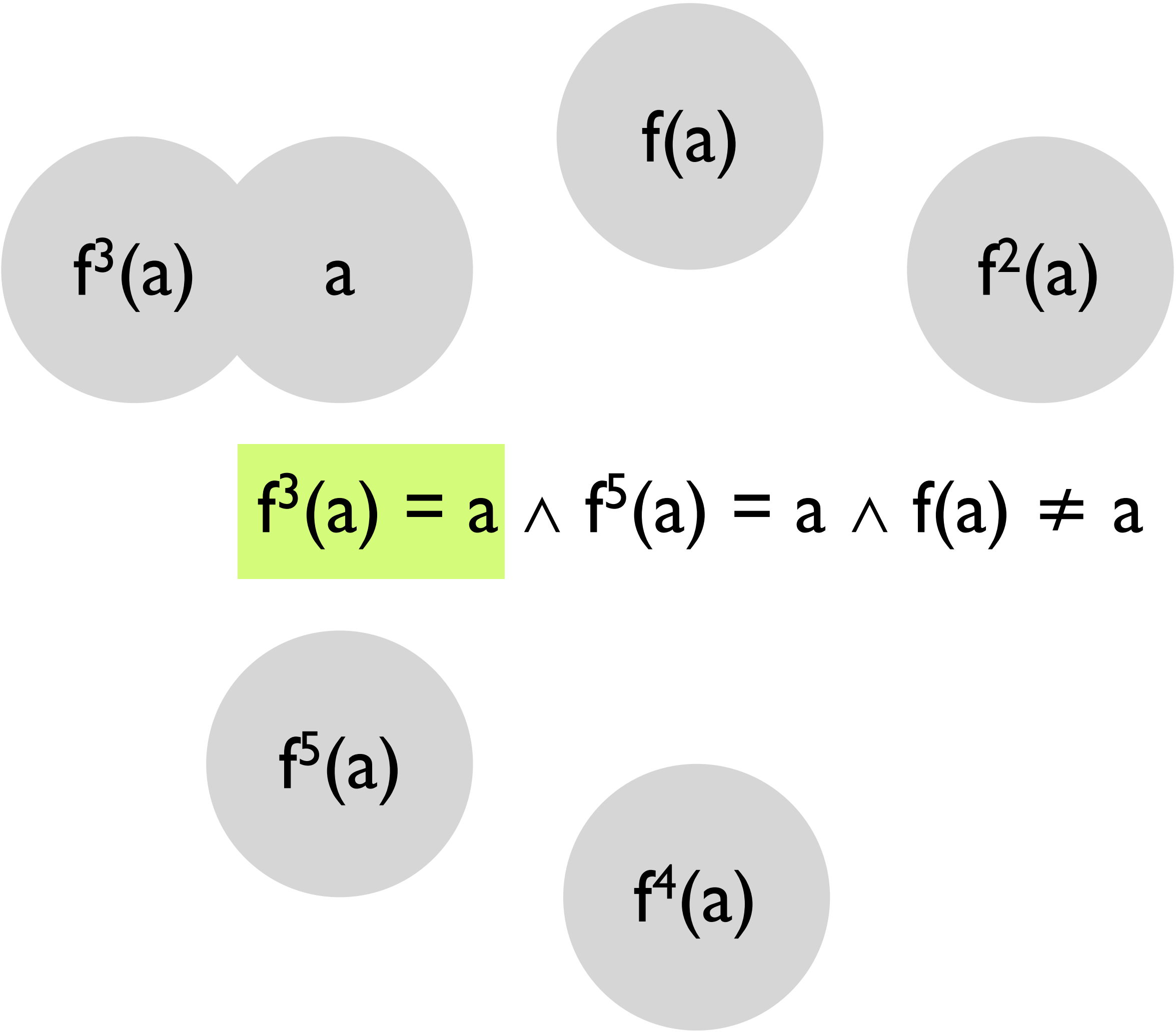

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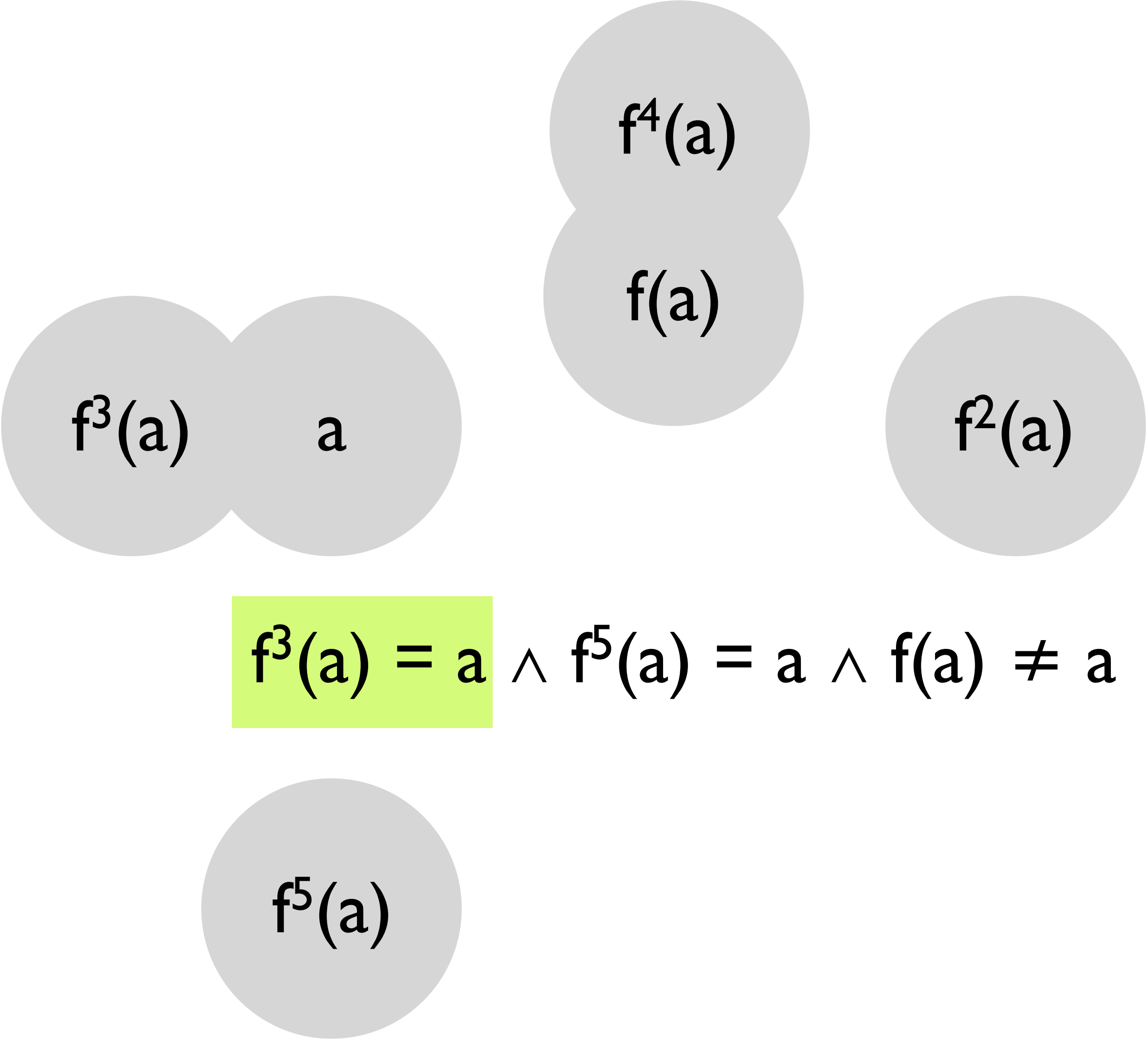

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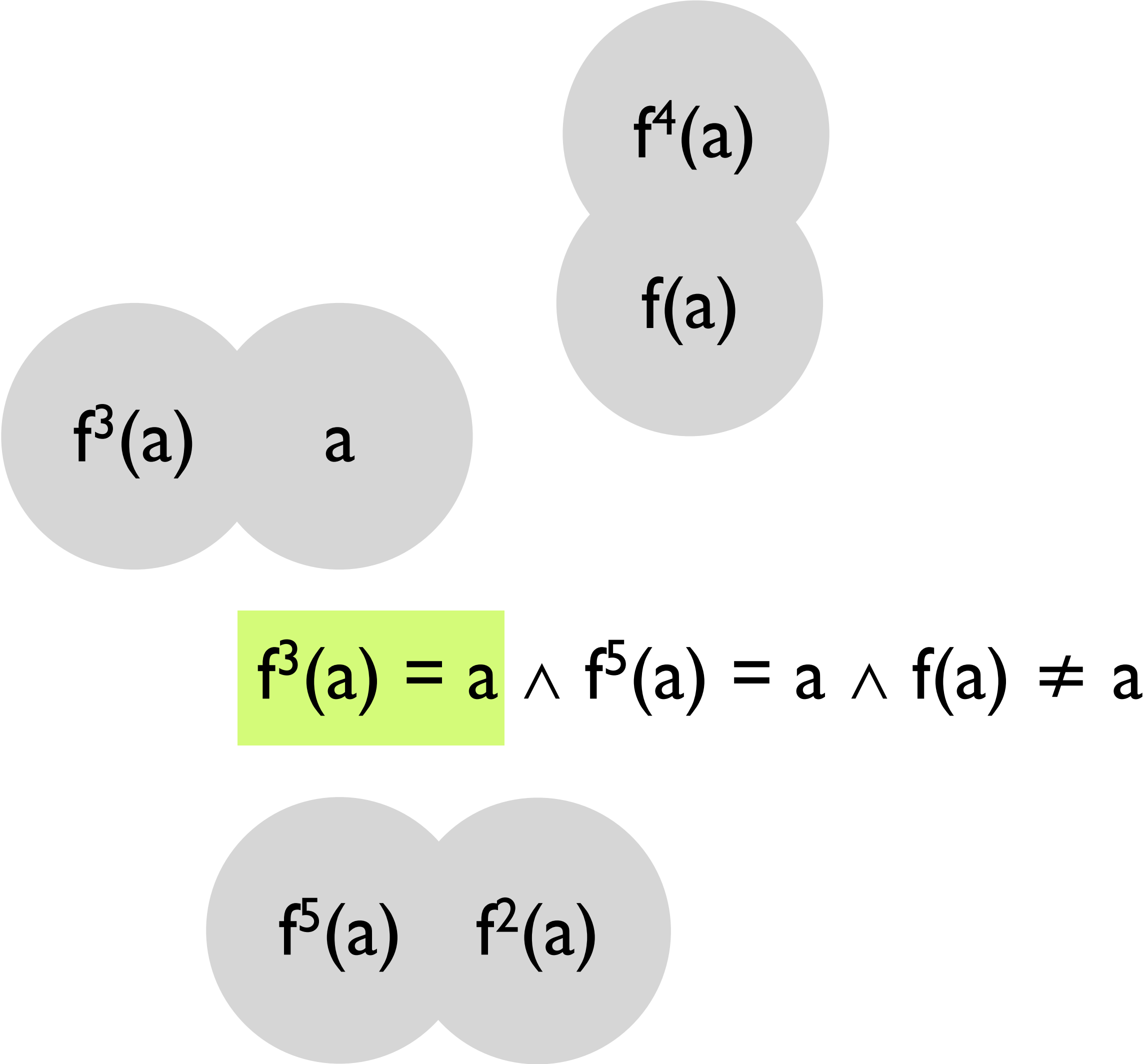

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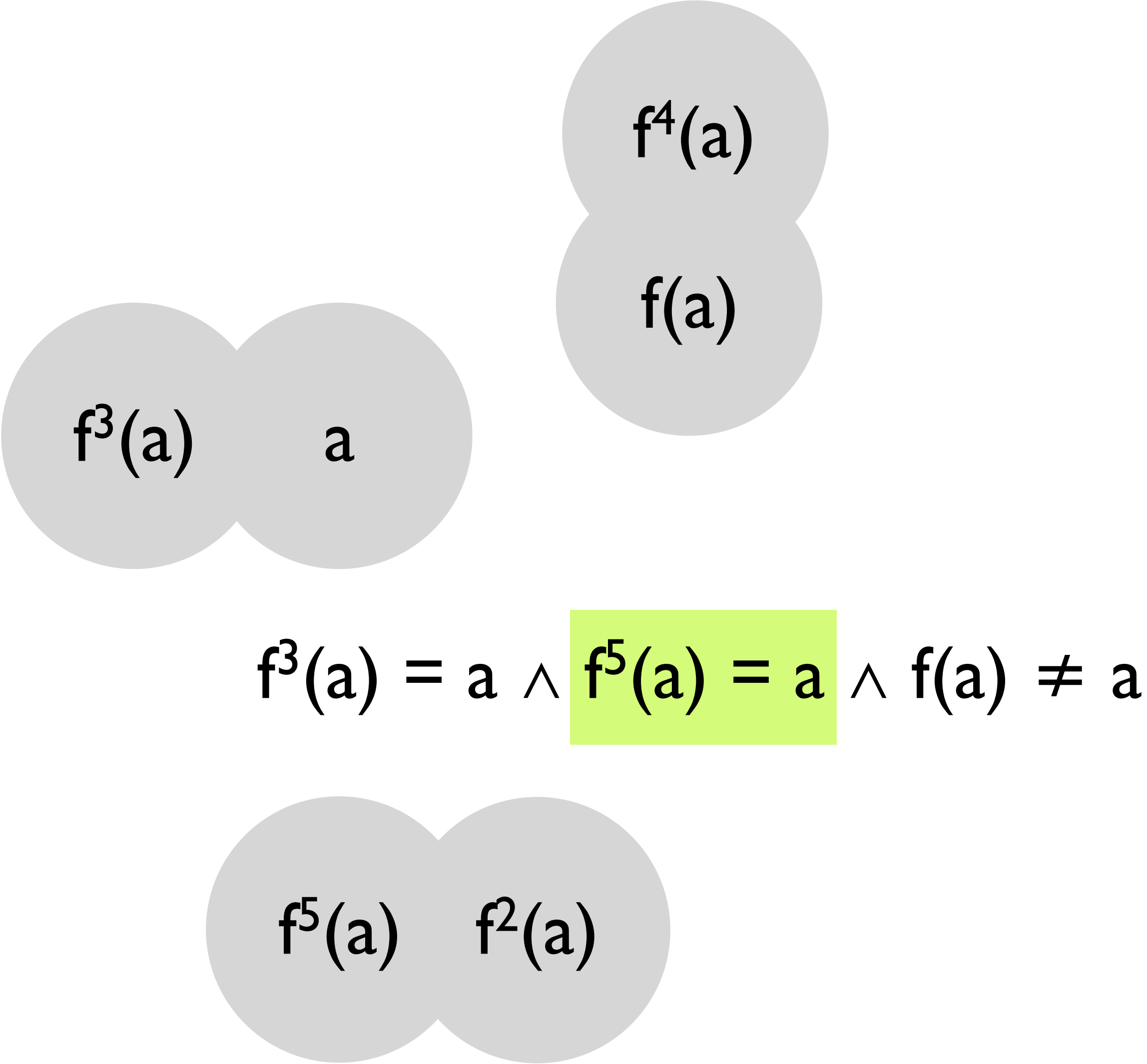

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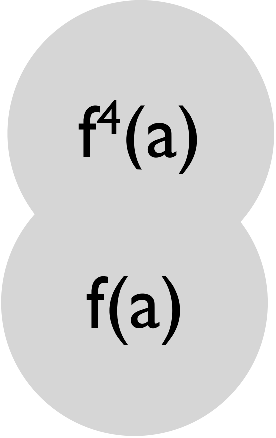

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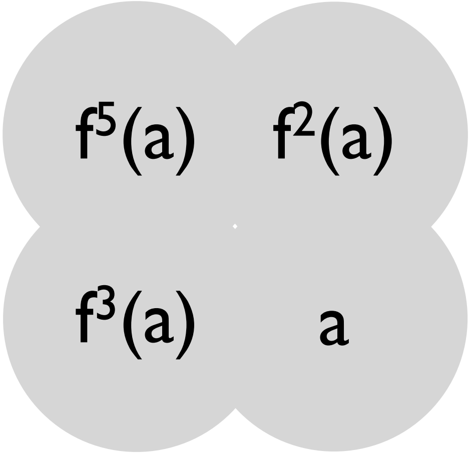

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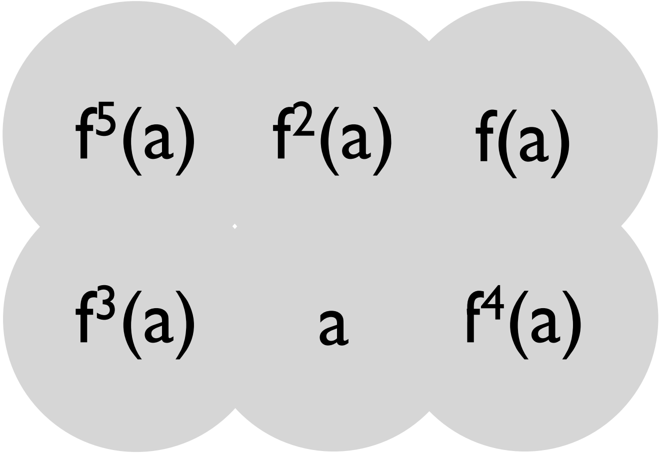

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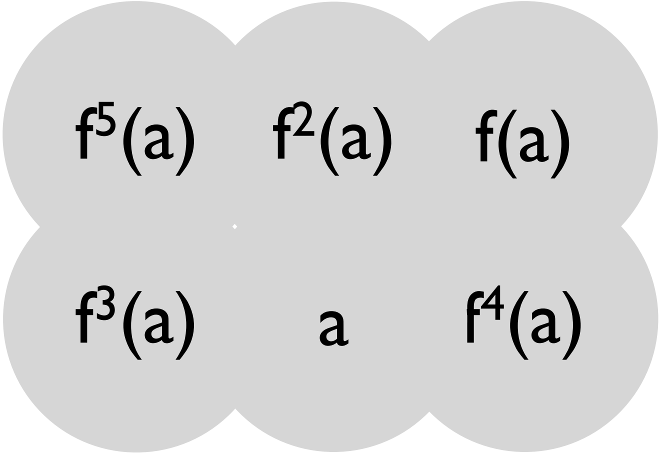

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UNSAT

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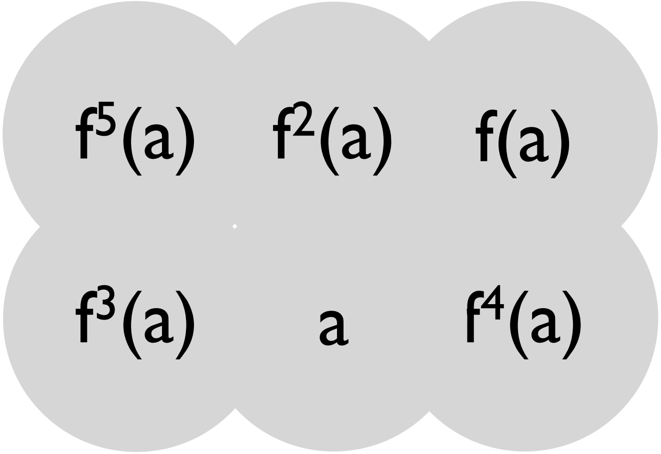

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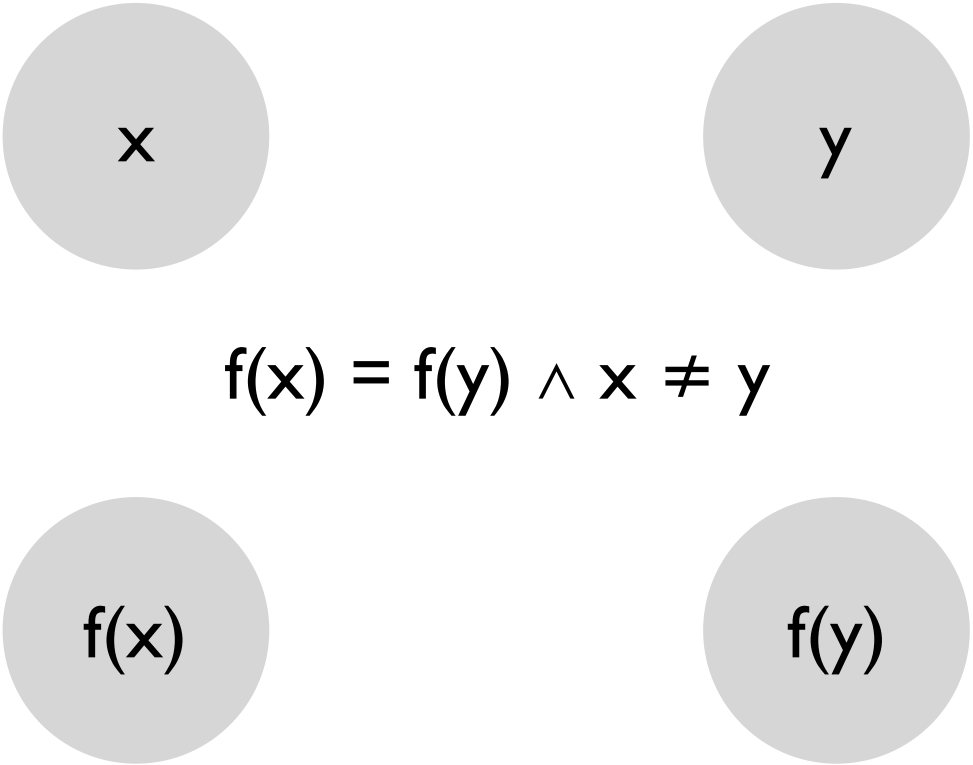

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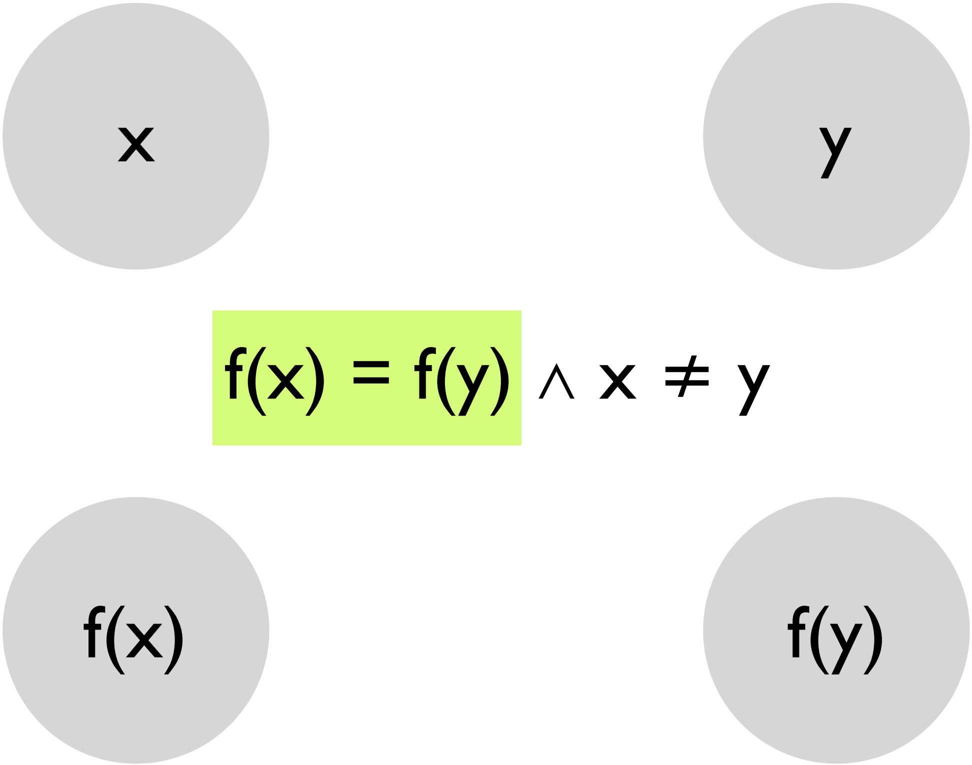

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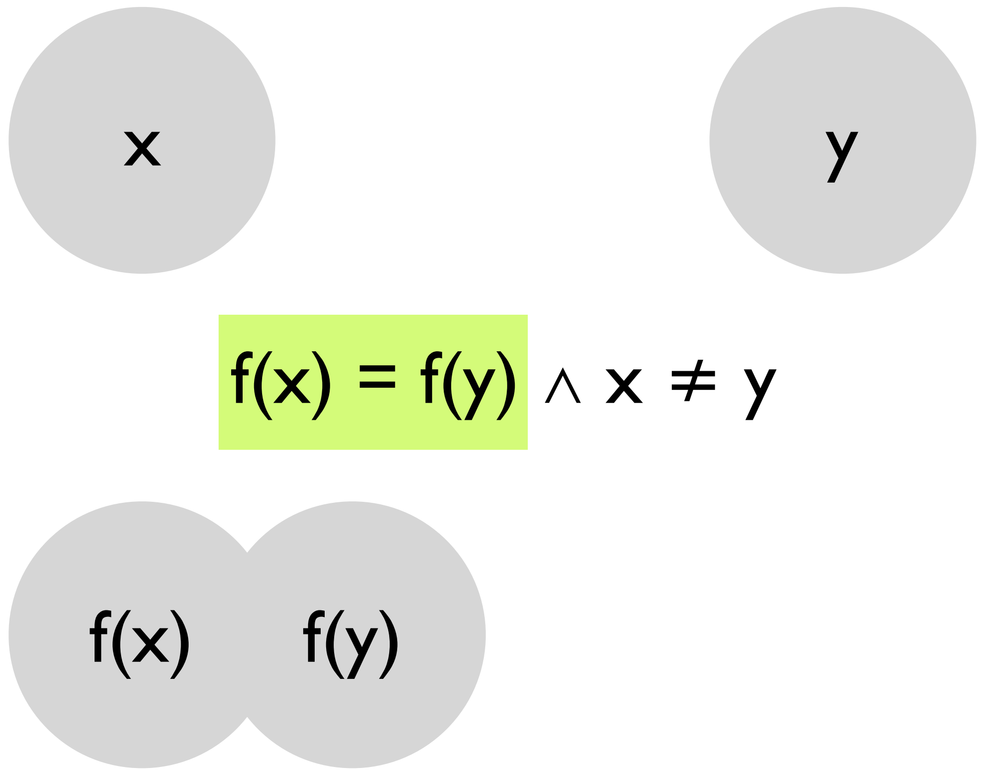

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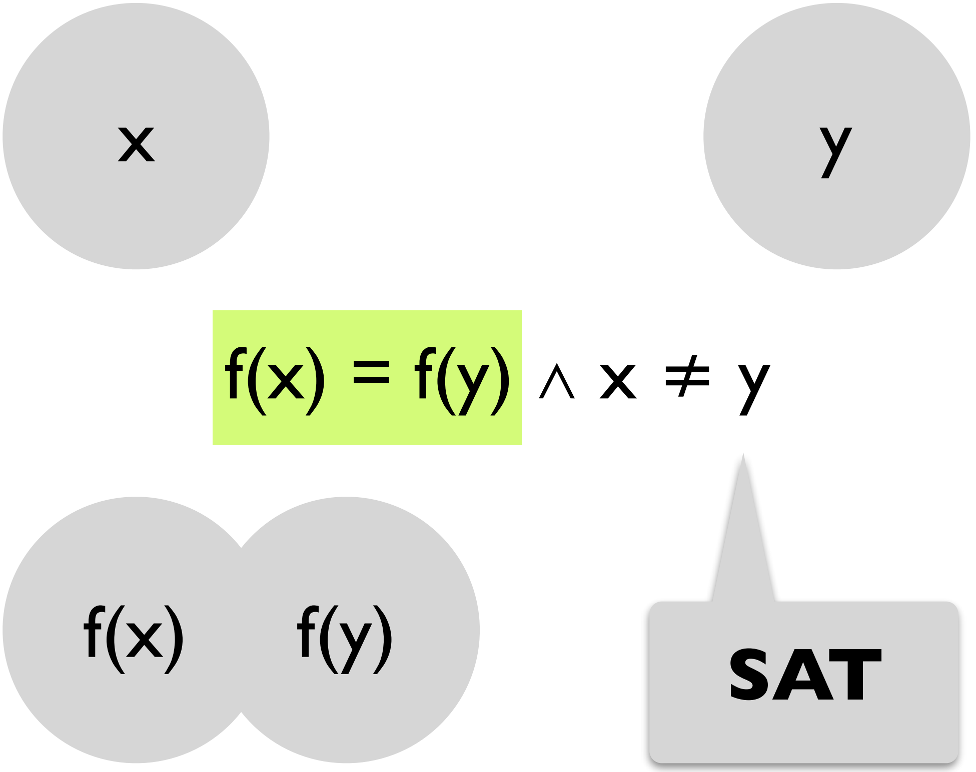

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SAT

Congruence closure algorithm: definitions

A binary relation R is an **equivalence relation** if it is reflexive, symmetric, and transitive.

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An equivalence relation R is a **congruence relation** if for every n-ary function f

$$\forall \overline{x}, \overline{y}.\ \bigwedge R(x_i, y_i) \rightarrow R(f(\overline{x}), f(\overline{y}))$$

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The **equivalence class** of an element s ∈ S under an equivalence relation R:

$$\{\ s' \in S \mid R(s, s')\ \}$$

What is the equivalence class of 9 under $\equiv_3$?

Congruence closure algorithm: definitions

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The **equivalence class** of an element s ∈ S under an equivalence relation R:

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An equivalence class is called a **congruence class** if R is a congruence relation.

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

What is the equivalence closure of $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle\}$?

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

What is the equivalence closure of $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle d, d\rangle\}$?

$$R^E = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle d, d\rangle \\ \langle a, b\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle c, b\rangle, \\ \langle a, c\rangle, \langle c, a\rangle\}$$

Congruence closure algorithm: definitions

The **equivalence closure** R^E of a binary relation R is the smallest equivalence relation that contains R.

The **congruence closure** R^C of a binary relation R is the smallest congruence relation that contains R.

The congruence closure algorithm computes the congruence closure of the equality relation over terms asserted by a conjunctive quantifier-free formula in $T_=$.

Congruence closure algorithm: data structure

$f(a, b) = a \wedge f(f(a, b), b) \neq a$

Congruence closure algorithm: data structure

- Represent subterms with a DAG

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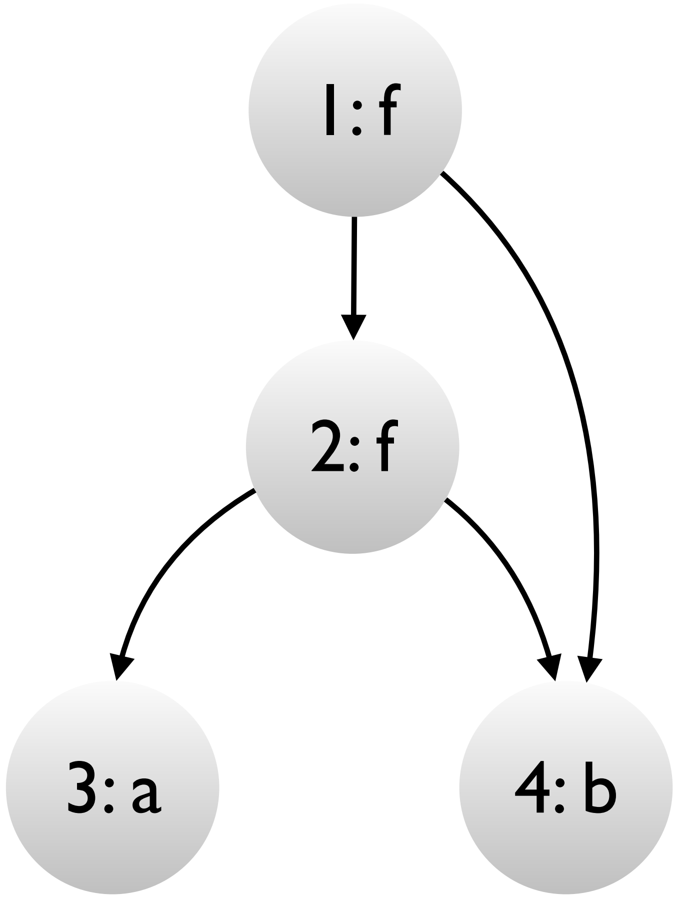

Congruence closure algorithm: data structure

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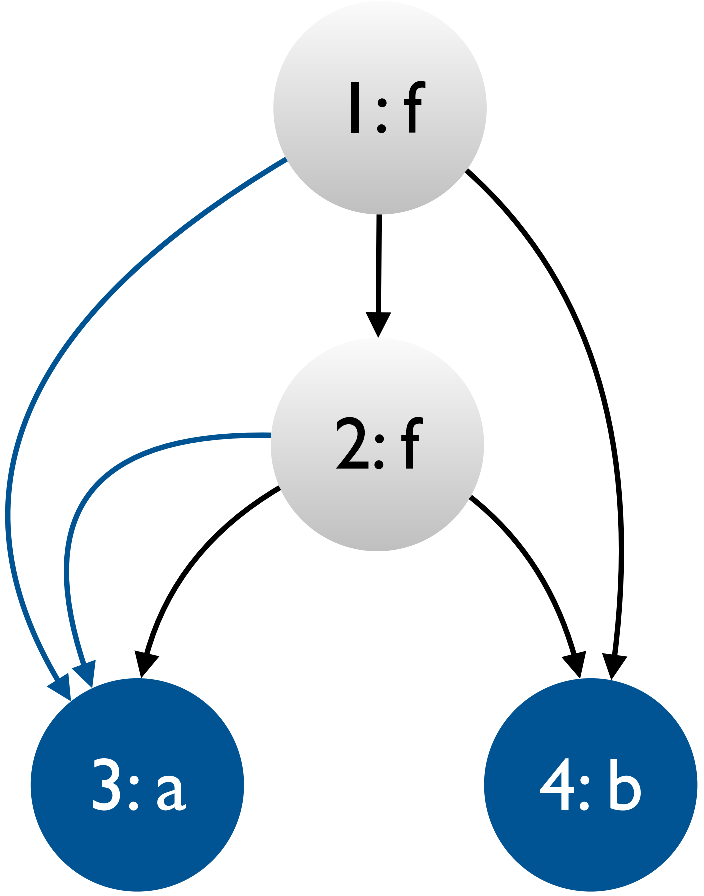
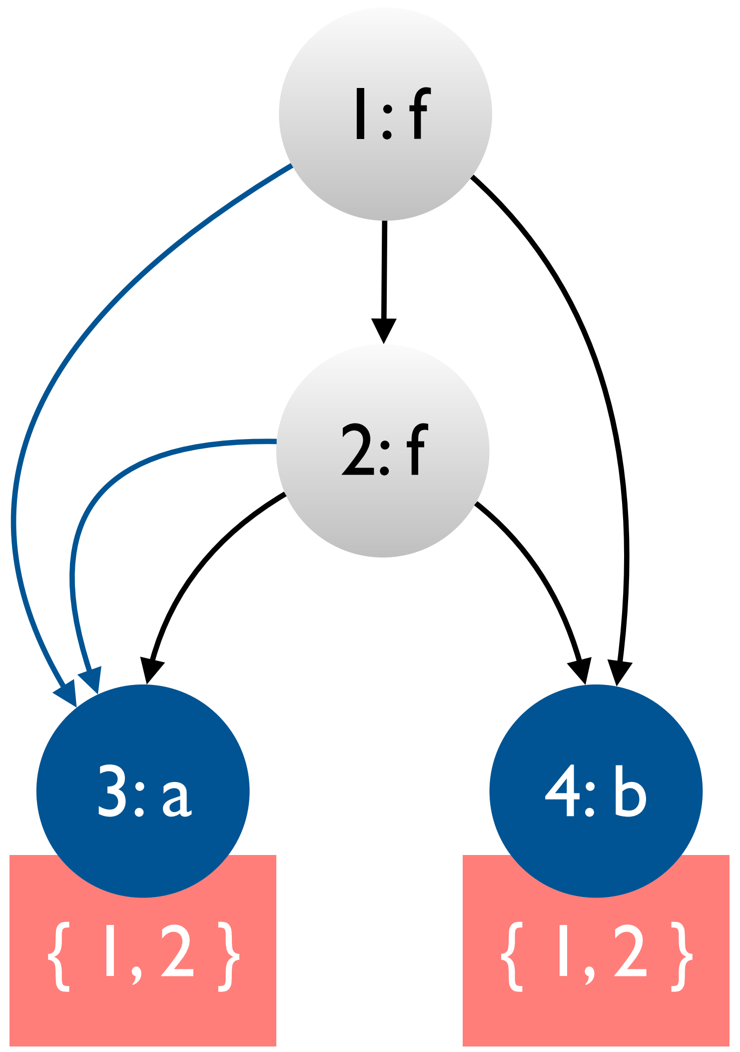
- Represent subterms with a DAG
- Each node has a **find** pointer to another node in its congruence class (or to itself if it is the **representative**)



Congruence closure algorithm: data structure

- Represent subterms with a DAG
- Each node has a **find** pointer to another node in its congruence class (or to itself if it is the **representative**)
- Each representative has a **ccp** field that stores all parents of all nodes in its congruence class.

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: union-find

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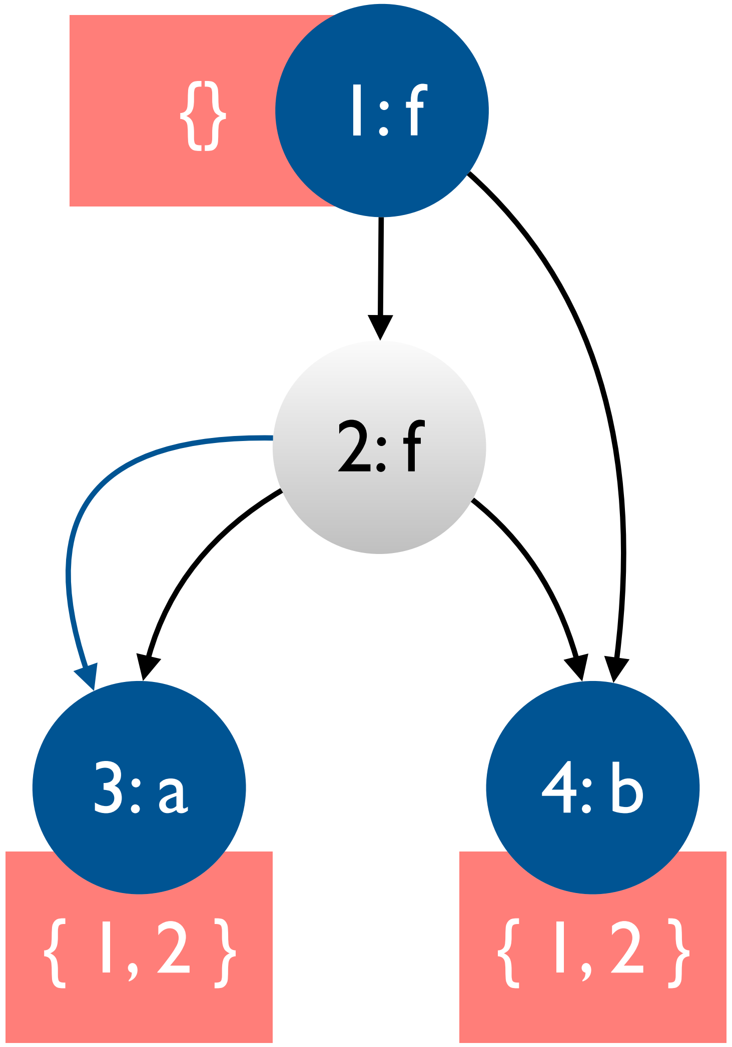

Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.

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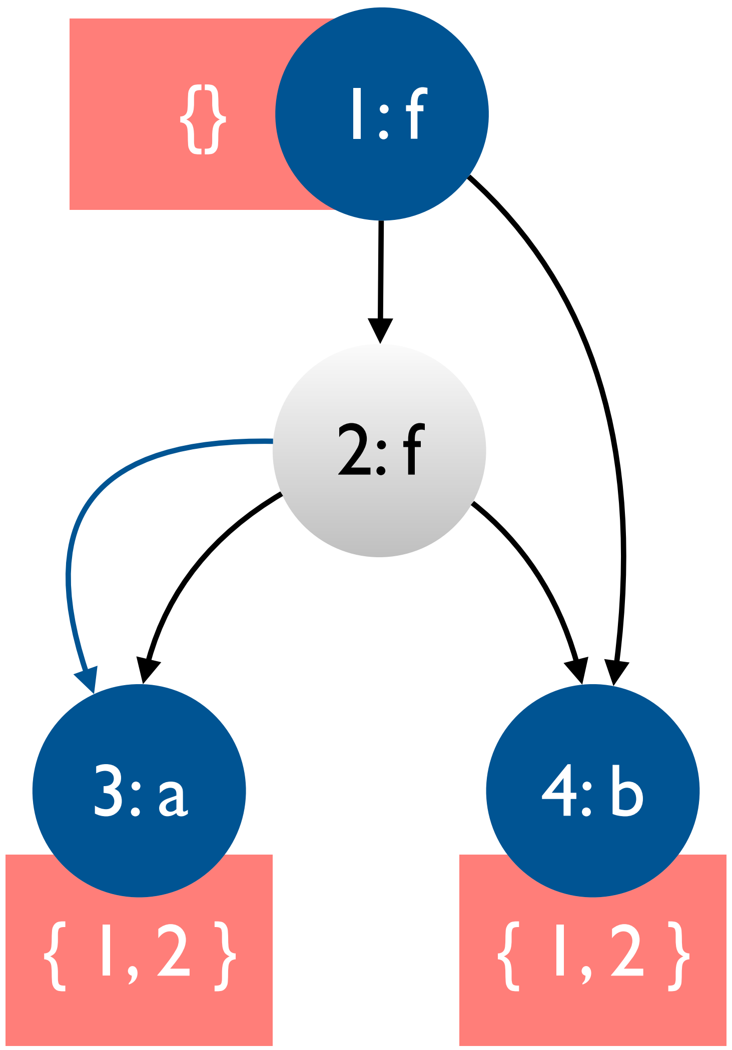

Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.
- UNION combines equivalence classes for nodes i_1 and i_2:
  - $n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)
  - n_1.find $\leftarrow n_2$
  - n_2.ccp $\leftarrow n_1$.ccp $\cup$ n_2.ccp
  - n_1.ccp $\leftarrow \varnothing$

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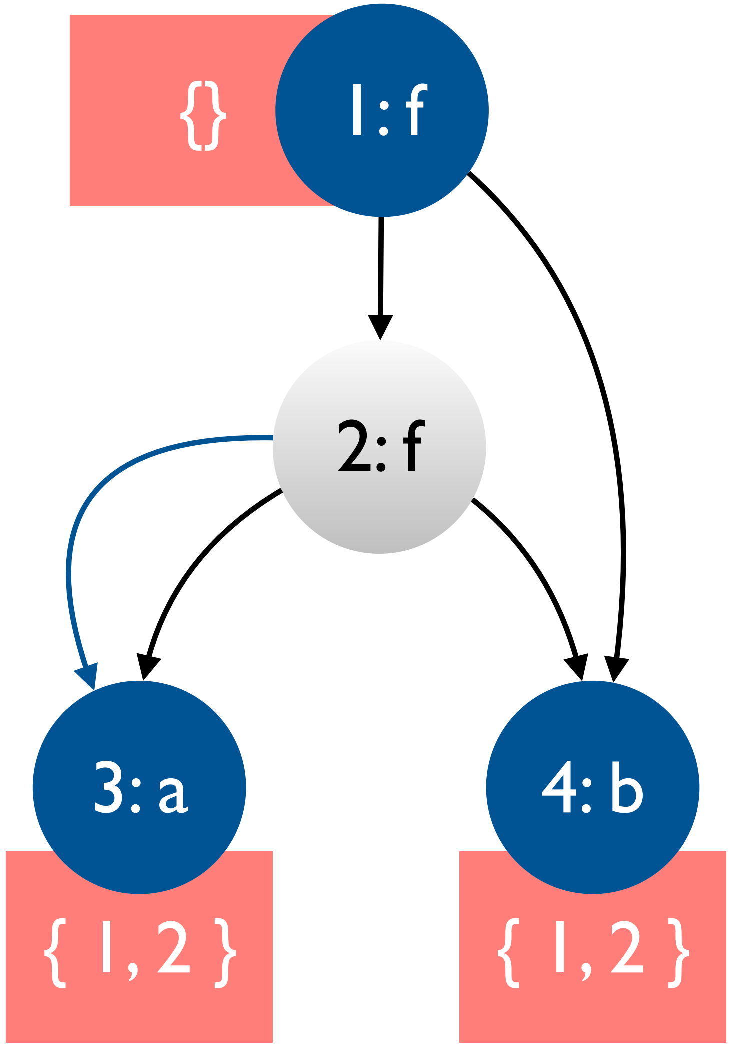

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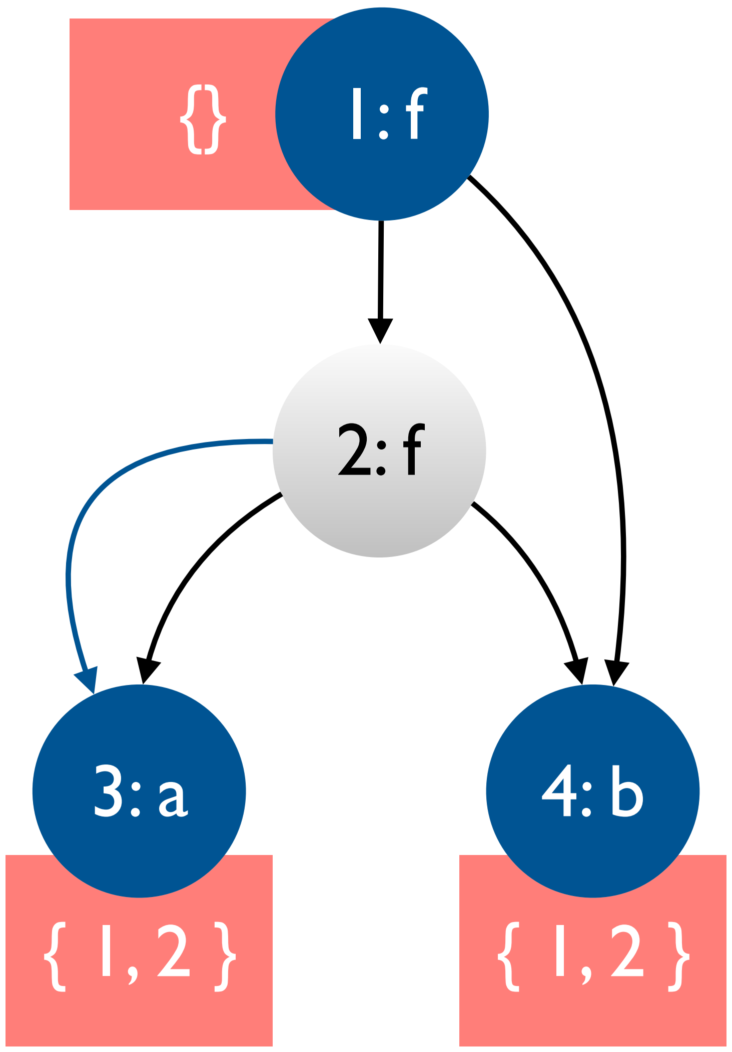
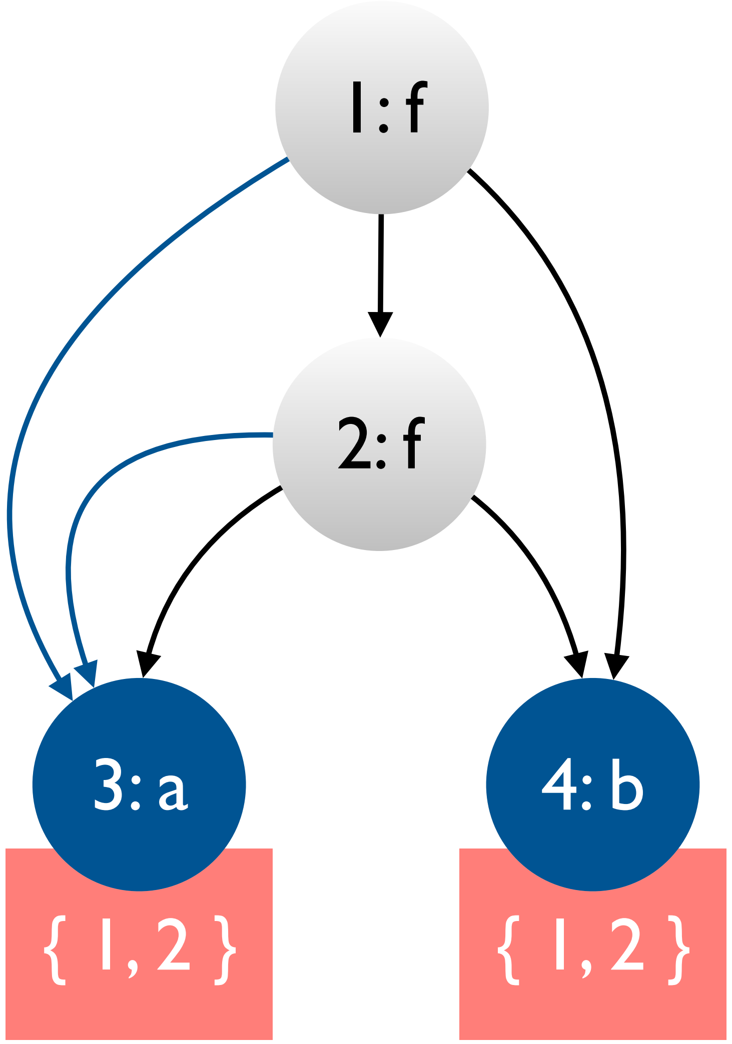


What is UNION(1, 2)?

Congruence closure algorithm: union-find

- FIND returns the representative of a node's equivalence class by following **find** pointers until it finds a self-loop.
- UNION combines equivalence classes for nodes i_1 and i_2:
  - $n_1, n_2 \leftarrow$ FIND(i_1), FIND(i_2)
  - n_1.find $\leftarrow n_2$
  - n_2.ccp $\leftarrow n_1$.ccp $\cup$ n_2.ccp
  - n_1.ccp $\leftarrow \varnothing$

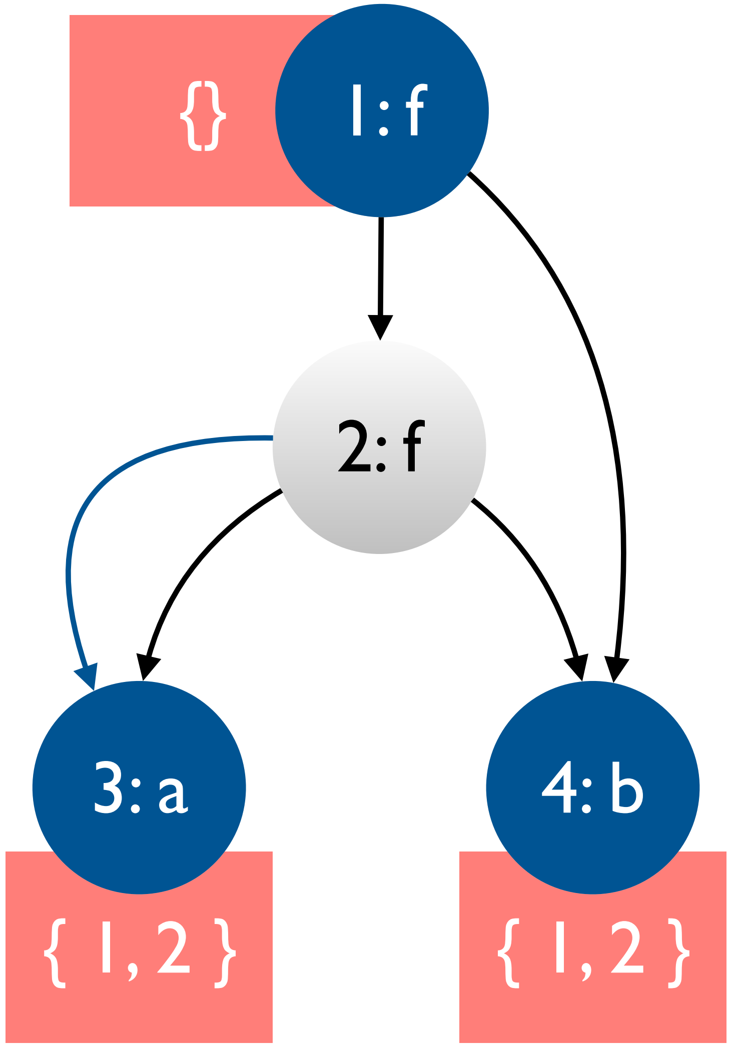
$f(a, b) = a \wedge f(f(a, b), b) \neq a$



Congruence closure algorithm: congruent

- CONGRUENT takes as input two nodes and returns true iff their
  - functions are the same
  - corresponding arguments are in the same congruence class

f(a, b) = a ∧ f(f(a, b), b) ≠ a



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CONGRUENT(1, 2)?

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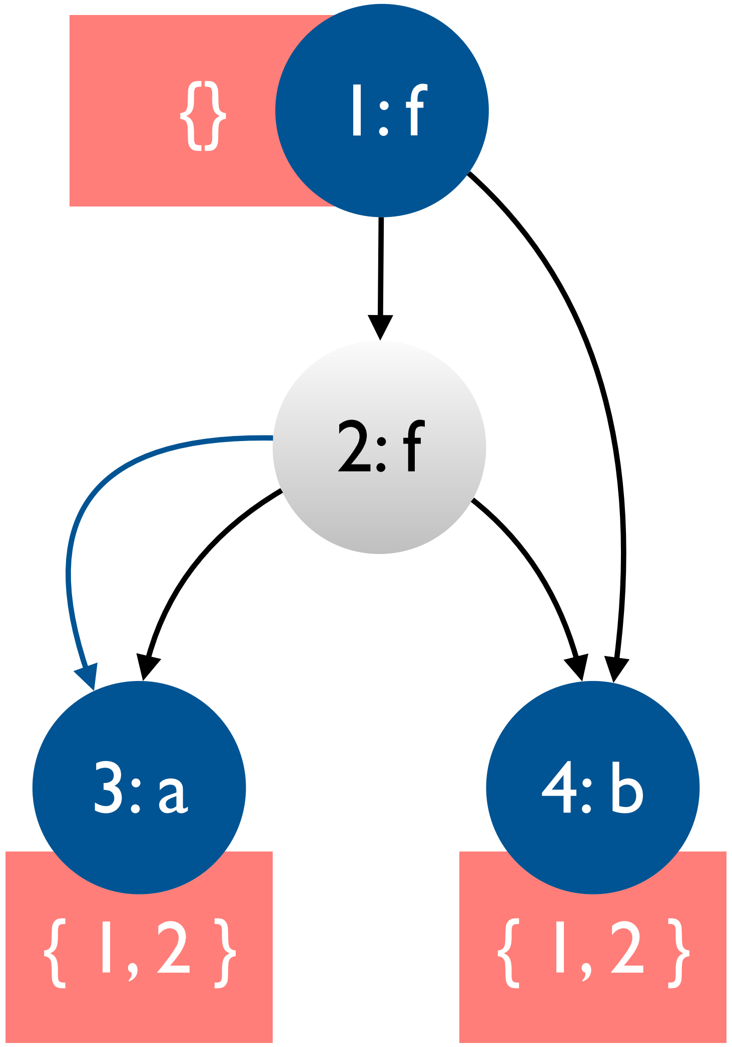
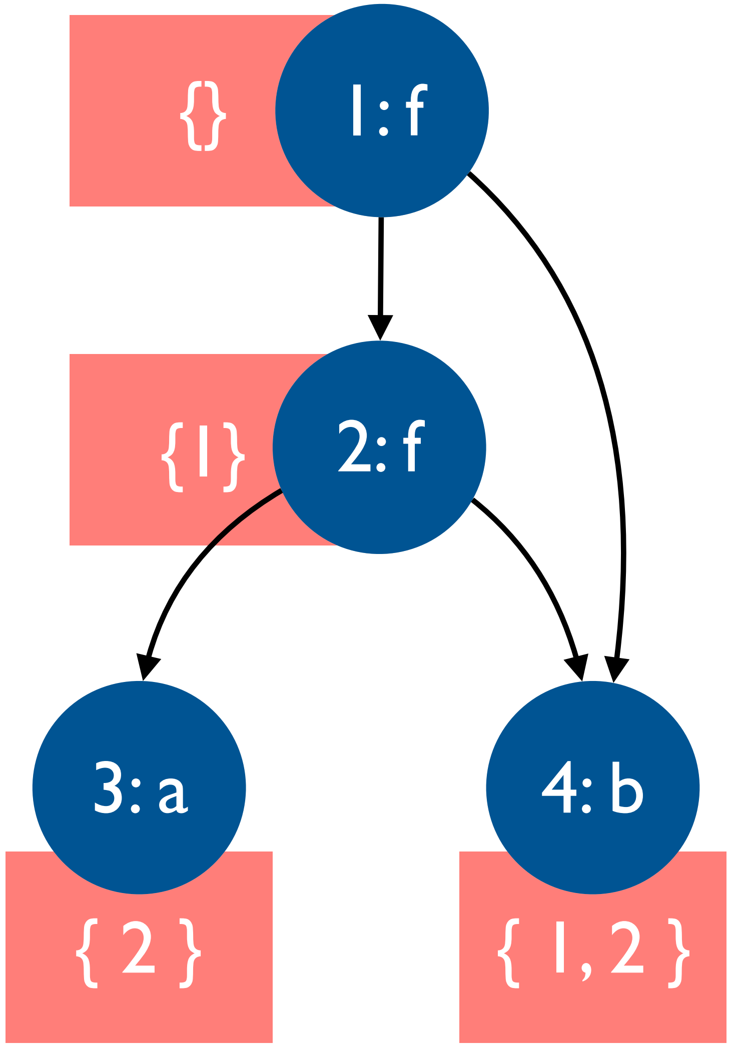


Congruence closure algorithm: merge

```
MERGE ($i_1$ , $i_2$)
    $n_1, n_2 \leftarrow$ FIND($i_1$), FIND($i_2$)
    **if** $n_1 = n_2$ **then return**
    $p_1, p_2 \leftarrow n_1$.ccp, $n_2$.ccp
    UNION($n_1, n_2$)
    **for** each $t_1, t_2 \in p_1 \times p_2$
        **if** FIND($t_1$) $\neq$ FIND($t_2$) $\wedge$ CONGRUENT($t_1, t_2$)
        **then** MERGE($t_1, t_2$)
```

$f(a, b) = a \wedge f(f(a, b), b) \neq a$



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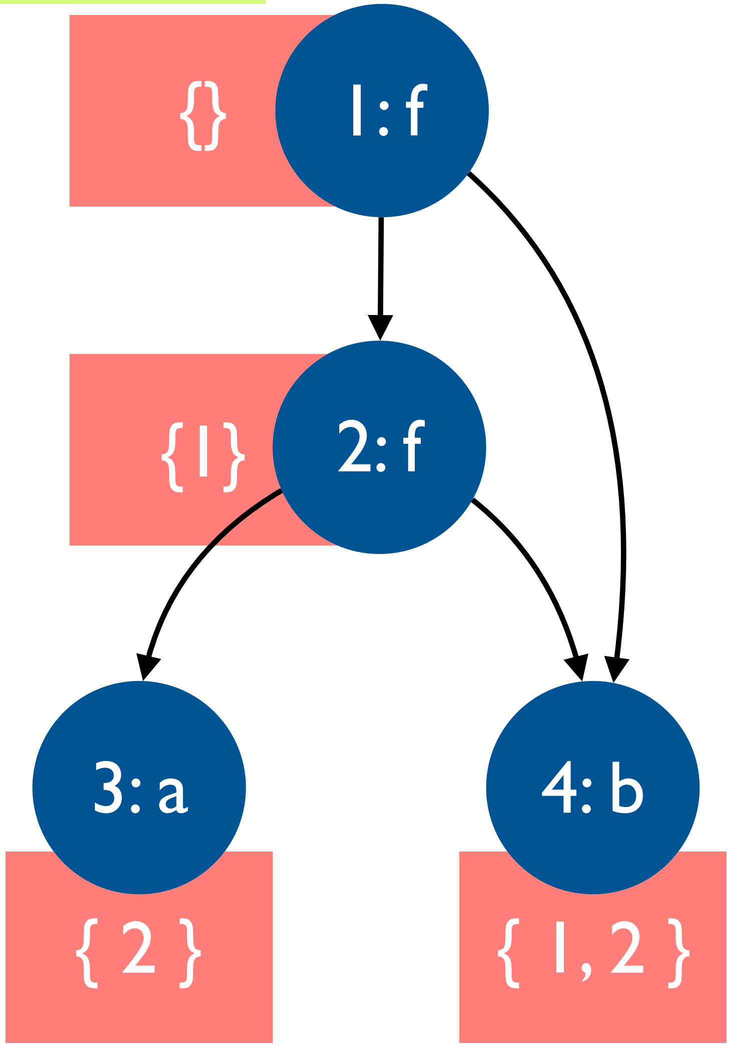

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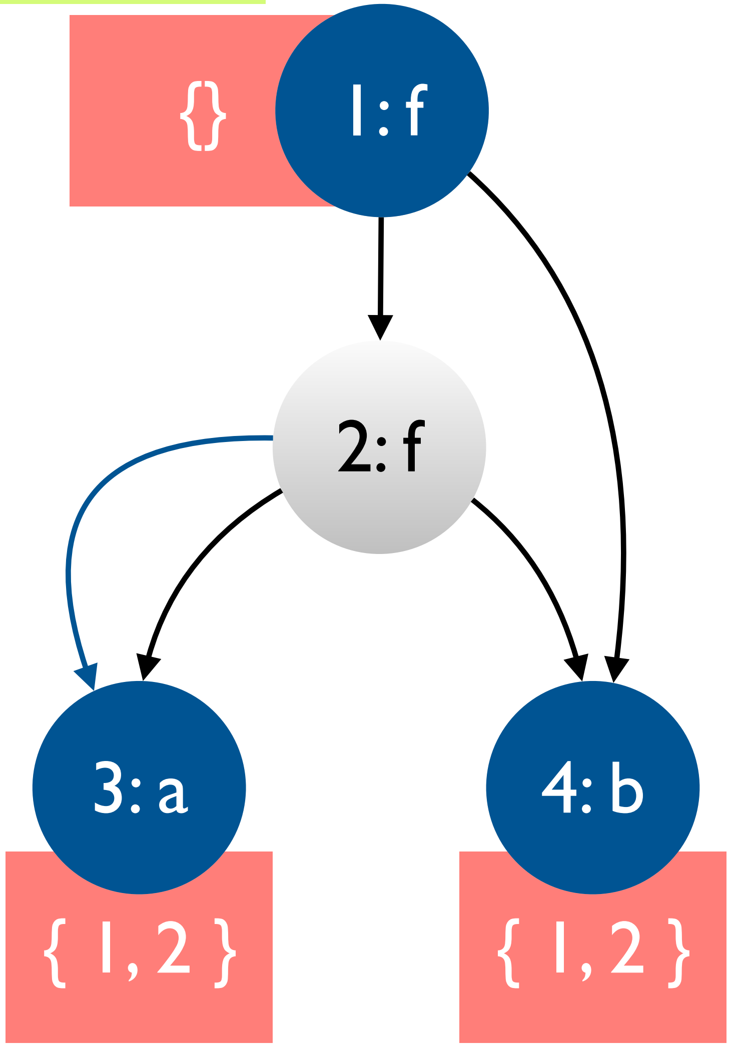

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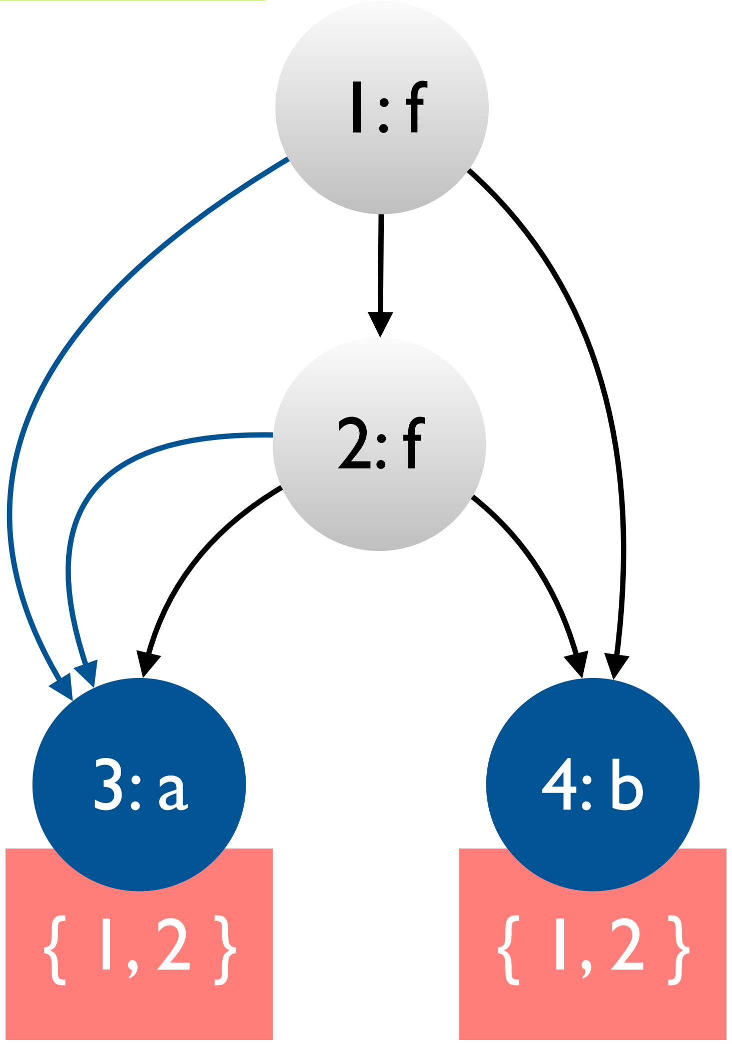

Congruence closure algorithm: deciding $T_=$

```
DECIDE (F)
    construct the DAG for F's subterms
    for $s_i = t_i \in F$
      MERGE($s_i$, $t_i$)
    for $s_i \neq t_i \in F$
      if FIND($s_i$) = FIND($t_i$) then return UNSAT
    return SAT
```

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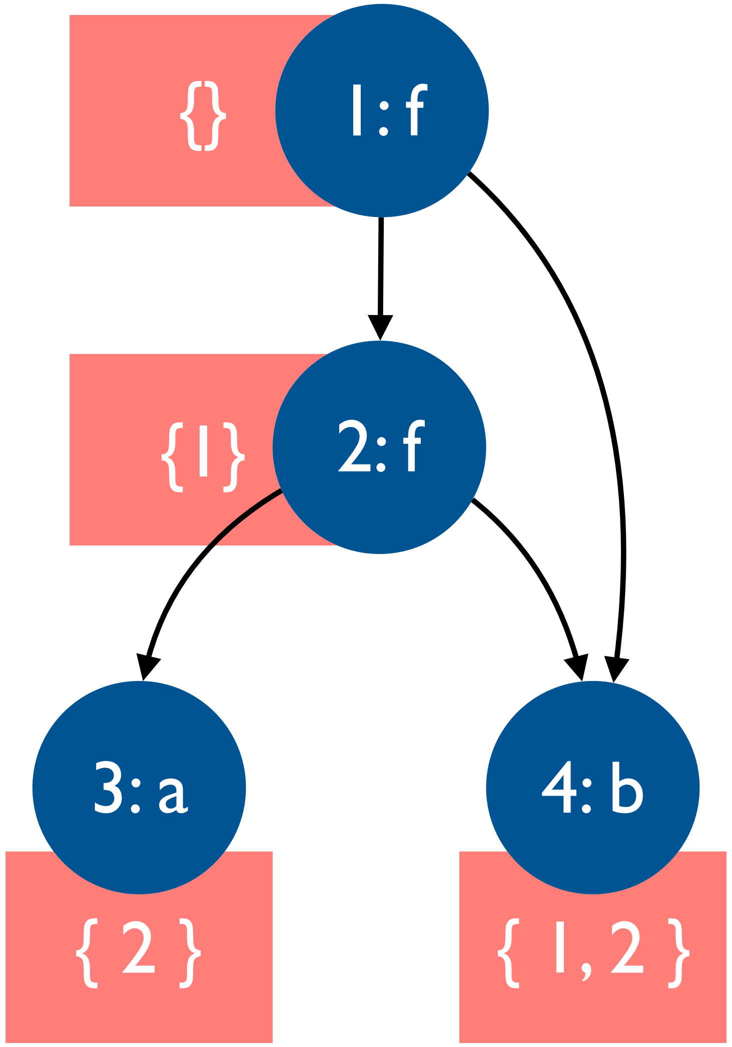

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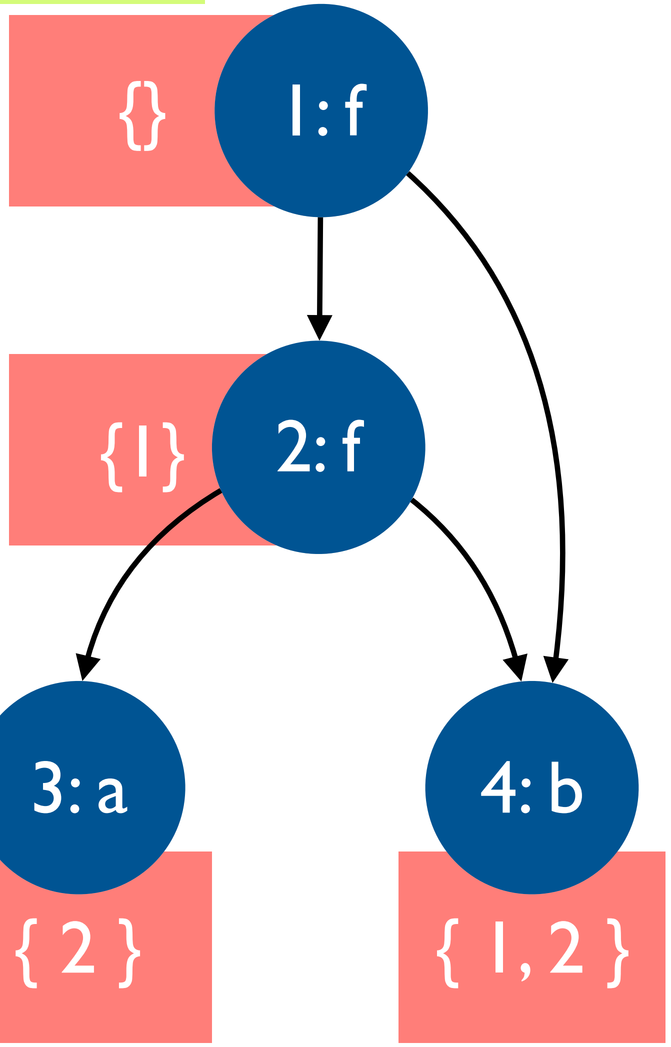

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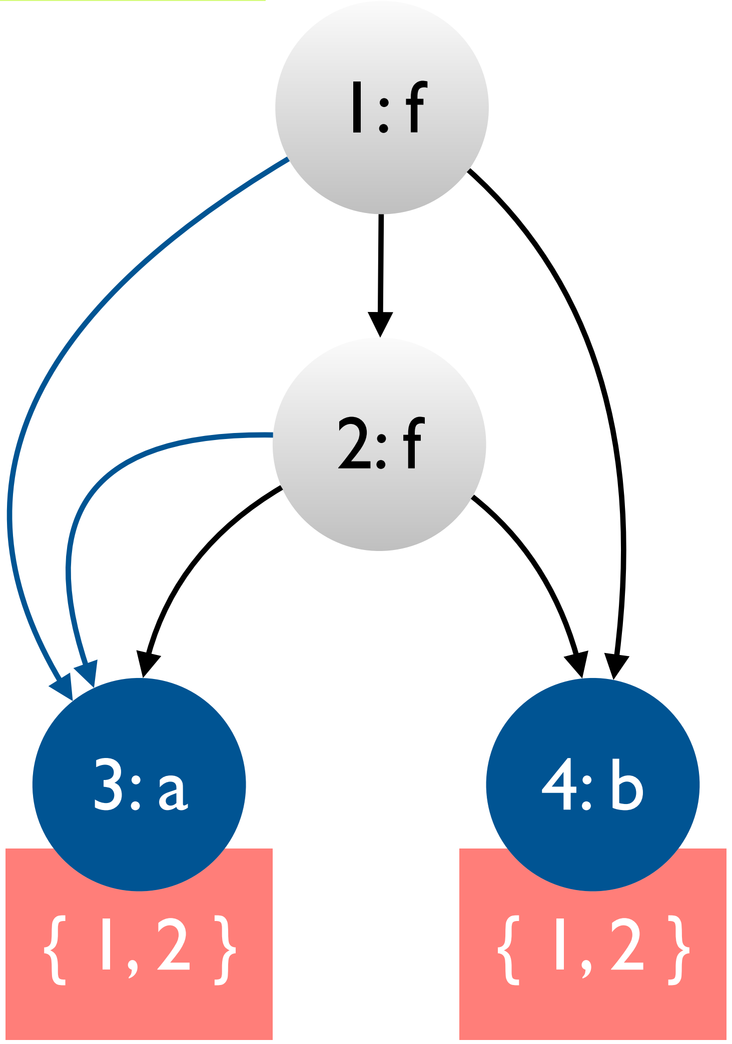

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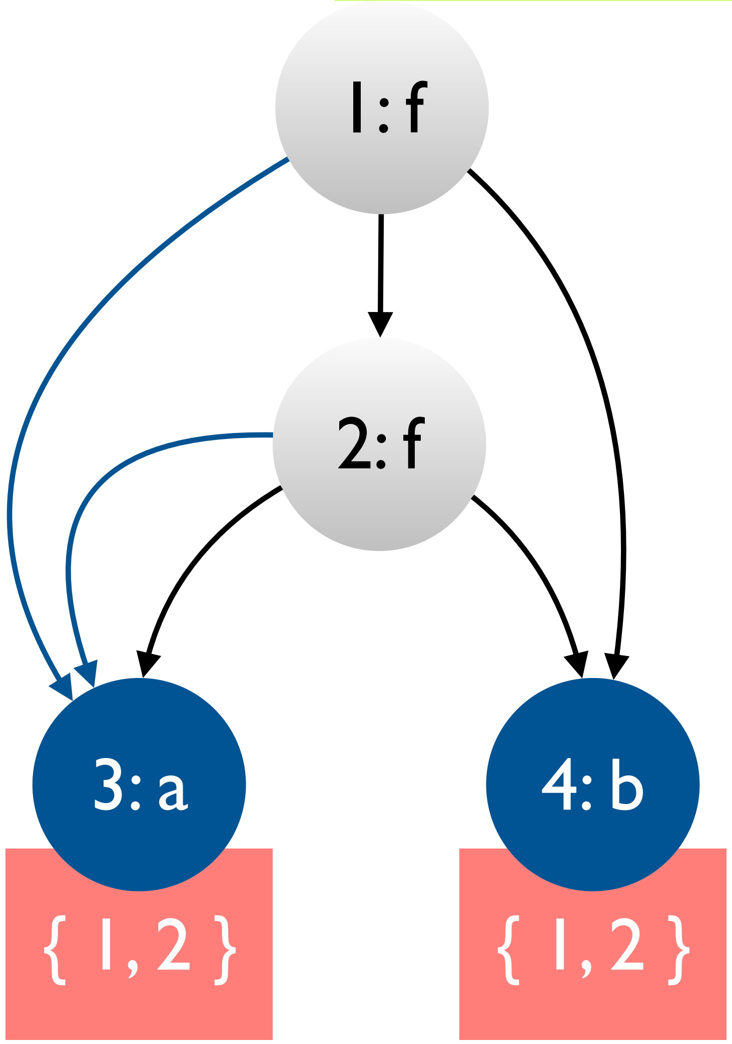

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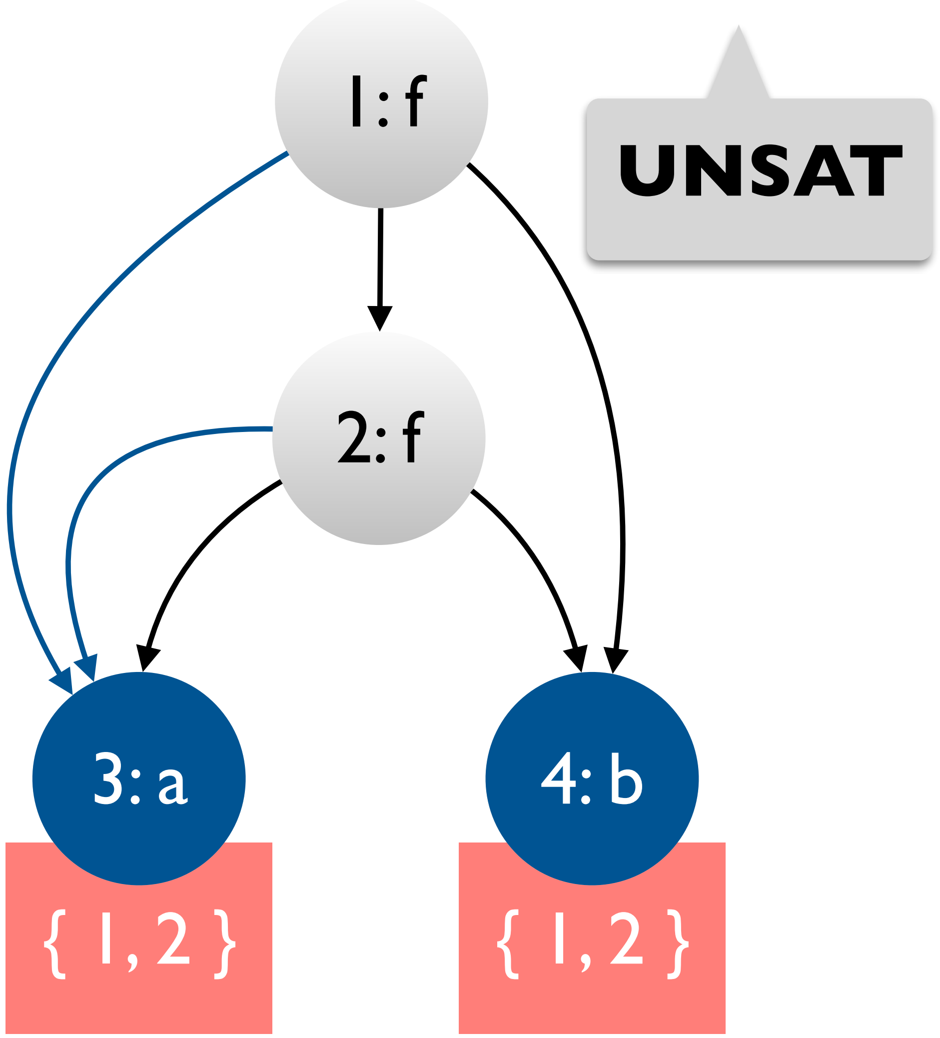

Summary

Today

- A brief survey of theory solvers
- Congruence closure algorithm for deciding conjunctive $T_=$ formulas

Next lecture

- Combining (decision procedures for different) theories