Computer-Aided Reasoning for Software

CSE507

Satisfiability Modulo Theories

courses.cs.washington.edu/courses/cse507/14au/

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Today

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Last lecture

- Practical applications of SAT and the need for a richer logic

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- Introduction to Satisfiability Modulo Theories (SMT)
- Syntax and semantics of (quantifier-free) first-order logic
- Overview of key theories

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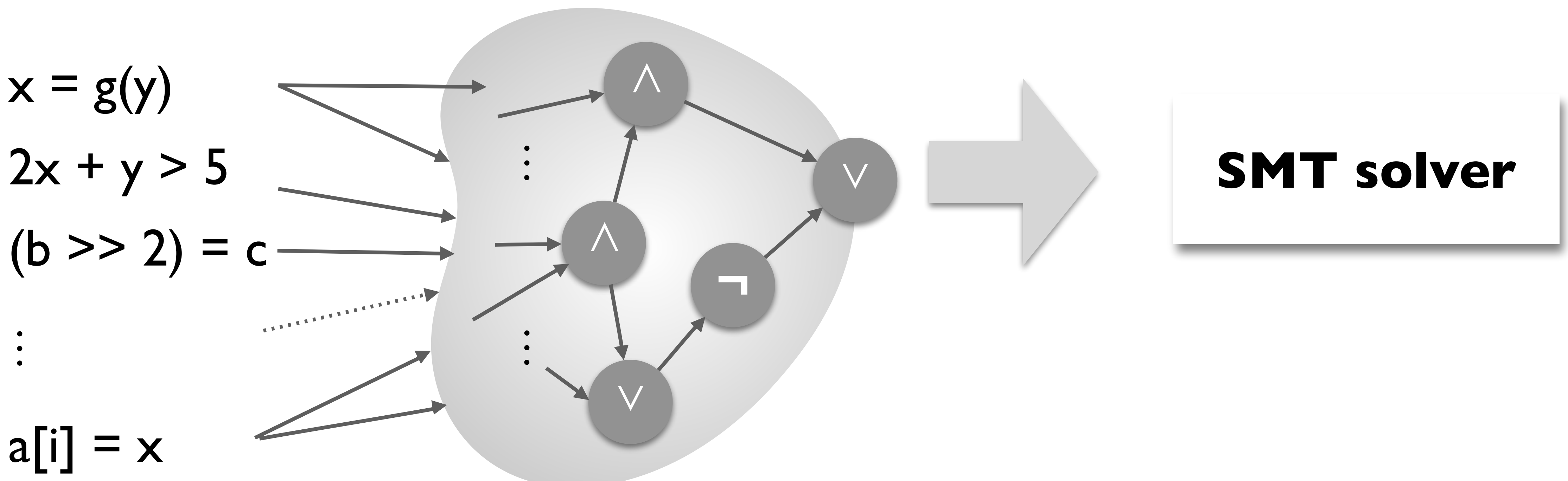
Reminder

- Email us the names of your team members by 11pm today

Satisfiability Modulo Theories (SMT)



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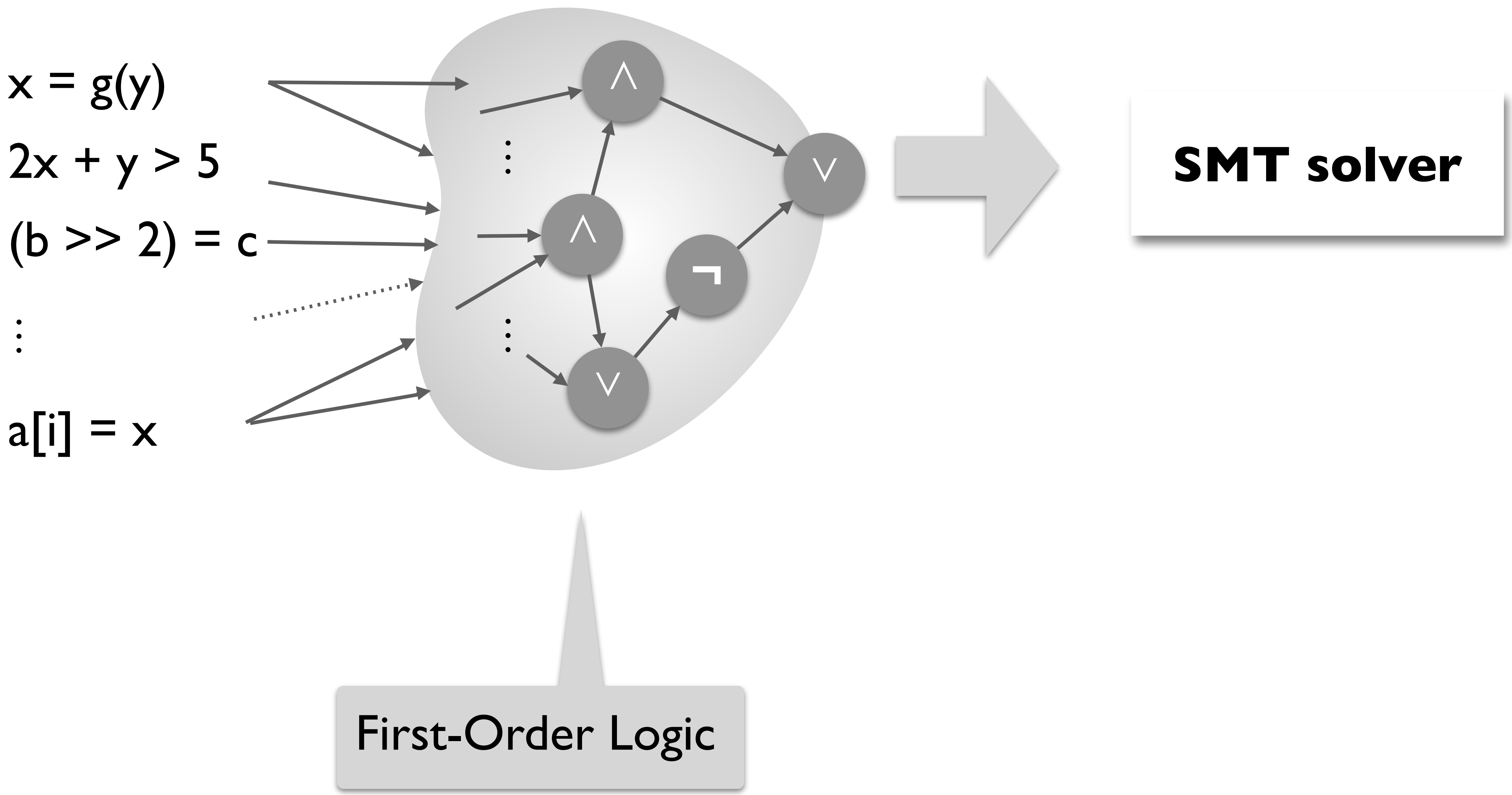

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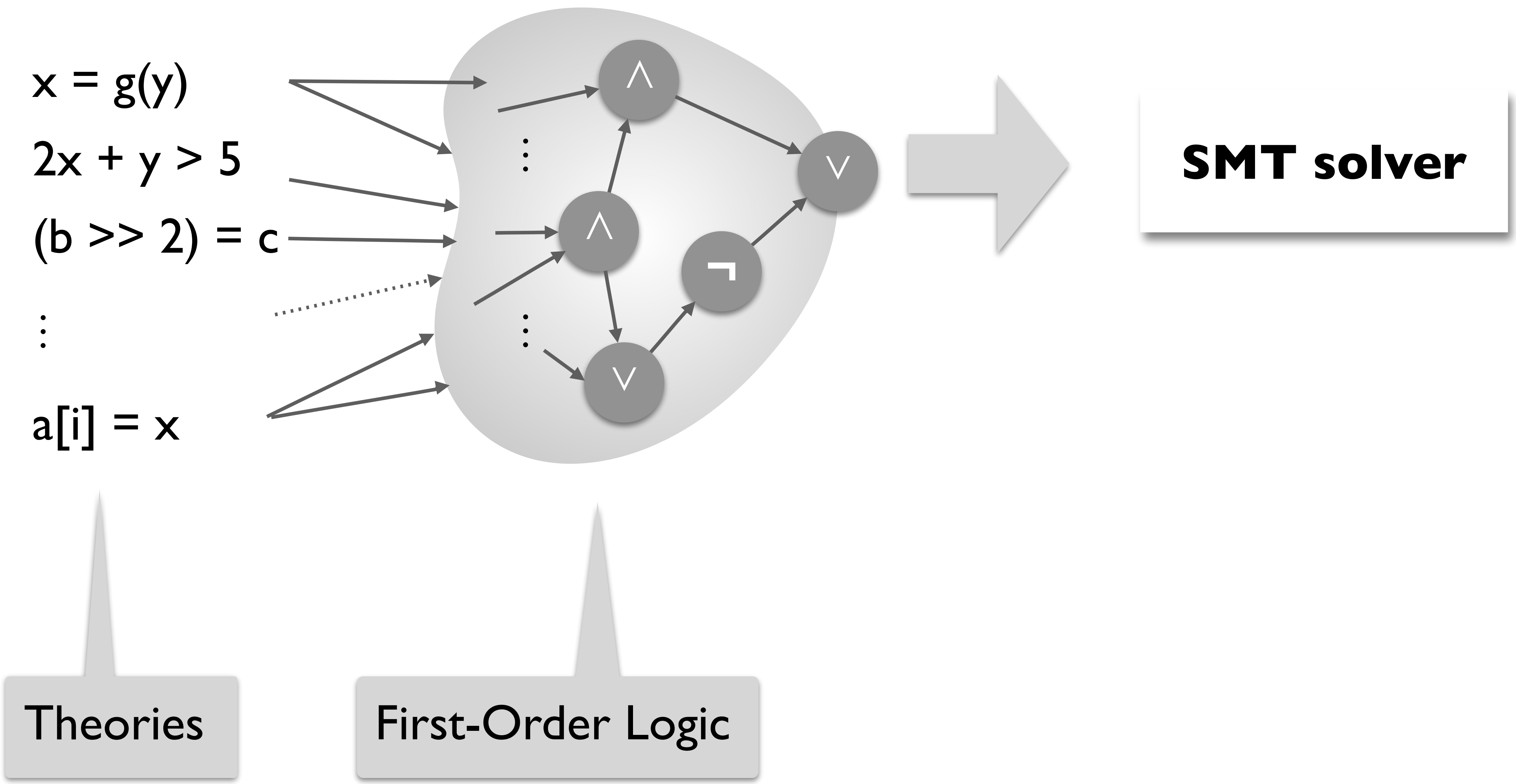

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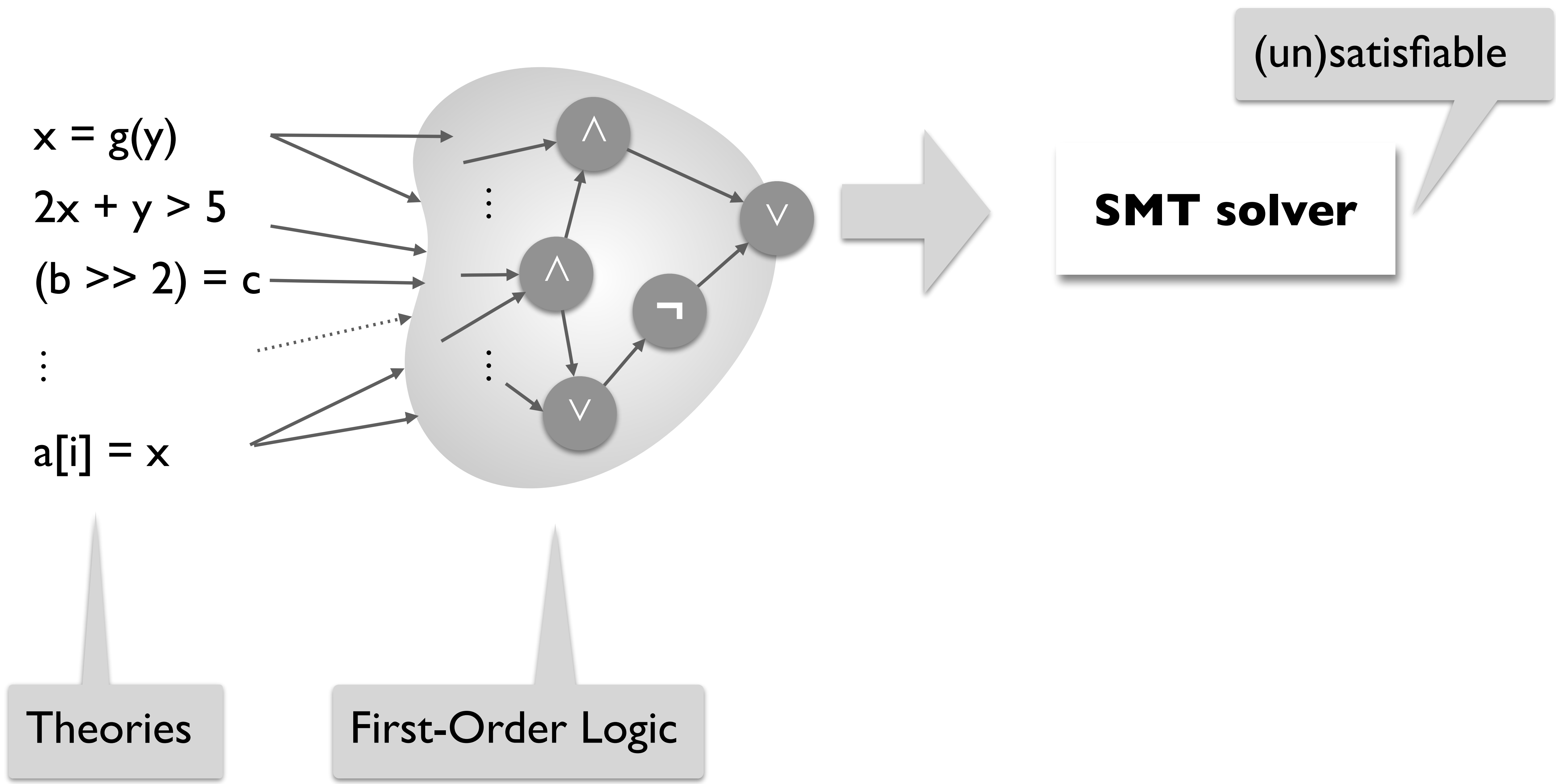

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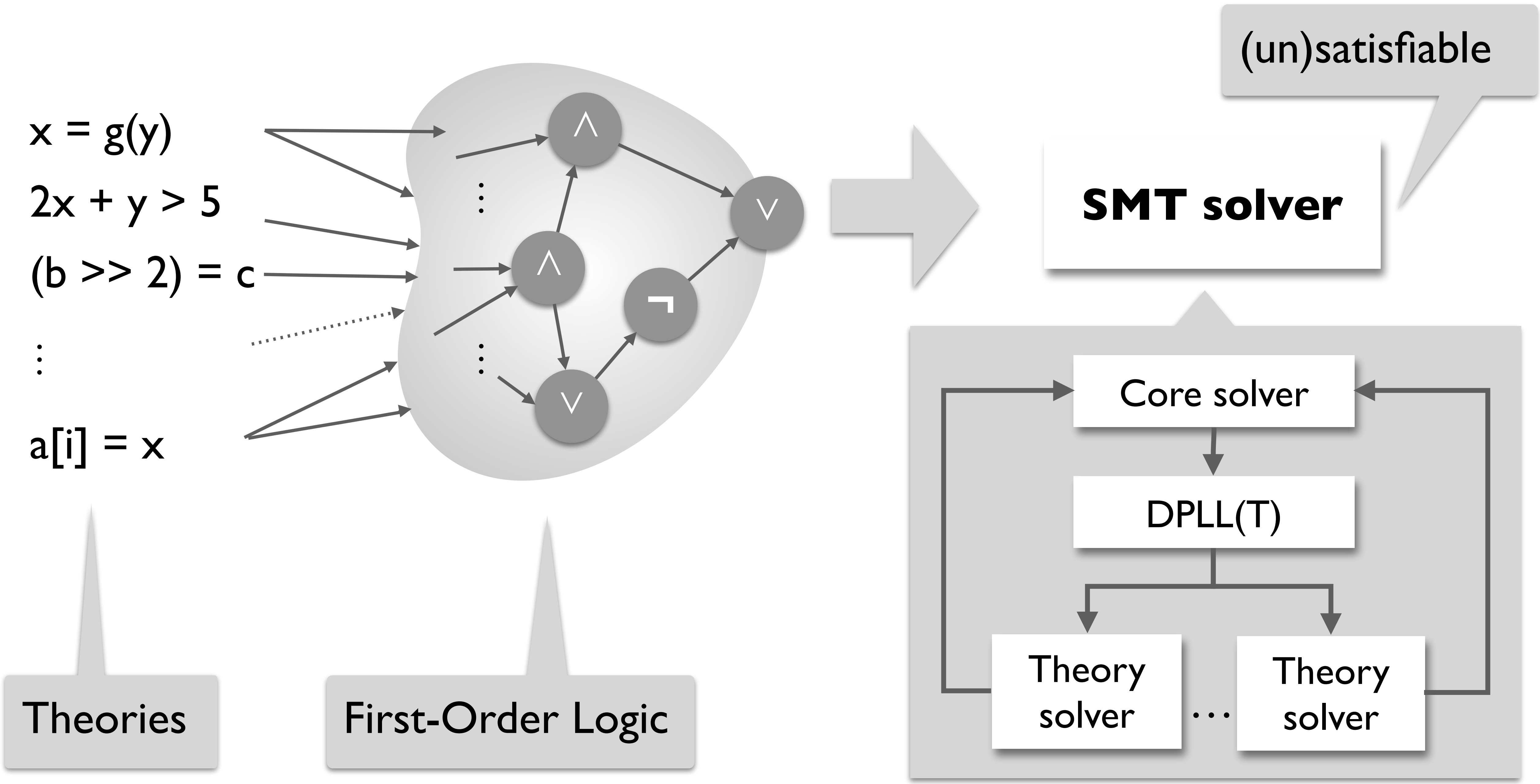

First-Order Logic (FOL)

Logical symbols

- Connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$
- Parentheses: ()
- Quantifiers: $\forall, \exists$

Non-logical symbols

- Constants: x, y, z
- N-ary functions: f, g
- N-ary predicates: p, q
- Variables: u, v, w

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We will only consider the quantifier free fragment of FOL.

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- ✗ Variables: u, v, w

No variables, just constants.

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A **term** is a constant, or an n-ary function applied to n terms.

An **atom** is $\top$, $\bot$, or an n-ary predicate applied to n terms.

A **literal** is an atom or its negation.

A (quantifier-free) **formula** is a literal or the application of logical connectives to formulas.

First-Order Logic (FOL)

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- Parentheses: ()

isPrime(x) $\Rightarrow$ $\neg$ isInteger(sqrt(x))

Non-logical symbols

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Semantics of FOL: first-order structures $\langle \mathbf{U}, \mathbf{I} \rangle$

Universe

Interpretation

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Universe

- A non-empty set of values
- Finite or (un)countably infinite

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Interpretation

- Maps a constant symbol c to an element of I: $I[c] \in U$
- Maps an n-ary function symbol f to a function $f_I : U^n \rightarrow U$
- Maps an n-ary predicate symbol p to an n-ary relation $p_I \subseteq U^n$

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$I[f(t_1, \ldots, t_n)] = I[f](I[t_1], \ldots, I[t_n])$

$I[p(t_1, \ldots, t_n)] = I[p](I[t_1], \ldots, I[t_n])$

$\langle U, I \rangle \models \top$

$\langle U, I \rangle \not\models \bot$

$\langle U, I \rangle \models p(t_1, \ldots, t_n)$ **iff** $I[p(t_1, \ldots, t_n)] = \text{true}$

$\langle U, I \rangle \models \neg F$ **iff** $\langle U, I \rangle \not\models F$

…

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U = {☀, ☁}

I(x) = ☀

I(y) = ☁

I(f) = {☀ ↦ ☁, ☁ ↦ ☀}

I(p) = {⟨☀,☀⟩, ⟨☀,☁⟩}

⟨U, I⟩ ⊨ p(f(y), f(f(x))) ?

FOL satisfiability and validity

F is **satisfiable** iff M ⊨ F for some structure M = ⟨U, I⟩.

F is **valid** iff M ⊨ F for all structures M = ⟨U, I⟩.

Duality of satisfiability and validity:

F is valid iff $\neg F$ is unsatisfiable.

Components of a first-order theory T

Signature Σ_T

Set of T-models

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- One or more (possibly infinitely many) models that fix the interpretation of the symbols in Σ_T
- Can also view a theory as a set of axioms over Σ_T (and Σ_T-models are the models of the theory axioms)

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- Can also view a theory as a set of axioms over Σ_T (and Σ_T-models are the models of the theory axioms)

A formula F is **satisfiable modulo T** iff $M \models F$ for some T-model M.

A formula F is **valid modulo T** iff $M \models F$ for all T-models M.

Common theories

Equality (and uninterpreted functions)

- $x = g(y)$

Fixed-width bitvectors

- $(b >> 1) = c$

Linear arithmetic (over R and Z)

- $2x + y > 5$

Arrays

- $a[i] = x$

Theory of equality (QF_UF)

Extends FOL with the equality predicate =

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Signature (all symbols)

- {=, x, y, z, …, f, g, …, p, q, …}

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- $\{=, x, y, z, \ldots, f, g, \ldots, p, q, \ldots\}$

Axioms

- = is reflexive, symmetric, transitive
- $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (x_1 = y_1 \land \ldots \land x_n = y_n) \rightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$
- $\forall x_1, \ldots, x_n, y_1, \ldots, y_n.\ (x_1 = y_1 \land \ldots \land x_n = y_n) \rightarrow (p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n))$

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Decidable in polynomial time

QF_UF example: checking program equivalence

```
int fun1(int y) {
  int x, z;
  z = y;
  y = x;
  x = z;
  return x*x;
}

int fun2(int y) {
  return y*y;
}
```

An QF_UF formula that is satisfiable iff programs are not equivalent:

Example from Sanjit Seshia

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An QF_UF formula that is satisfiable iff programs are not equivalent:

$(z_1 = y_0 \wedge y_1 = x_0 \wedge x_1 = z_1 \wedge r_1 = x_1 * x_1) \wedge$

$(r_2 = y_0 * y_0) \wedge$

$\neg(r_2 = r_1)$

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Using 32-bit integers, a SAT solver fails to return an answer in 5 min.

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An QF_UF formula that is satisfiable iff programs are not equivalent:

$(z_1 = y_0 \land y_1 = x_0 \land x_1 = z_1 \land r_1 = sq(x_1)) \land$

$(ret_2 = sq(y_0)) \land$

$\neg(ret_2 = ret_1)$

QF_UF example: checking program equivalence

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$(ret_2 = sq(y_0)) \wedge$

$\neg(ret_2 = ret_1)$

Using QF_UF, an SMT solver proves unsatisfiability in a fraction of a second.

Example from Sanjit Seshia

QF_UF example: checking program equivalence

```
int fun1(int y) {
  int x;
  x = x ^ y;
  y = x ^ y;
  x = x ^ y;
  return x*x;
}

int fun2(int y) {
  return y*y;
}
```

Is the uninterpreted function abstraction going to work in this case?

Example from Sanjit Seshia

QF_UF example: checking program equivalence

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int fun1(int y) {
  int x;
  x = x ^ y;
  y = x ^ y;
  x = x ^ y;
  return x*x;
}

int fun2(int y) {
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```

Is the uninterpreted function abstraction going to work in this case?

No, we need the theory of fixed-width bitvectors to reason about ^ (xor).

Example from Sanjit Seshia

Theory of fixed-width bitvectors (QF_BV)

Signature

- constants
- fixed-width words (modeling machine ints, longs, etc.)
- arithmetic operations (+, -, *, /, etc.)
- bitwise operations (&, |, ^, etc.)
- comparison operators (<, >, etc.)
- equality (=)

Satisfiability problem: NP-complete

Theory of linear arithmetic (QF_LIA)

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Signature

- $\{\ldots, -1, 0, 1, \ldots, -2\cdot, 2\cdot, \ldots, +, -, =, >, x, y, z, \ldots\}$
- Constants, integers, multiplication by an integer constant, addition, subtraction, equality, greater-than

Theory of linear arithmetic (QF_LIA)

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Satisfiability problem: NP-complete

Theory of reals (QF_LRA) can be decided in polynomial time.

Theory of linear arithmetic (QF_LIA)

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Satisfiability problem: NP-complete

Theory of reals (QF_LRA) can be decided in polynomial time.

Difference Logic (QF_DIA) can also be decided in polynomial time

- Conjunctions of the form $x - y \leq c$, where c is an integer constant

QF_LIA example: compiler optimization

```
for (i=1; i<=10; i++) {
  a[j+i] = a[j];
}
```

```
int v = a[j];
for (i=1; i<=10; i++) {
  a[j+i] = v;
}
```

An QF_LIA formula that is satisfiable iff this transformation is invalid:

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An QF_LIA formula that is satisfiable iff this transformation is invalid:

$(i \geq 1) \wedge (i \leq 10) \wedge$

$(j + i = j)$

Polyhedral model

Theory of arrays

Theory of arrays

Signature

- {read, write, =, x, y, z, . . .}

Theory of arrays

Signature

- {read, write, =, x, y, z, …}

Axioms

- $\forall$i. read(write(a, i, v), i) = v
- $\forall$i, j. $\neg$(i = j) $\rightarrow$ (read(write(a, i, v), j) = read(a, j))
- ($\forall$i. read(a, i) = read(b, i)) $\rightarrow$ a = b

Theory of arrays

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- {read, write, =, x, y, z, …}

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- $\forall i, j.\ \neg(i = j) \rightarrow (\mathrm{read}(\mathrm{write}(a, i, v), j) = \mathrm{read}(a, j))$
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Satisfiability problem: NP-complete

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- $(\forall i.\ read(a, i) = read(b, i)) \rightarrow a = b$

Satisfiability problem: NP-complete

Used in many software verification tools to model memory (e.g., Dafny)

Summary

Today

- Introduction to SMT
- Quantifier-free FOL (syntax & semantics)
- Overview of common theories

Next lecture

- Survey of theory solvers