Computer-Aided Reasoning for Software

CSE507

A Modern SAT Solver

courses.cs.washington.edu/courses/cse507/14au/

Emina Torlak
emina@cs.washington.edu

Today

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Last lecture

- The DPLL algorithm for deciding satisfiability of propositional formulas.

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- The CDCL algorithm at the core of modern SAT solvers:
  - 3 important extensions of DPLL
  - Engineering matters

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- The CDCL algorithm at the core of modern SAT solvers:
  - 3 important extensions of DPLL
  - Engineering matters

Announcements

- Homework 1 is out. Due on Oct 14. Start early.
- Form your project team by Oct 09.

A brief review of DPLL

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// Returns true if the CNF formula F is
// satisfiable; otherwise returns false.

DPLL(F)
  G ← BCP(F)
  if G = ⊤ then return true
  if G = ⊥ then return false
  p ← choose(vars(F’))
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Boolean constraint propagation applies unit resolution until fixed point:

$$\frac{\text{lit} \quad \text{clause}[\neg\text{lit}]}{\text{clause}[\bot]}$$

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Okay for randomly generated CNFs, but not on practical ones. Why?

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No learning: throws away all the work performed to conclude that the current partial assignment (PA) is bad. Revisits bad PAs that lead to conflict due to the same root cause.

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Naive decisions: picks an arbitrary variable and assumes it is true. Fails to consider the state of the search to make heuristically better decisions.

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Conflict-Driven Clause Learning (CDCL)

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Learning: F augmented with a **conflict clause** that summarizes the root cause of the conflict.

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Decision heuristics choose the next literal to add to the current partial assignment based on the state of the search.

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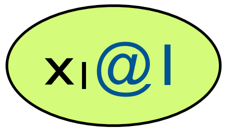

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x_1@1

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$x_8@2$ $x_1@1$

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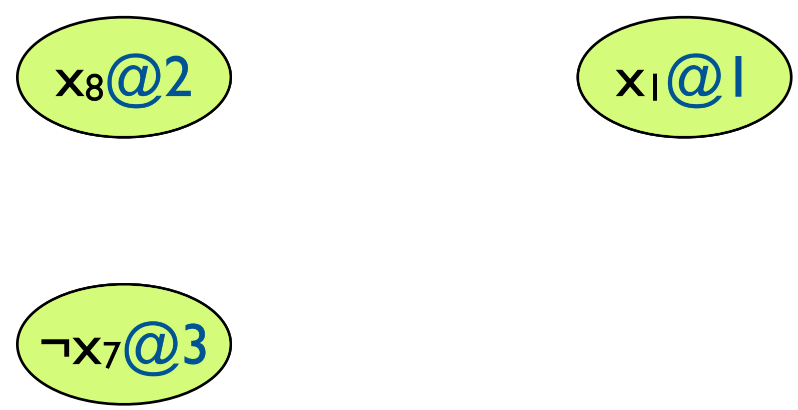

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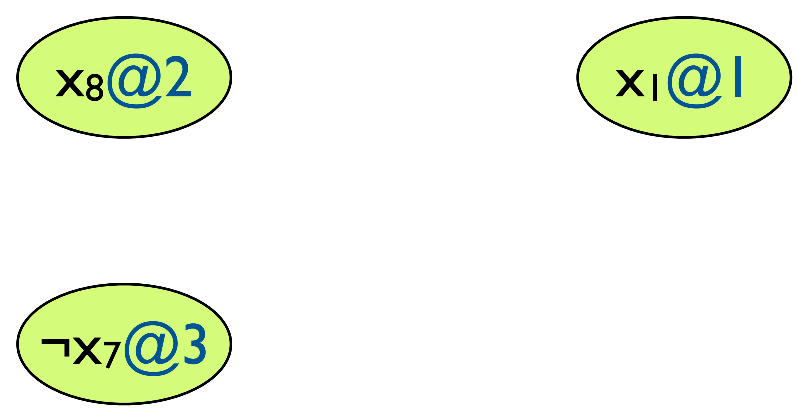

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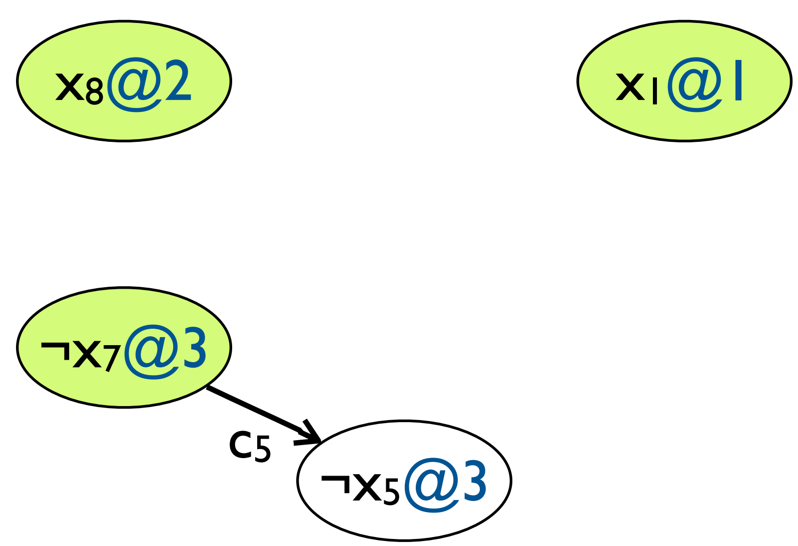

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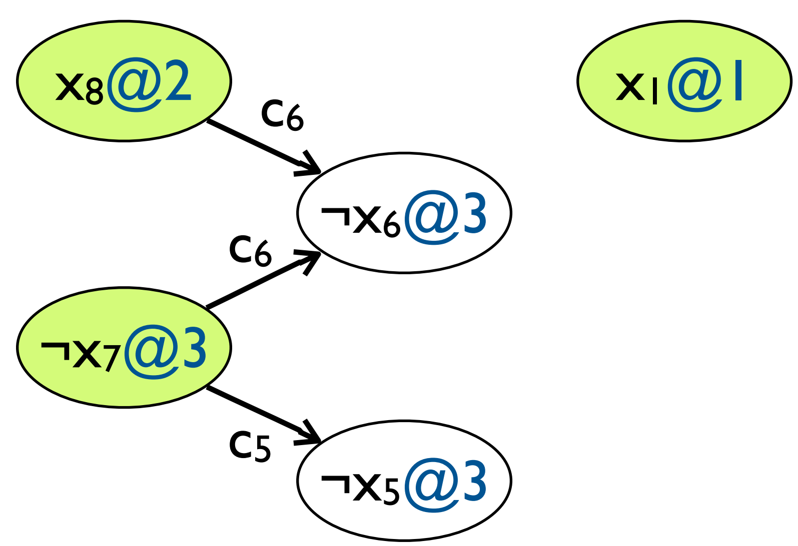

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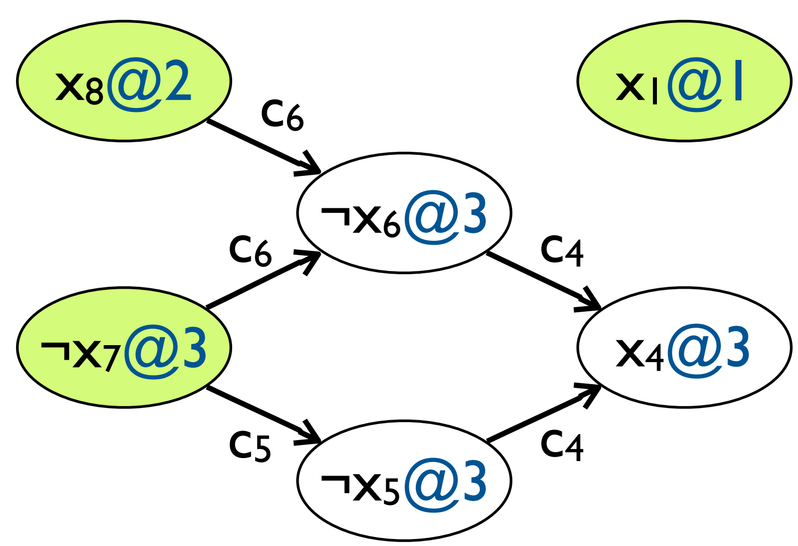

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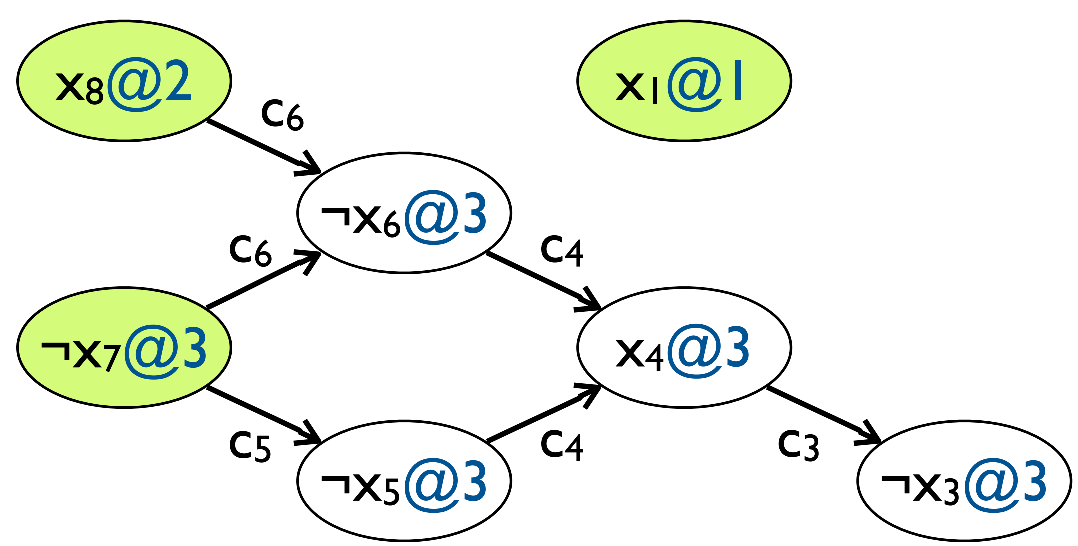

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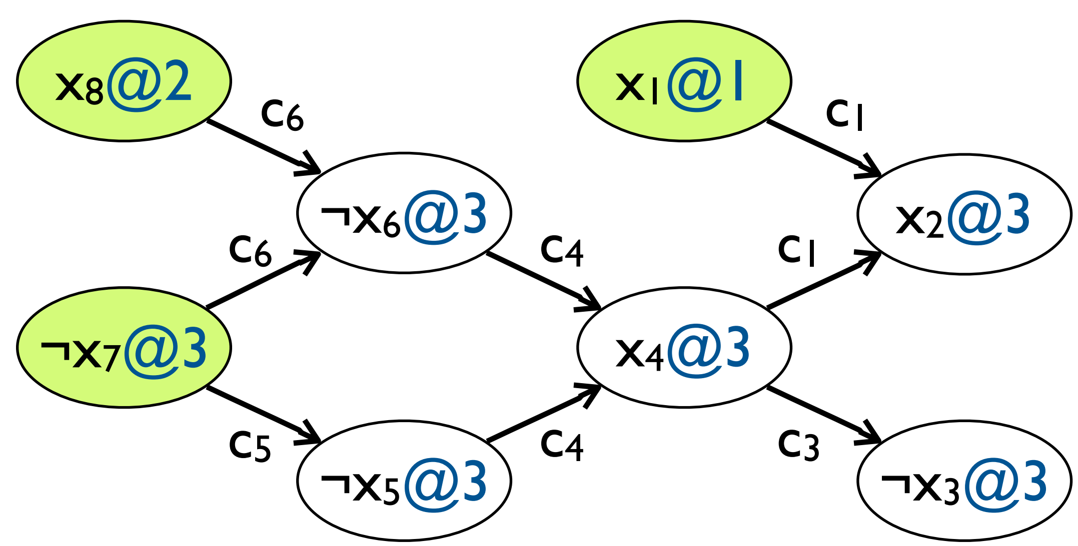

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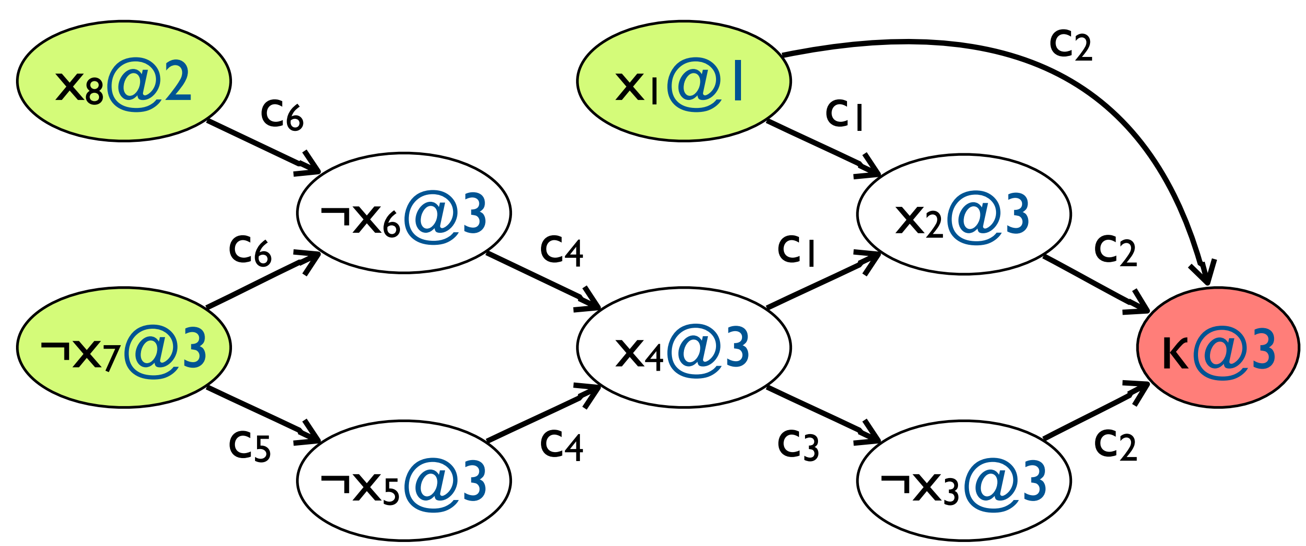

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$\langle 1, \neg x_1 \vee \neg x_4 \rangle$

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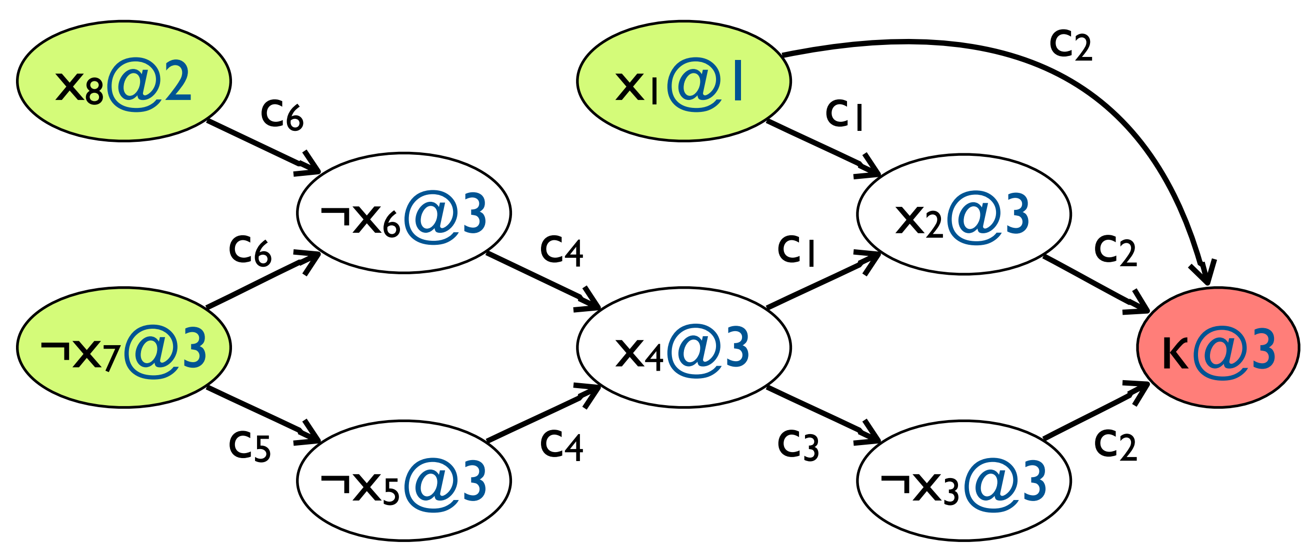

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⟨1, ¬x₁ ∨ ¬x₄⟩

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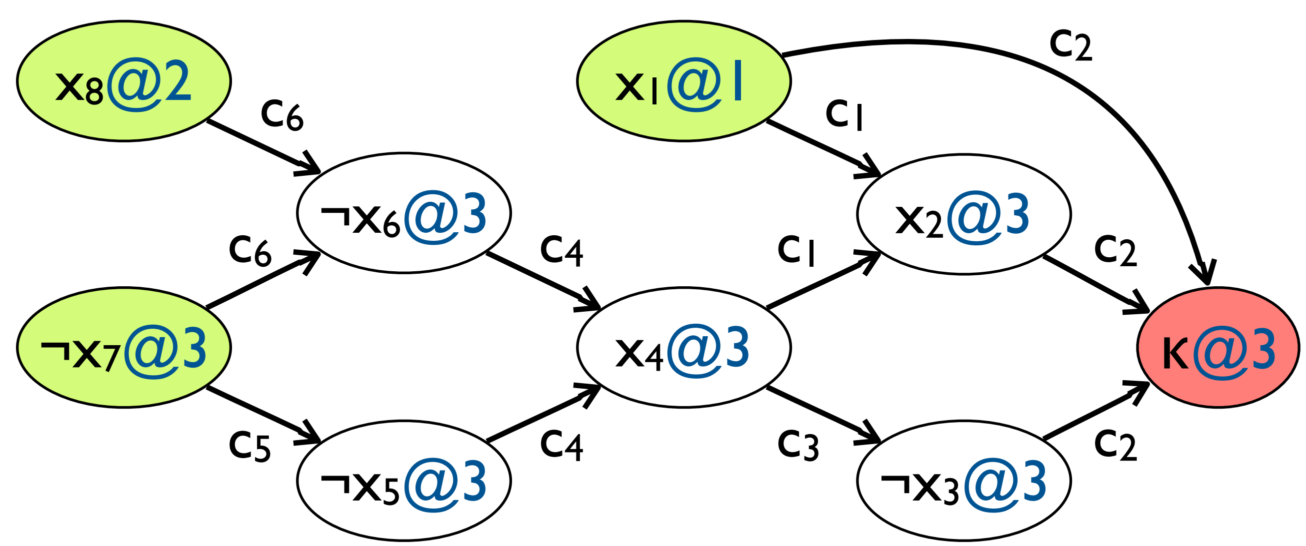

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     F ← F ∪ {c}
     if b < 0 then return false
     else BACKTRACK(F,A, b)
          level ← b
 return true
```

$\langle 1, \neg x_1 \lor \neg x_4 \rangle$

$F = \{ c_1, c_2, c_3, c_4, c_5, c_6, \ldots, c_9, c \}$

$c_1 : \neg x_1 \lor x_2 \lor \neg x_4$

$c_2 : \neg x_1 \lor \neg x_2 \lor x_3$

$c_3 : \neg x_3 \lor \neg x_4$

$c_4 : x_4 \lor x_5 \lor x_6$

$c_5 : \neg x_5 \lor x_7$

$c_6 : \neg x_6 \lor x_7 \lor \neg x_8$

…

$c : \neg x_1 \lor \neg x_4$

$x_1@1$

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$F = \{ c_1, c_2, c_3, c_4, c_5, c_6, \ldots, c_9, c \}$

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$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

$c_3 : \neg x_3 \vee \neg x_4$

$c_4 : x_4 \vee x_5 \vee x_6$

$c_5 : \neg x_5 \vee x_7$

$c_6 : \neg x_6 \vee x_7 \vee \neg x_8$

…

$c : \neg x_1 \vee \neg x_4$

Conflict clause is unit after backtracking!

x_1@1

CDCL in depth

```
CDCL(F)
 A ← {}
 if BCP(F,A) = conflict then return false
 level ← 0
 while hasUnassignedVars(F)
   level ← level + 1
   A ← A ∪ { DECIDE(F,A) }
   while BCP(F,A) = conflict
     ⟨b, c⟩ ← ANALYZECONFLICT()
     F ← F ∪ {c}
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 return true
```

The Plan

- Definitions
- ANALYZECONFLICT
- DECIDE heuristics
- Implementation

Basic definitions

Under a given partial assignment (PA), a variable may be

- **assigned** (true/false literal)
- **unassigned**.

$F = \{ c_1, c_2, c_3, c_4, c_5, c_6, \ldots, c_9 \}$

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

$c_2 : \neg x_1 \vee \neg x_2 \vee x_3$

…

$c_8 : x_9 \vee \neg x_2$

$c_9 : x_9 \vee x_{10} \vee x_3$

True literals highlighted in green; false literals highlighted in red.

Basic definitions

Under a given partial assignment (PA), a variable may be

- **assigned** (true/false literal)
- **unassigned**.

A clause may be

- **satisfied** (≥ 1 true literal)
- **unsatisfied** (all false literals)
- **unit** (one unassigned literal, rest false)
- **unresolved** (otherwise)

$F = \{ c_1, c_2, c_3, c_4, c_5, c_6, \ldots, c_9 \}$

$c_1 : \neg x_1 \vee x_2 \vee \neg x_4$

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…

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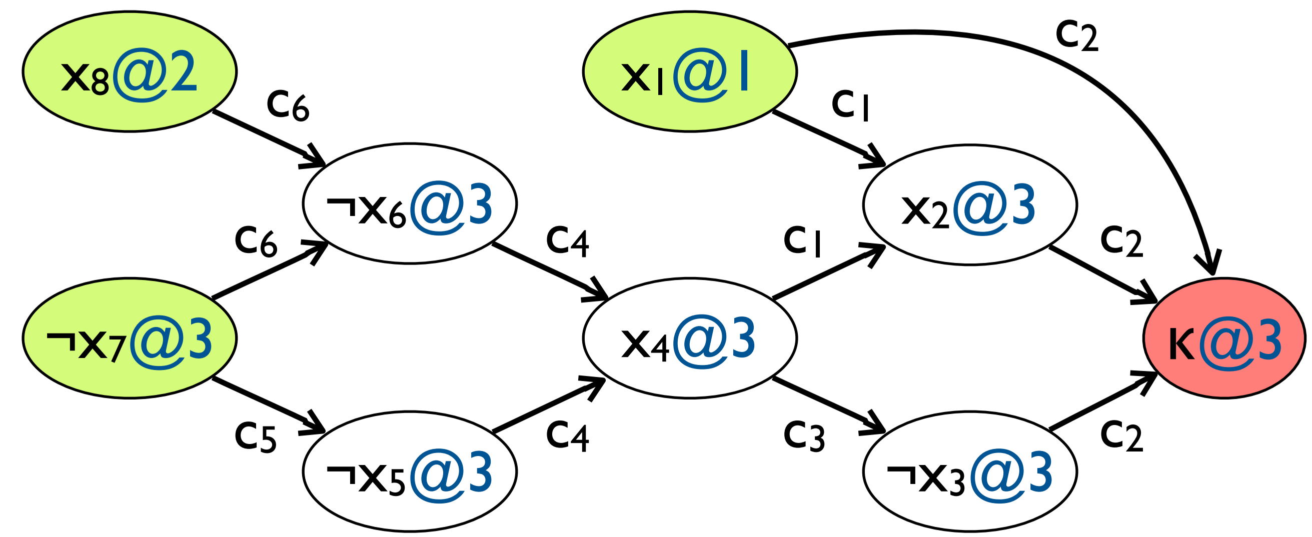
$c_9 : x_9 \vee x_{10} \vee x_3$

True literals highlighted in green; false literals highlighted in red.

Implication graph

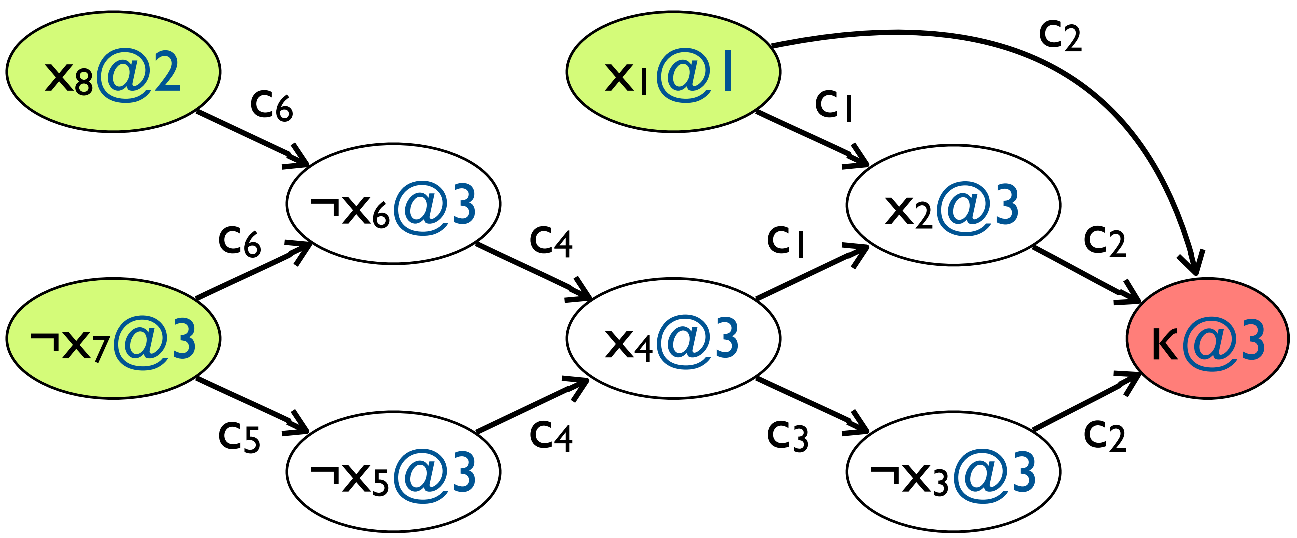
An **implication graph** G = (V, E) is a DAG that records the history of decisions and the resulting deductions derived with BCP.



Implication graph

An **implication graph** G = (V, E) is a DAG that records the history of decisions and the resulting deductions derived with BCP.

- v ∈ V is a literal (or κ) and the decision level at which it entered the current PA.



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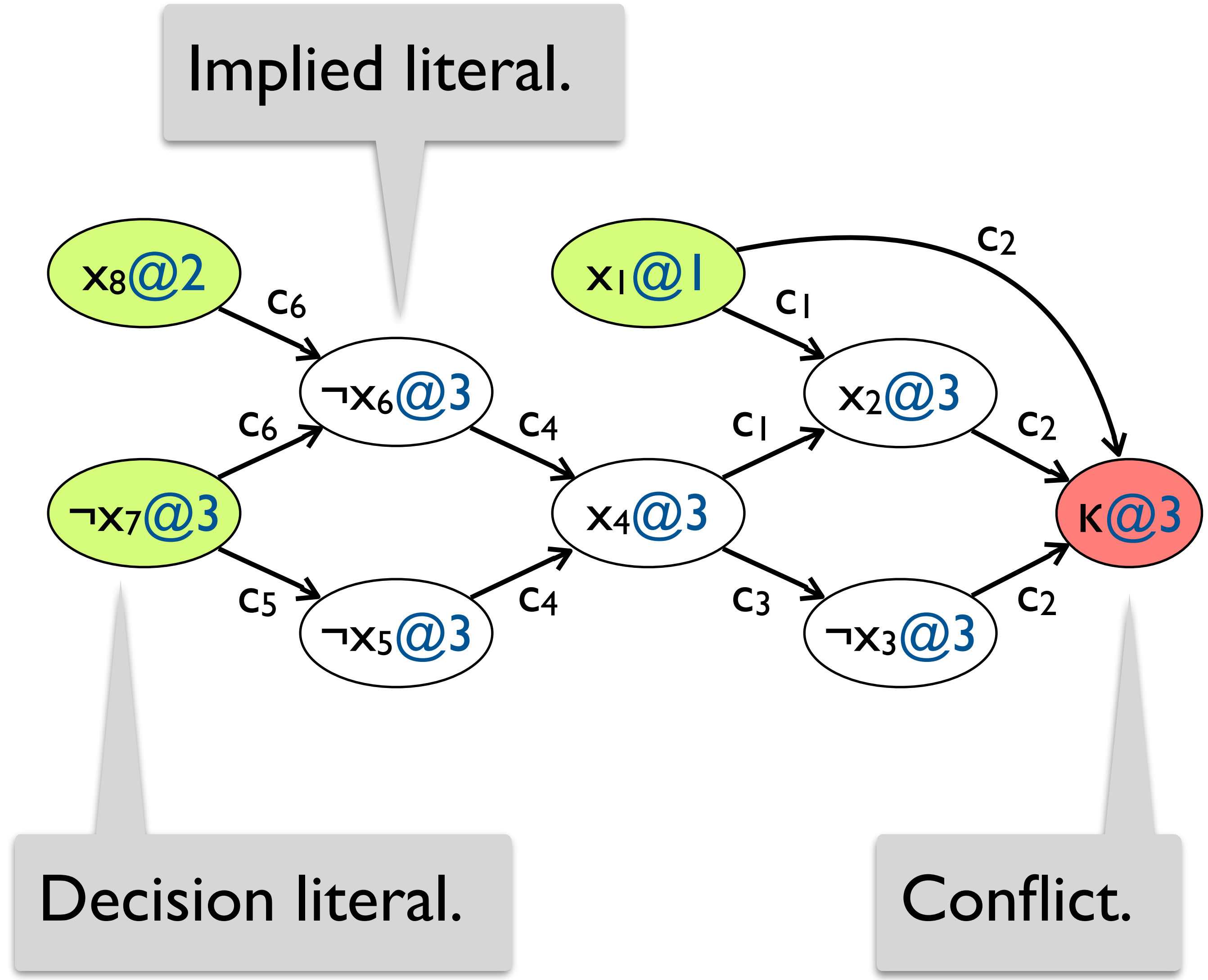

Implication graph

An **implication graph** G = (V, E) is a DAG that records the history of decisions and the resulting deductions derived with BCP.

- v ∈ V is a literal (or κ) and the decision level at which it entered the current PA.
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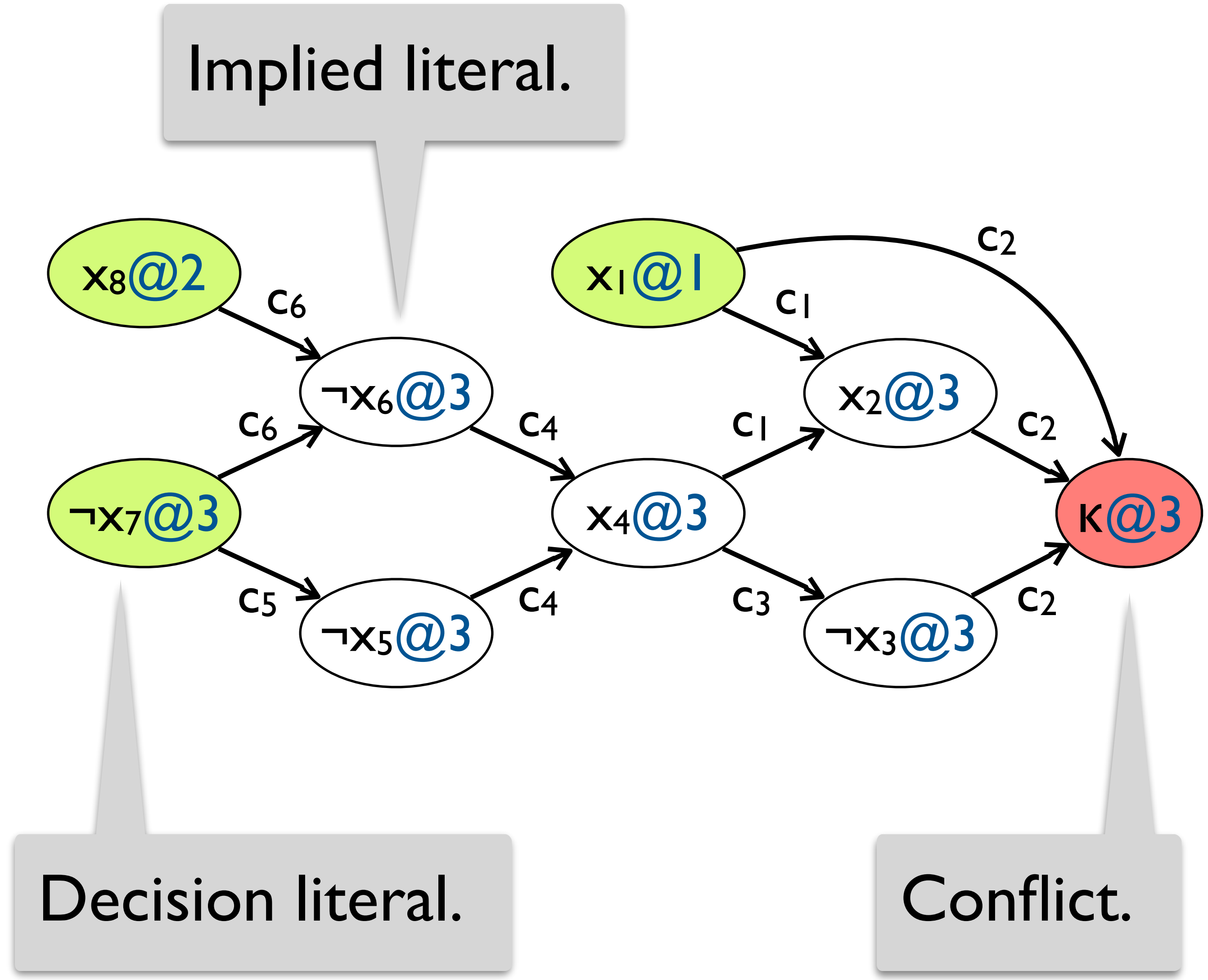

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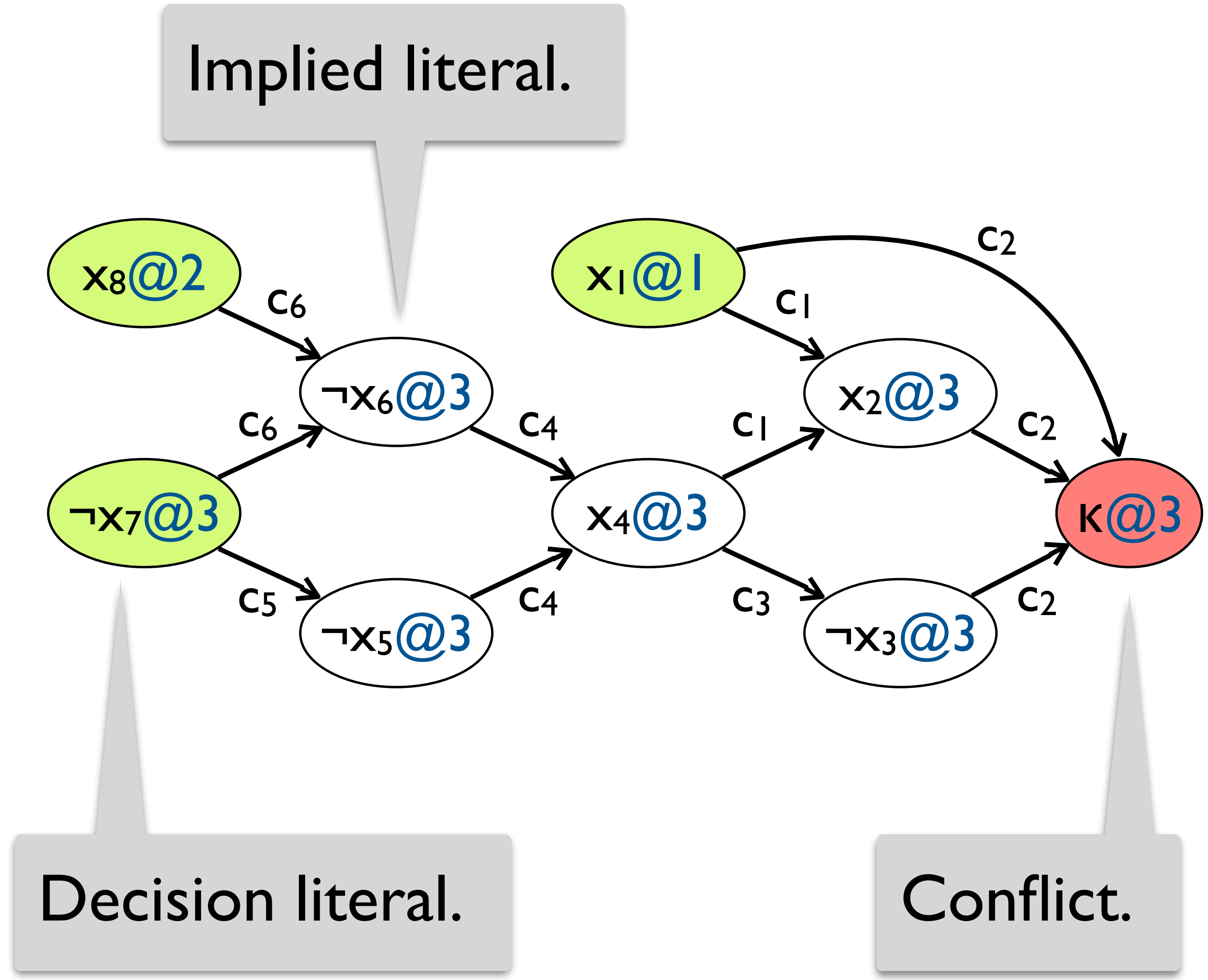
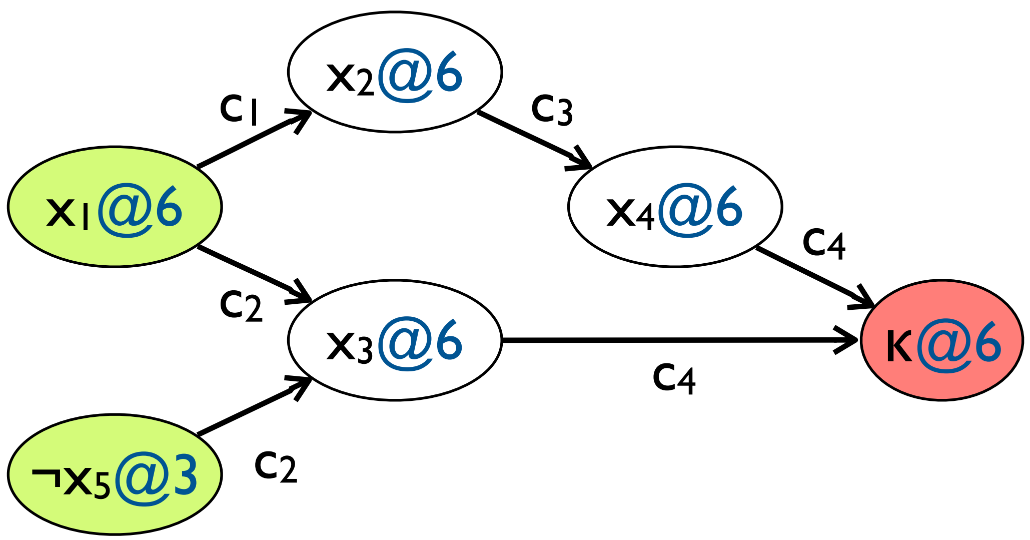
A unit clause c is an antecedent of its sole unassigned literal.



Implication graph: a quick exercise



What clauses gave rise to this implication graph?

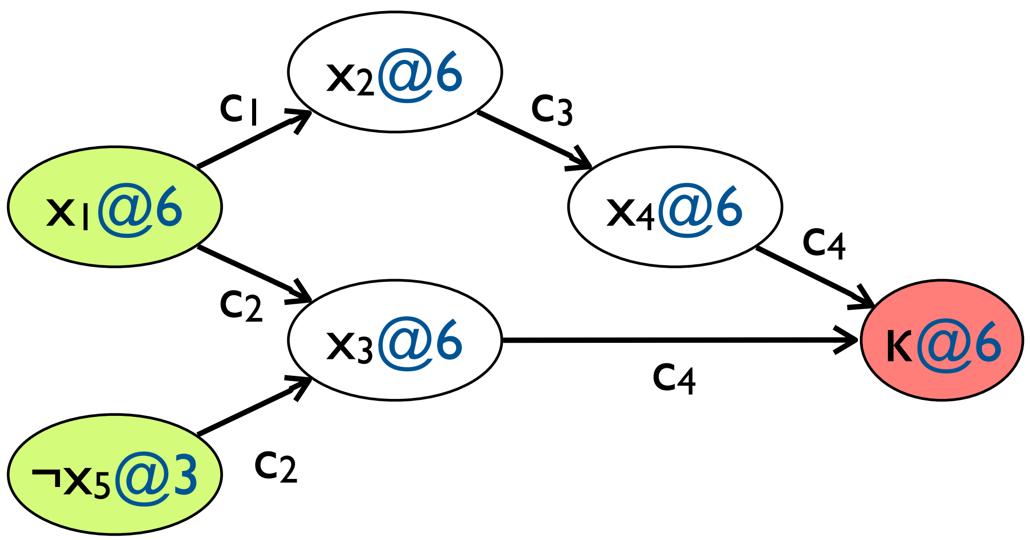
c_1 :

c_2 :

c_3 :

c_4 :

Implication graph: a quick exercise



What clauses gave rise to this implication graph?

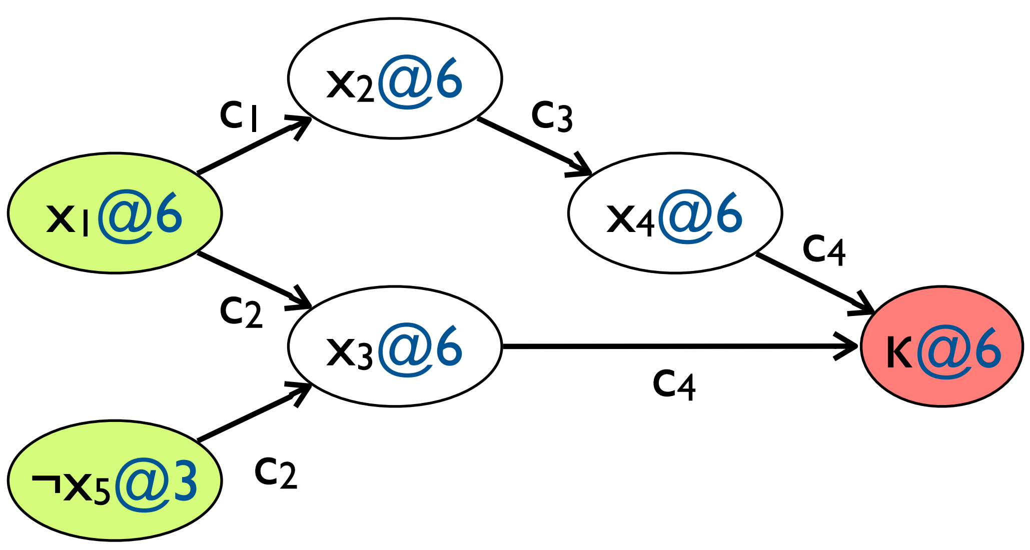
c_1 : $\neg x_1 \vee x_2$

c_2 :

c_3 :

c_4 :

Implication graph: a quick exercise



What clauses gave rise to this implication graph?

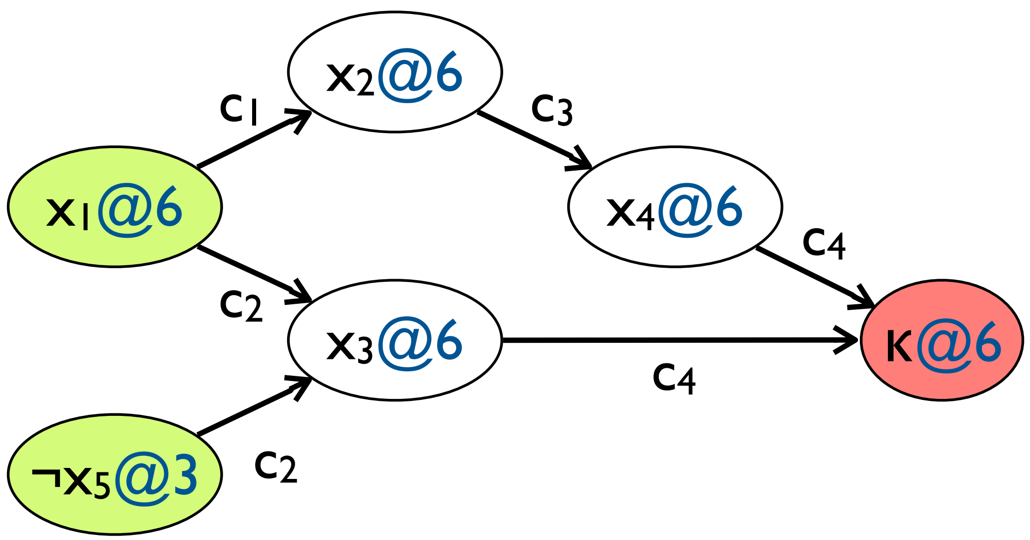
c_1 : $\neg x_1 \vee x_2$

c_2 : $\neg x_1 \vee x_3 \vee x_5$

c_3 :

c_4 :

Implication graph: a quick exercise



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c_1 : $\neg x_1 \lor x_2$

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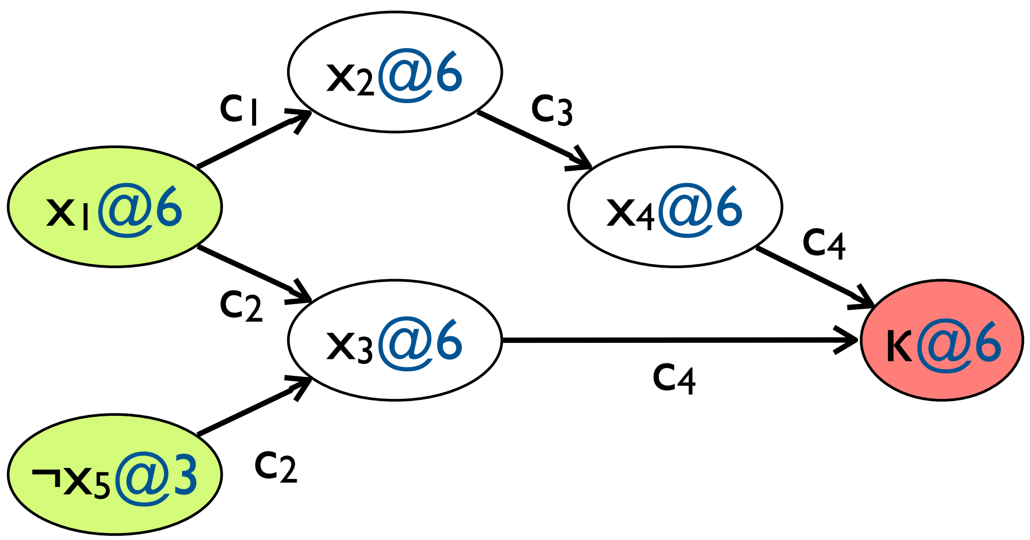
c_3 : $\neg x_2 \lor x_4$

c_4 :

Implication graph: a quick exercise



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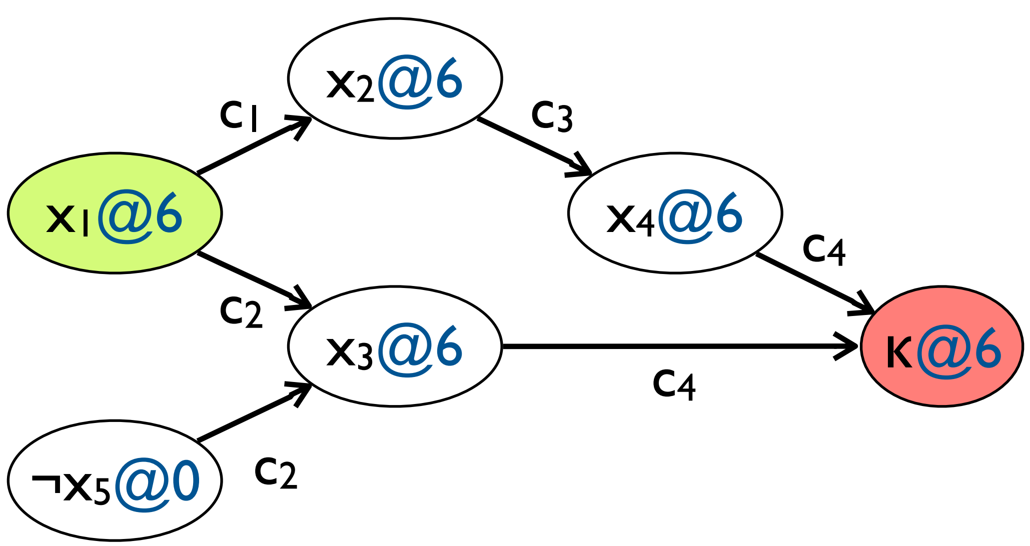
Implication graph: an even quicker exercise



What clauses gave rise to this implication graph?

c_1 :

c_2 :

c_3 :

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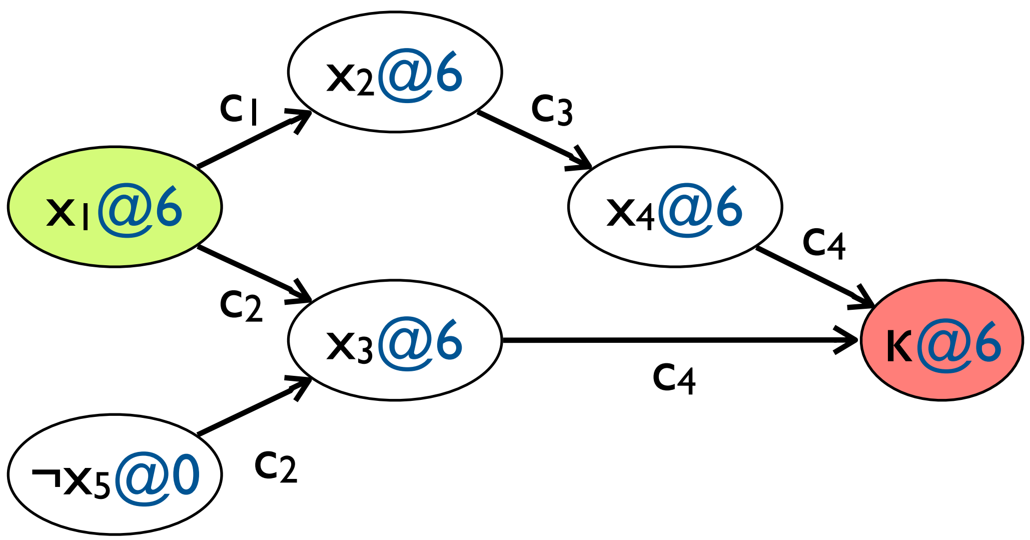
Implication graph: an even quicker exercise



Assignments at ground (0) level are implied by unary clauses.

What clauses gave rise to this implication graph?

$c_1 : \neg x_1 \lor x_2$

$c_2 : \neg x_1 \lor x_3 \lor x_5$

$c_3 : \neg x_2 \lor x_4$

$c_4 : \neg x_3 \lor \neg x_4$

$c_k : \neg x_5$

Using an implication graph to analyze a conflict

```
CDCL(F)
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 level ← 0
 while hasUnassignedVars(F)
   level ← level + 1
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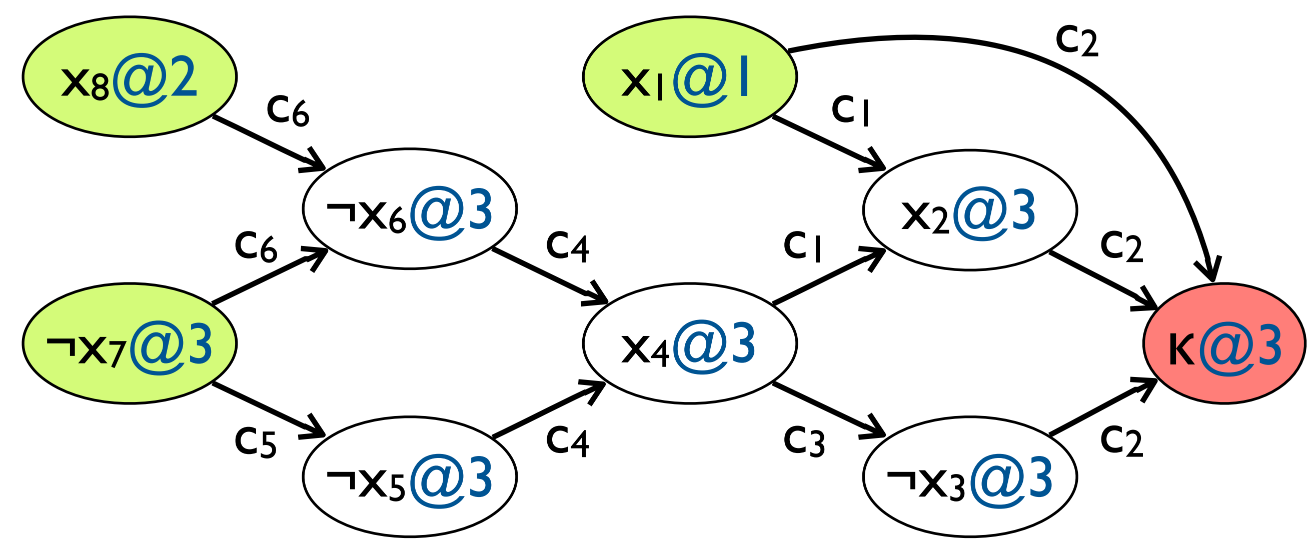

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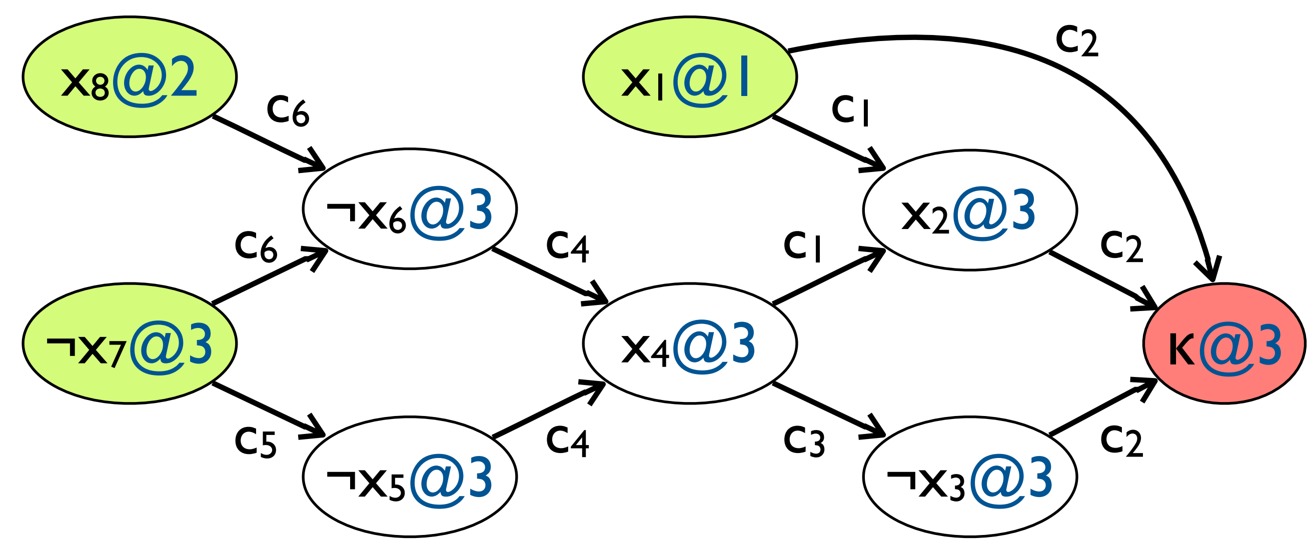

A **conflict clause** is implied by F and it blocks PAs that lead to the current conflict.

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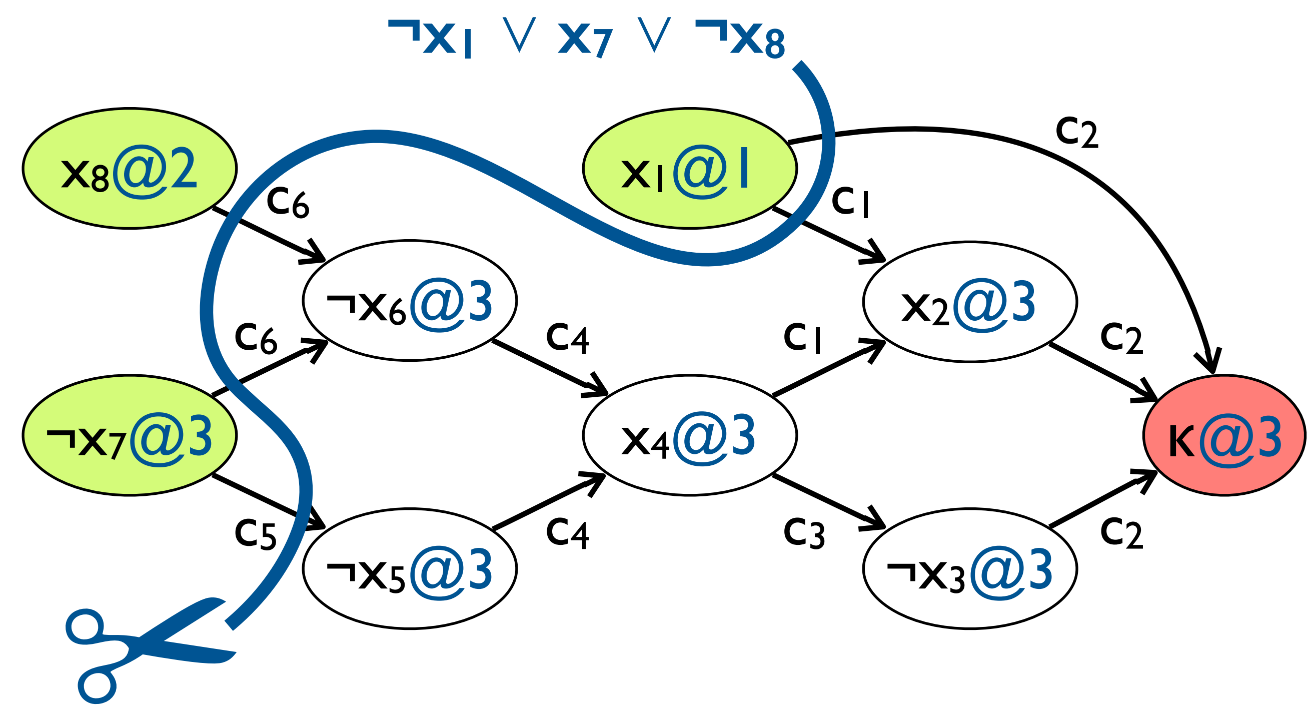

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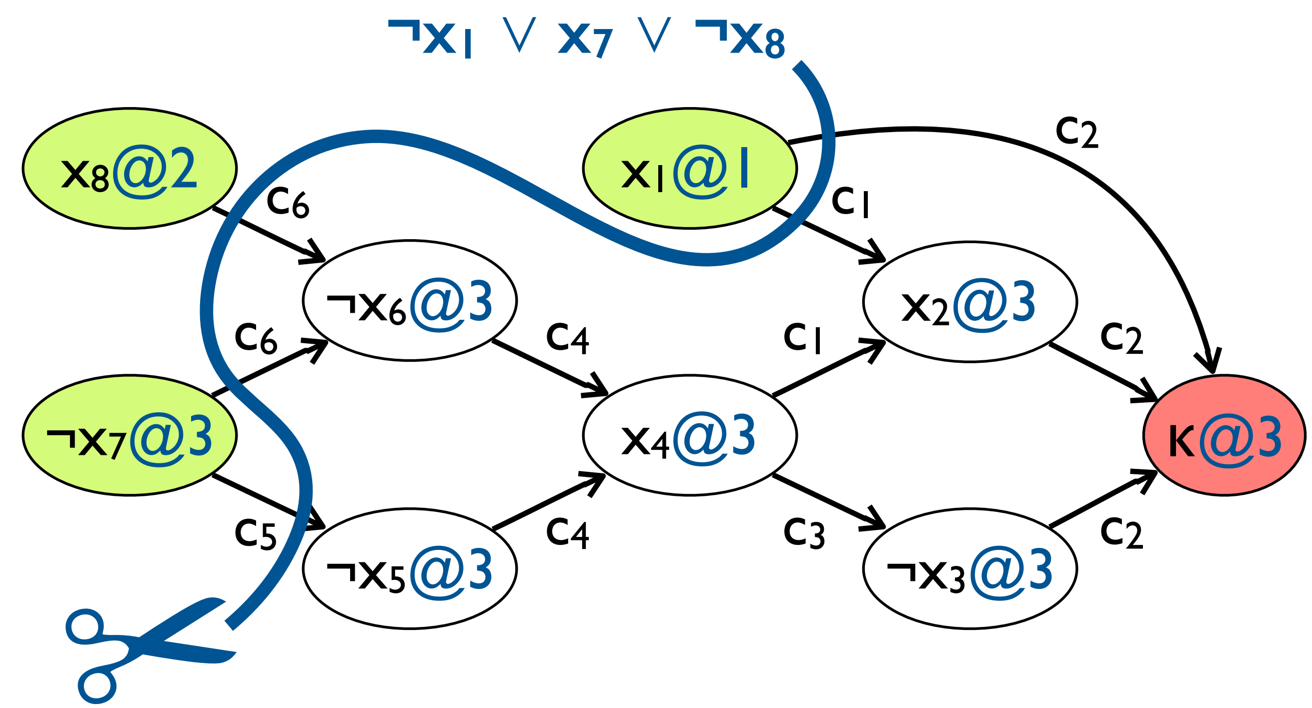

A **conflict clause** is implied by F and it blocks PAs that lead to the current conflict.

Every cut that separates sources from the sink defines a valid conflict clause.

Using an implication graph to analyze a conflict

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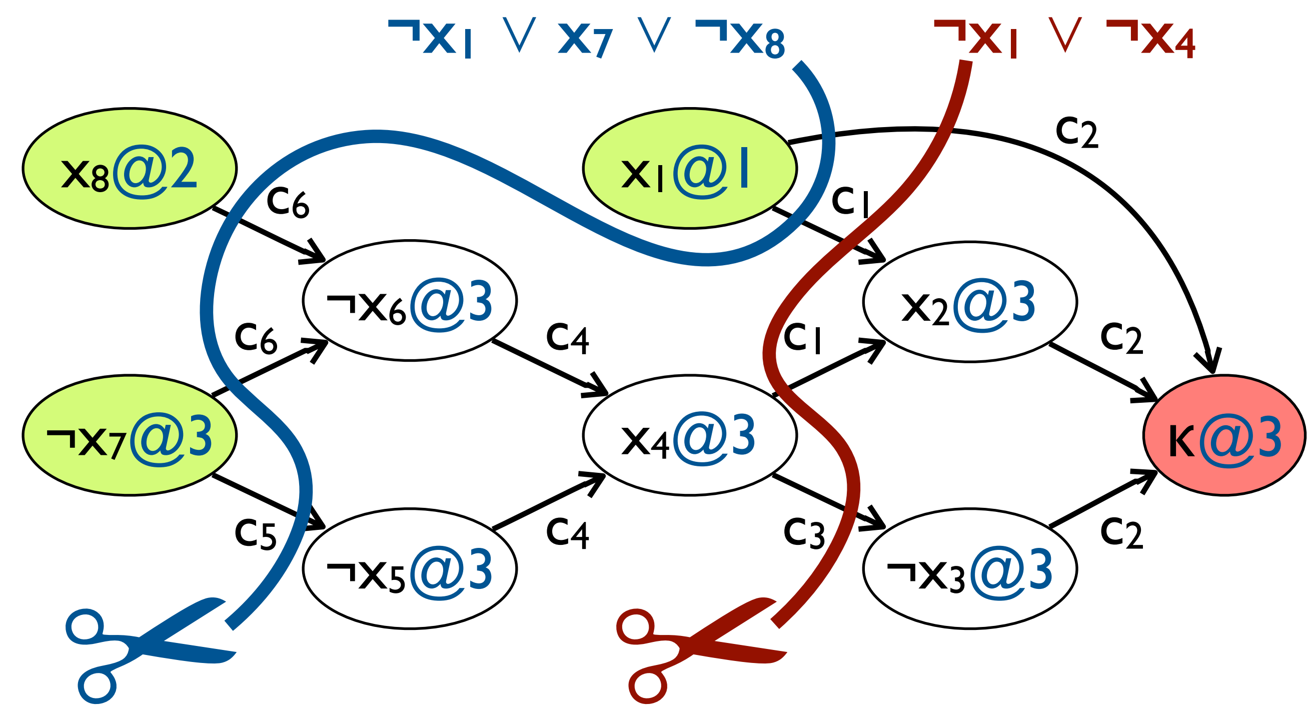

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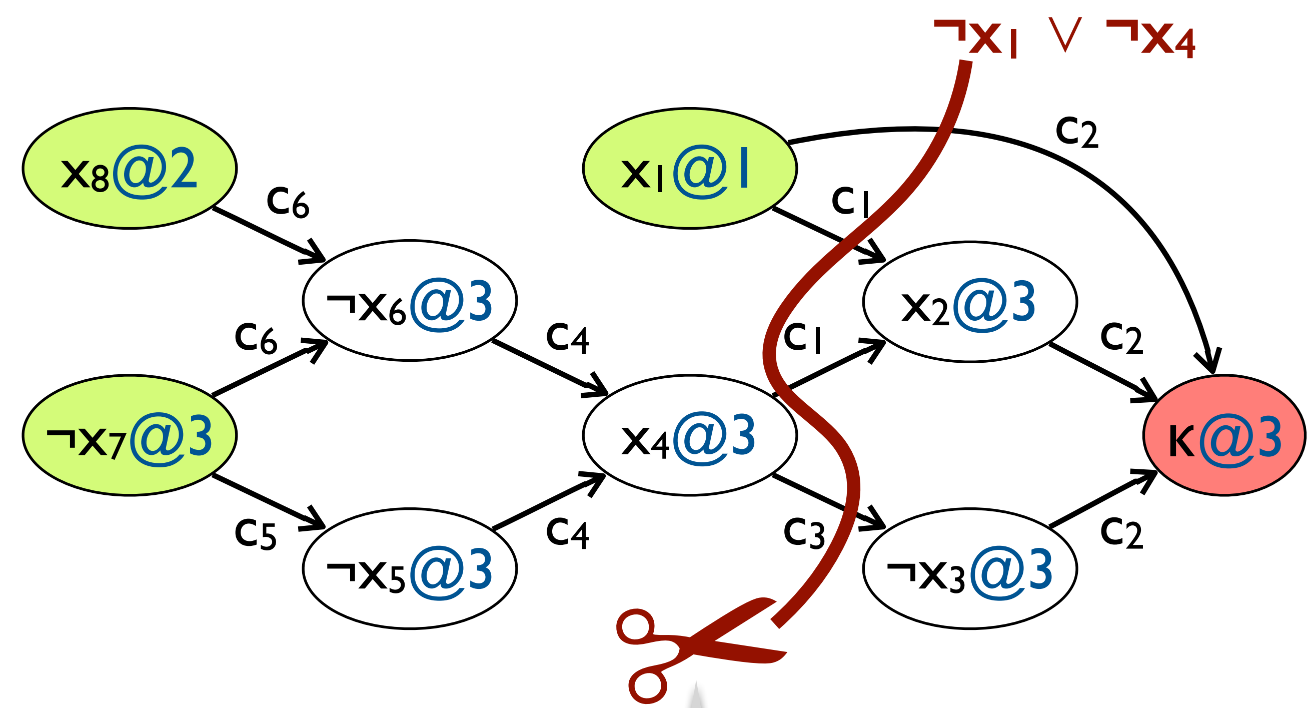
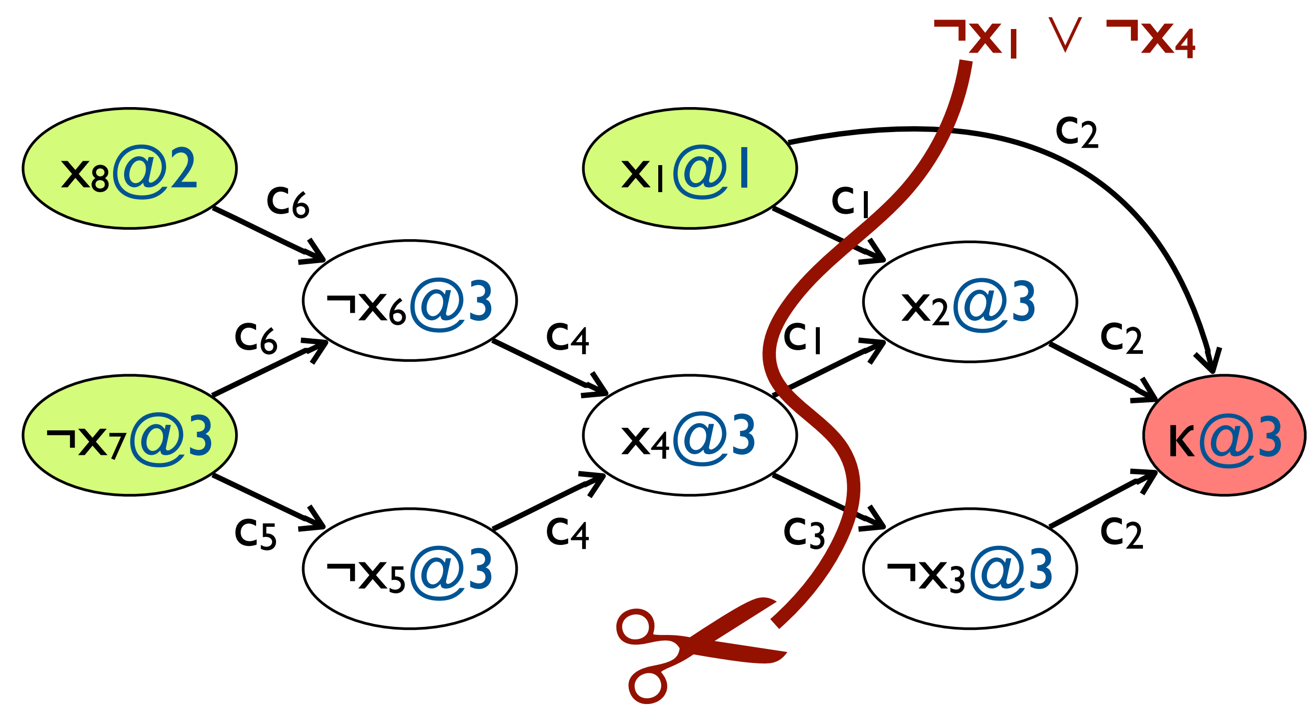


Cut after the **first unique implication point** to get the shortest conflict clause.

Unique implication points (UIPs)

A **unique implication point** (UIP) is any node in the implication graph other than the conflict that is on all paths from the current decision literal (lit@d) to the conflict (κ@d).

A **first UIP** is the UIP that is closest to the conflict.



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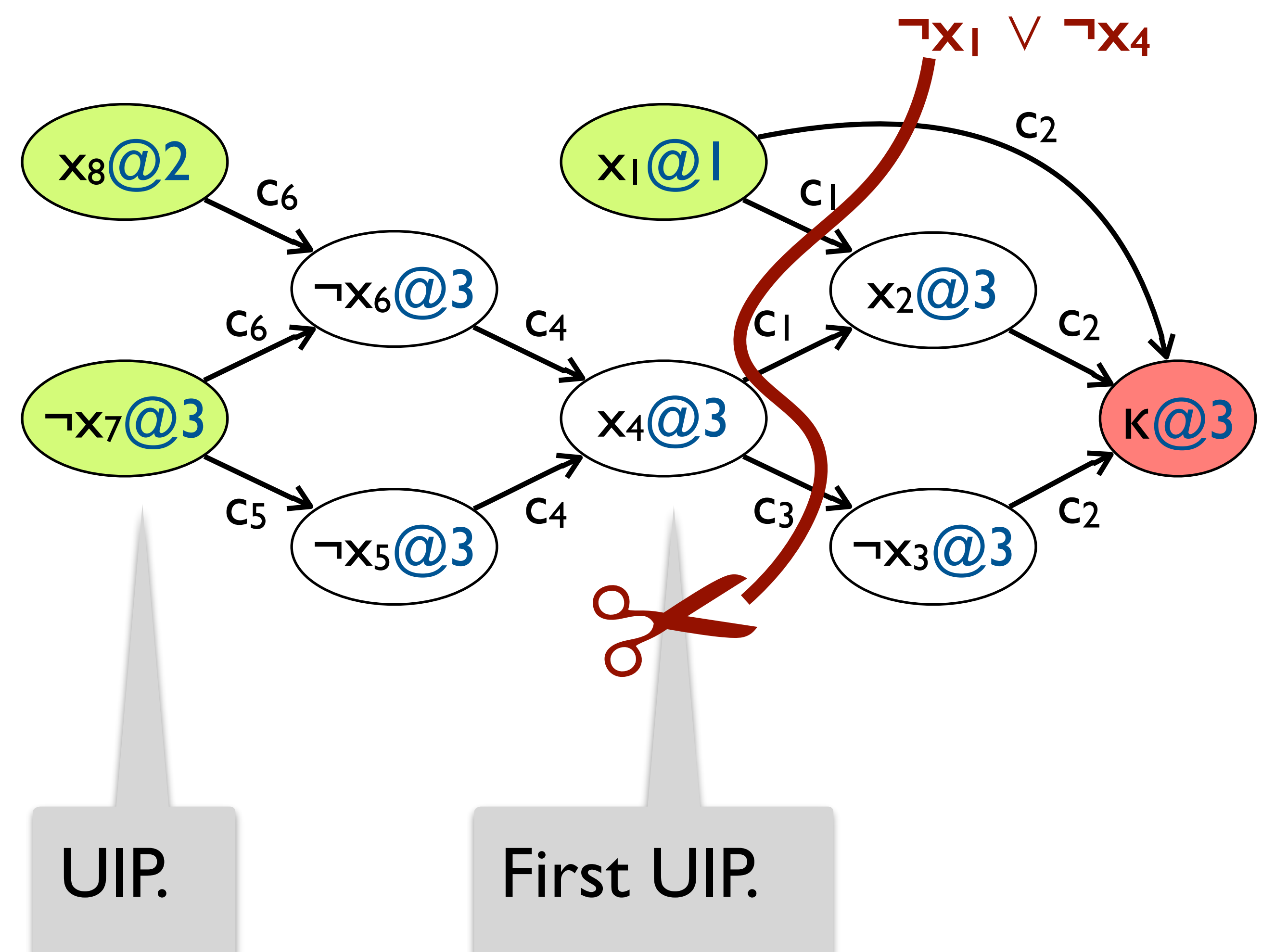
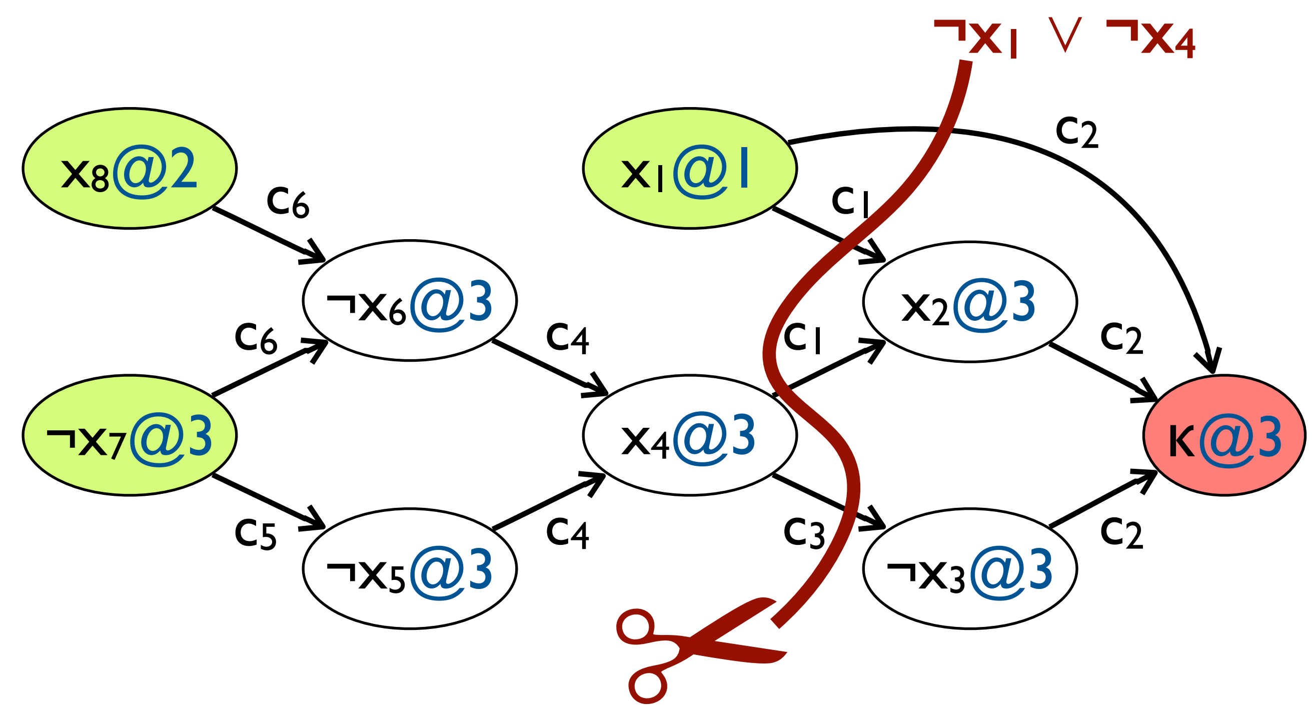


ANALYZECONFLICT: computing the conflict clause

```
ANALYZECONFLICT()
  d ← level(conflict)
  if d = 0 then return -1
  c ← antecedent(conflict)
  s ← ¬firstUIP()
  repeat
    t ← lastAssignedLitAtLevel(c, d)
    v ← varOfLit(t)
    ante ← antecedent(t)
    c ← resolve(ante, c, v)
  until soleLitAtLevel(s, c, d)
  b ← …
  return ⟨b, c⟩
```



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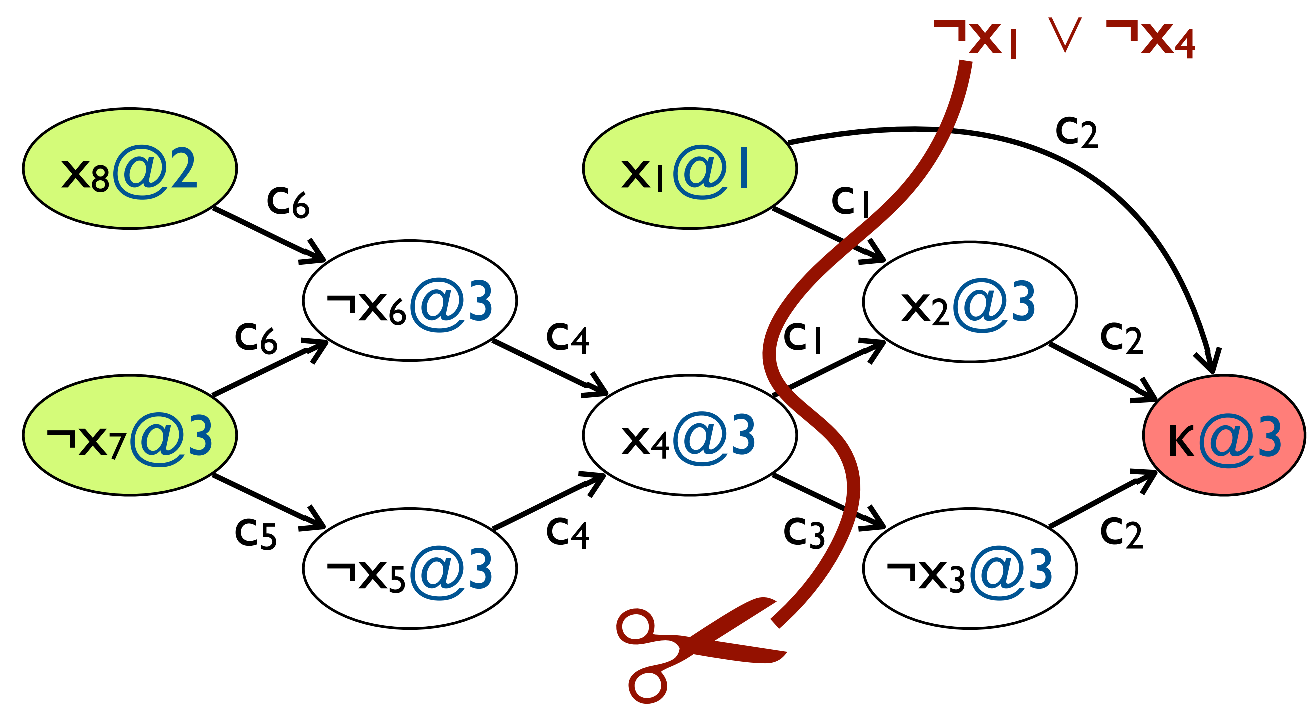
```



Binary resolution rule

$$\frac{(a_1 \vee \ldots \vee a_n \vee \beta) \quad (b_1 \vee \ldots \vee b_m \vee \neg\beta)}{(a_1 \vee \ldots \vee a_n \vee b_1 \vee \ldots \vee b_m)}$$

ANALYZECONFLICT: computing the conflict clause

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    c ← resolve(ante, c, v)
  until soleLitAtLevel(s, c, d)
  b ←…
  return ⟨b, c⟩
```

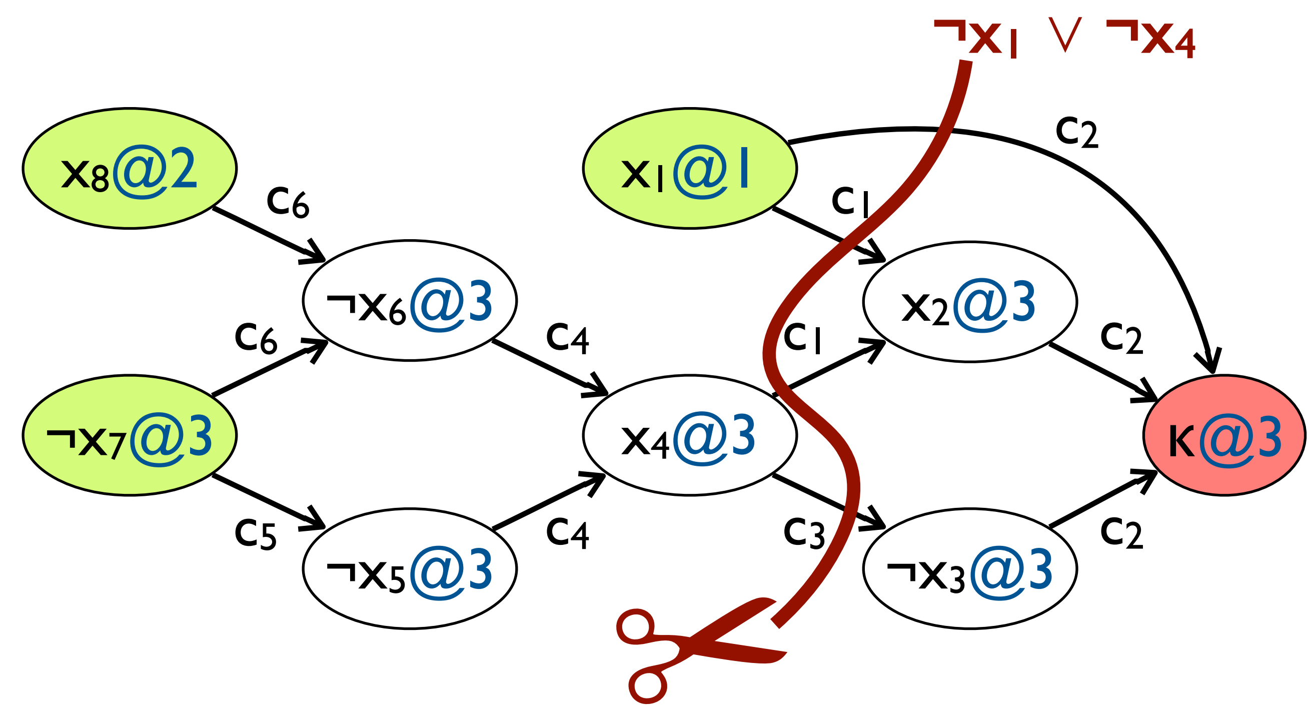

Example:

AnalyzeConflict: computing the conflict clause

```
ANALYZECONFLICT()
  d ← level(conflict)
  if d = 0 then return -1
  c ← antecedent(conflict)
  s ← ¬firstUIP()
  repeat
    t ← lastAssignedLitAtLevel(c, d)
    v ← varOfLit(t)
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    c ← resolve(ante, c, v)
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  b ←…
  return ⟨b, c⟩
```
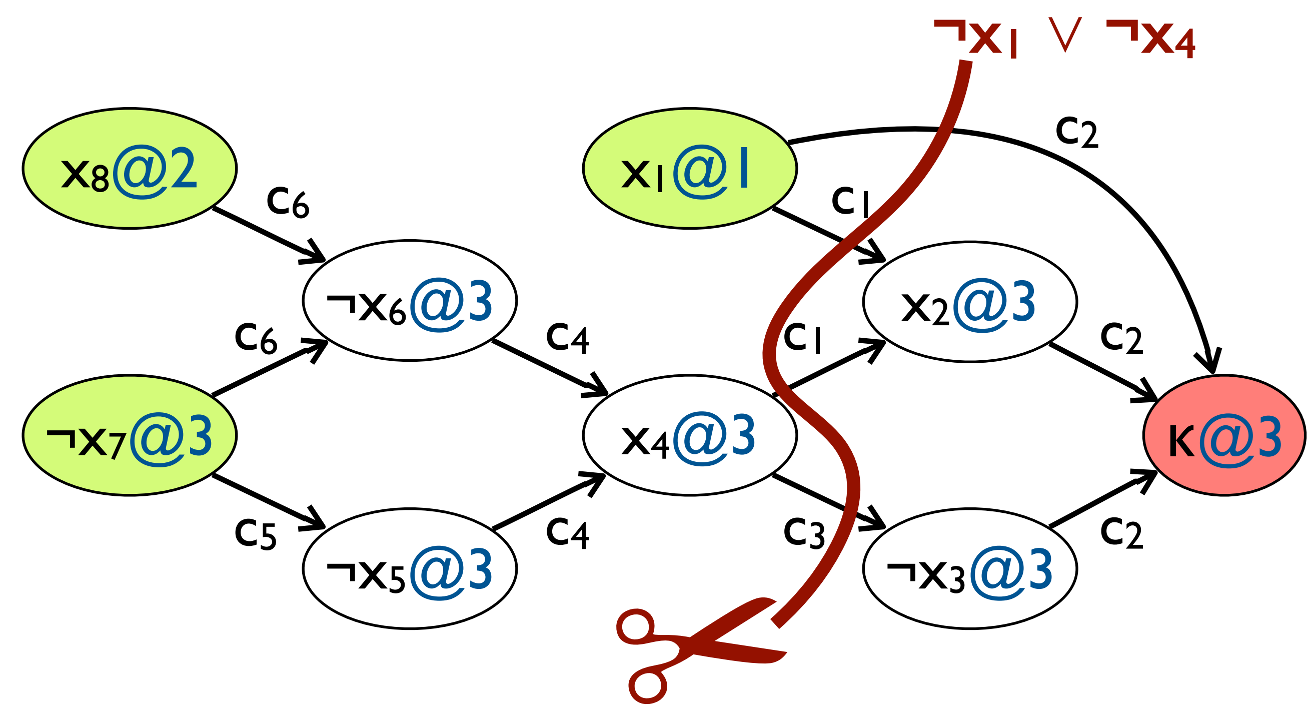
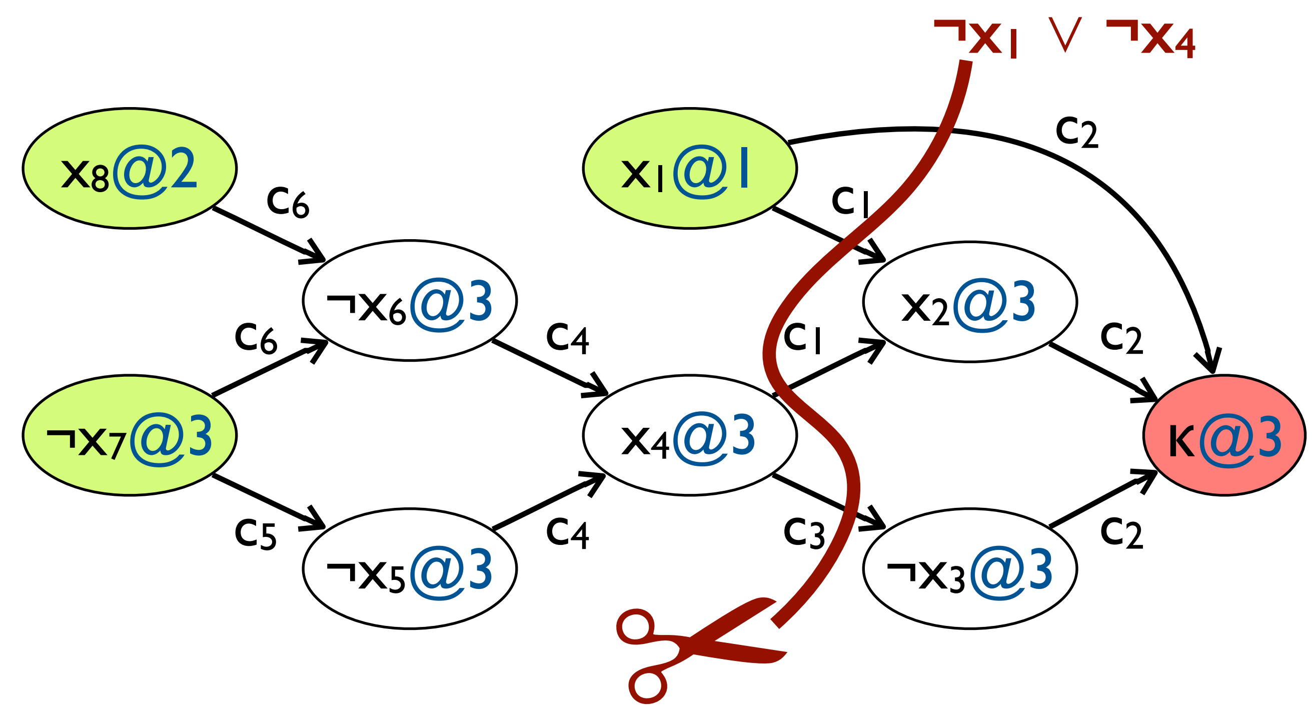



Example:

- $c = c_2$, $t = x_2$, $v = x_2$, ante $= c_1$

ANALYZECONFLICT: computing the conflict clause

```
ANALYZECONFLICT()
  d ← level(conflict)
  if d = 0 then return -1
  c ← antecedent(conflict)
  s ← ¬firstUIP()
  repeat
    t ← lastAssignedLitAtLevel(c, d)
    v ← varOfLit(t)
    ante ← antecedent(t)
    c ← resolve(ante, c, v)
  until soleLitAtLevel(s, c, d)
  b ← …
  return ⟨b, c⟩
```



Example:

- $c = c_2$, $t = x_2$, $v = x_2$, ante $= c_1$
- $c = \neg x_1 \lor x_3 \lor \neg x_4$, $t = x_3$, $v = x_3$, ante $= c_3$

ANALYZECONFLICT: computing the conflict clause

```
ANALYZECONFLICT()
  d ← level(conflict)
  if d = 0 then return -1
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  b ←…
  return ⟨b, c⟩
```
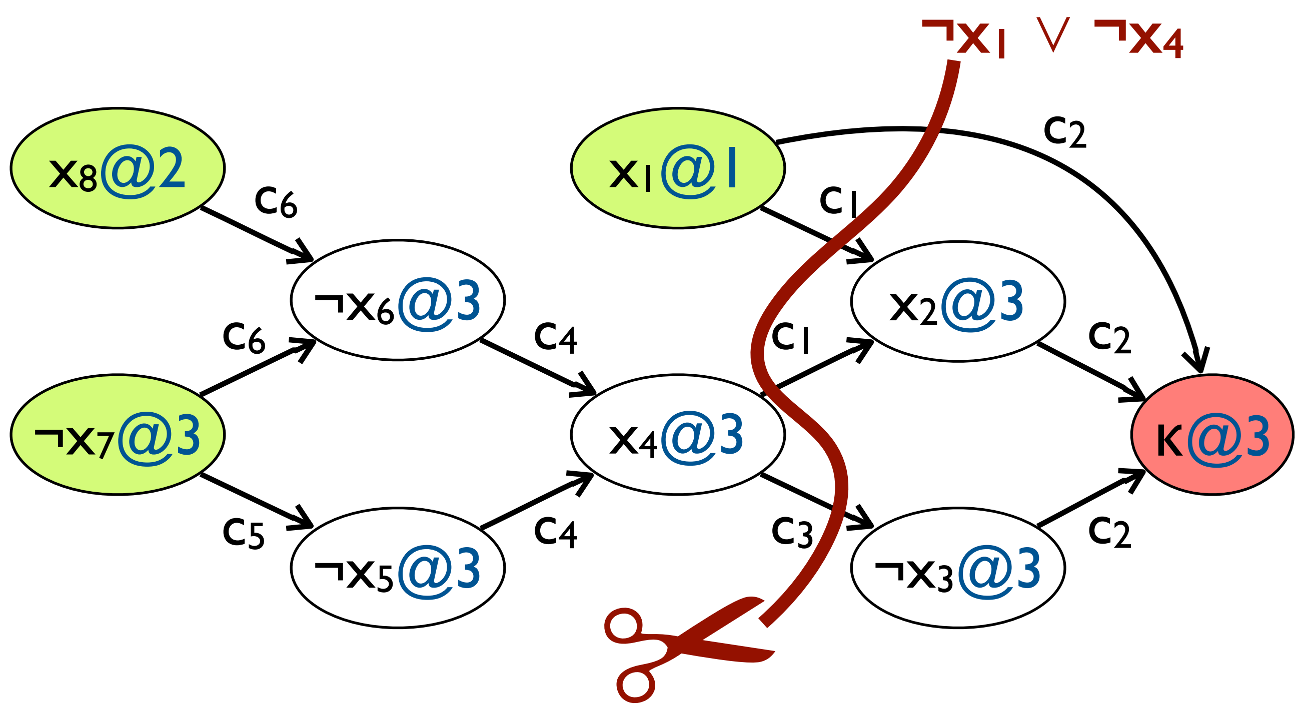
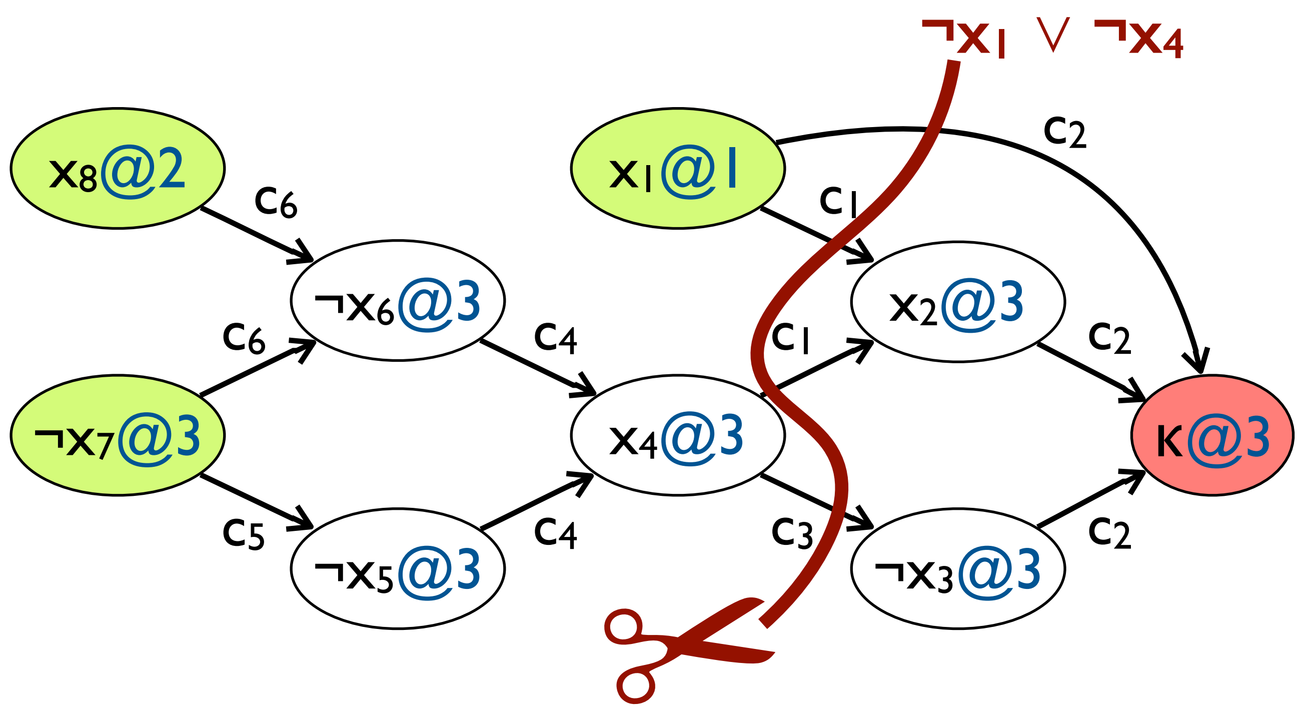



Example:

- $c = c_2$, $t = x_2$, $v = x_2$, ante $= c_1$
- $c = \neg x_1 \vee x_3 \vee \neg x_4$, $t = x_3$, $v = x_3$, ante $= c_3$
- $c = \neg x_1 \vee \neg x_4$, done!

ANALYZECONFLICT: computing backtracking level

```
ANALYZECONFLICT()
  d ← level(conflict)
  if d = 0 then return -1
  c ← antecedent(conflict)
  s ← ¬firstUIP()
  repeat
    t ← lastAssignedLit(c)
    v ← varOfLit(t)
    ante ← antecedent(t)
    c ← resolve(ante, c, v)
  until soleLitAtLevel(s, c, d)
  b ← assertingLevel(c)
  return ⟨b, c⟩
```



ANALYZECONFLICT: computing backtracking level

```
ANALYZECONFLICT()
  d ← level(conflict)
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  c ← antecedent(conflict)
  s ← ¬firstUIP()
  repeat
    t ← lastAssignedLit(c)
    v ← varOfLit(t)
    ante ← antecedent(t)
    c ← resolve(ante, c, v)
  until soleLitAtLevel(s, c, d)
  b ← assertingLevel(c)
  return ⟨b, c⟩
```
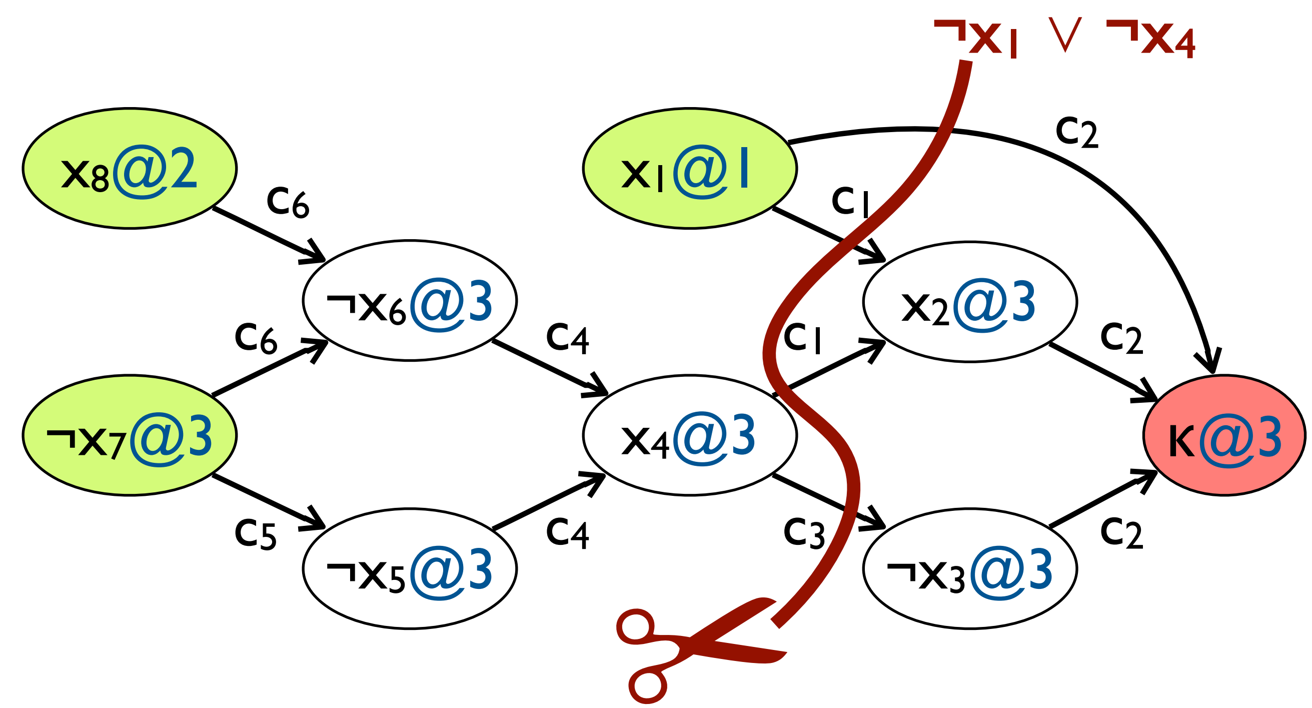
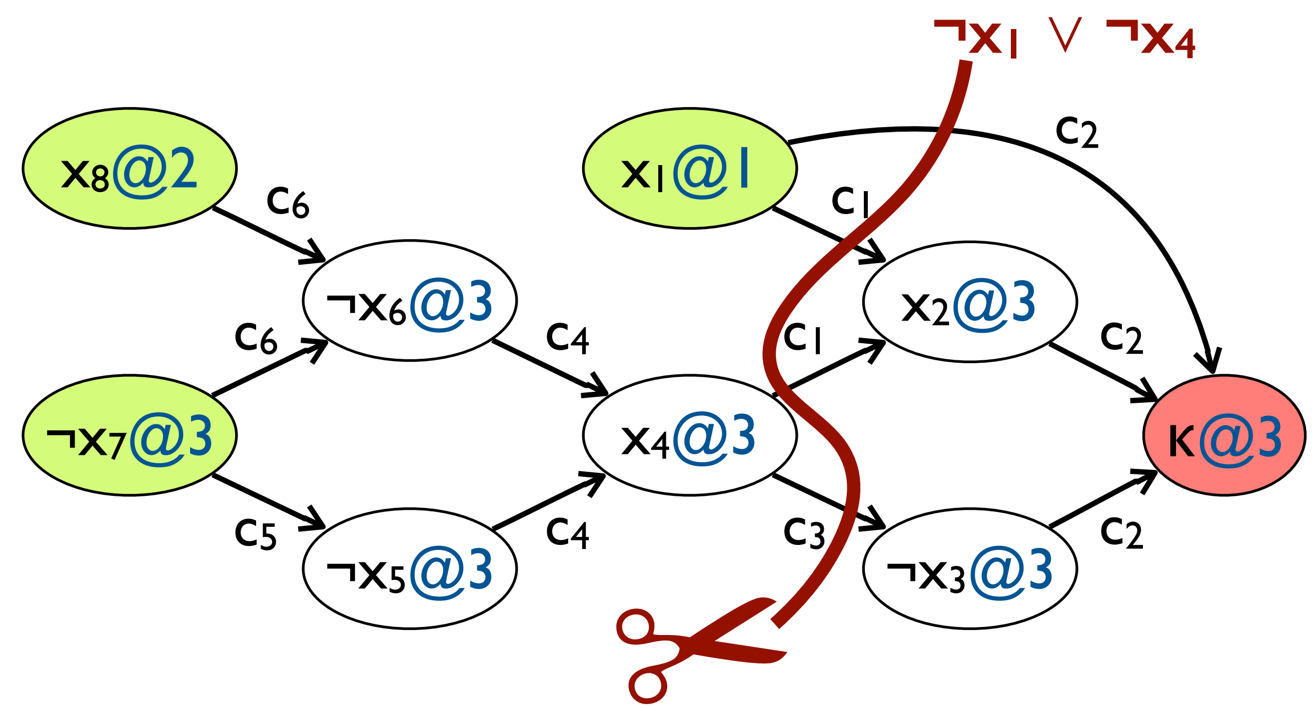



Second highest decision level for any literal in c, unless c is unary. In that case, its asserting level is zero.

ANALYZECONFLICT: computing backtracking level

```
ANALYZECONFLICT()
  d ← level(conflict)
  if d = 0 then return -1
  c ← antecedent(conflict)
  s ← ¬firstUIP()
  repeat
    t ← lastAssignedLit(c)
    v ← varOfLit(t)
    ante ← antecedent(t)
    c ← resolve(ante, c, v)
  until soleLitAtLevel(s, c, d)
  b ← assertingLevel(c)
  return ⟨b, c⟩
```



By construction, c is unit at b (since it has only one literal at the current level d).

Decision heuristics

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CDCL(F)
  A ← {}
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  level ← 0
  while hasUnassignedVars(F)
    level ← level + 1
    A ← A ∪ { DECIDE(F,A) }
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Example heuristics:

- Dynamic Largest Individual Sum (DLIS)
- Variable State Independent Decaying Sum (VSIDS)

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- Choose the literal that satisfies the most unresolved clauses.

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- Simple and intuitive.

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- Choose the literal that satisfies the most unresolved clauses.
- Simple and intuitive.
- But expensive: complexity of making a decision proportional to the number of clauses.

Decision heuristics: VSIDS (zChaff)

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Most solvers heuristically discard conflict clauses that are old, long, irrelevant, etc. (Why won't this cause the solver to run forever?)

BCP with watched literals (zChaff)

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- So, pick two unassigned literals per clause to watch.
- If a watched literal is assigned, pick another unassigned literal to watch in its place.
- If there is only one unassigned literal, it is implied by BCP.

Summary

Today

- The CDCL algorithm at the core of modern solvers extends DPLL with
  - Non-chronological backtracking
  - Learning
  - Decision heuristics
  - Engineering matters

Next lecture

- Practical applications of SAT solving to software engineering.