CSE-505: Programming Languages

Lecture 9 — Simply Typed Lambda Calculus

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Types

Major new topic worthy of several lectures: Type systems
▶ Continue to use (CBV) Lambda Calculus as our core model
▶ But will soon enrich with other common primitives

This lecture:
▶ Motivation for type systems
▶ What a type system is designed to do and not do
  ▶ Definition of stuckness, soundness, completeness, etc.
▶ The Simply-Typed Lambda Calculus
  ▶ A basic and natural type system
  ▶ Starting point for more expressiveness later

Next lecture:
▶ Prove Simply-Typed Lambda Calculus is sound

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Review: L-R CBV Lambda Calculus

e ::= λx.e | x | ee
v ::= λx.e

Implicit systematic renaming of bound variables
▶ α-equivalence on expressions (“the same term”)

\[
\begin{align*}
  e & \rightarrow e' \\
  (\lambda x. e) v & \rightarrow e[v/x] \\
  e_1 & \rightarrow e'_1 \\
  e_2 & \rightarrow e'_2 \\
  v e_2 & \rightarrow v e'_2 \\
  e_1[e_2/x] & = e_3 \\
  y \neq x & \quad e_1[e/x] = e'_1 \\
  y e/x & = y \\
  e_2[e/x] & = e'_2 \\
  (e_1 e_2)[e/x] & = e'_1 e'_2 \\
  e_1[e/x] & = e'_1 \\
  y \neq x & \quad y \not\in \text{FV}(e) \\
  (\lambda y. e_1)[e/x] & = \lambda y. e'_1
\end{align*}
\]

Introduction to Types

Naive thought: More powerful PLs are always better
▶ Be Turing Complete (e.g., Lambda Calculus or x86 Assembly)
▶ Have really flexible features (e.g., lambdas)
▶ Have conveniences to keep programs short

If this is the only metric, types are a step backward
▶ Whole point is to allow fewer programs
▶ A “filter” between abstract syntax and compiler/interpreter
  ▶ Fewer programs in language means less for a correct implementation
▶ So if types are a great idea, they must help with other desirable properties for a PL...
Why types? (Part 1)

1. Catch “simple” mistakes early, even for untested code
   ▶ Example: “if” applied to “mkpair”
   ▶ Even if some too-clever programmer meant to do it
   ▶ Even though decidable type systems must be conservative

2. (Safety) Prevent getting stuck (e.g., \( x \ v \))
   ▶ Ensure execution never gets to a “meaningless” state
   ▶ But “meaningless” depends on the semantics
   ▶ Each PL typically makes some things type errors (again being conservative) and others run-time errors

3. Enforce encapsulation (an abstract type)
   ▶ Clients can’t break invariants
   ▶ Clients can’t assume an implementation
   ▶ Requires safety, meaning no “stuck” states that corrupt run-time (e.g., C/C++)
   ▶ Can enforce encapsulation without static types, but types are a particularly nice way

Why types? (Part 2)

4. Assuming well-typedness allows faster implementations
   ▶ Smaller interfaces enable optimizations
   ▶ Don’t have to check for impossible states
   ▶ Orthogonal to safety (e.g., C/C++)

5. Syntactic overloading
   ▶ Have symbol lookup depend on operands’ types
   ▶ Only modestly interesting semantically
   ▶ Late binding (lookup via run-time types) more interesting

6. Detect other errors via extensions
   ▶ Often via a “type-and-effect” system
   ▶ Deep similarities in analyses suggest type systems a good way to think-about/define/prove what you’re checking
   ▶ Uncaught exceptions, tainted data, non-termination, IO performed, data races, dangling pointers, ...

We’ll focus on (1), (2), and (3) and maybe (6)
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   performed, data races, dangling pointers, ...

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What is a type system?

Er, uh, you know it when you see it. Some clues:
▶ A decidable (?) judgment for classifying programs
   ▶ E.g., \(e_1 + e_2\) has type int if \(e_1, e_2\) have type int (else no type)
▶ A sound (?) abstraction of computation
   ▶ E.g., if \(e_1 + e_2\) has type int, then evaluation produces an int
   (with caveats!)
▶ Fairly syntax directed
   ▶ Non-example (?): \(e\) terminates within 100 steps
▶ Particularly fuzzy distinctions with abstract interpretation
▶ Possible topic for a later lecture
▶ Often a more natural framework for flow-sensitive properties
▶ Types often more natural for higher-order programs

This is a CS-centric, PL-centric view. Foundational type theory has
more rigorous answers
▶ Later lecture: Typed PLs are like proof systems for logics
Plan for 3ish weeks

- Simply typed λ calculus
- (Syntactic) Type Soundness (i.e., safety)
- Extensions (pairs, sums, lists, recursion)

Break for the Curry-Howard isomorphism; continuations; midterm

- Subtyping
- Polymorphic types (generics)
- Recursive types
- Effect systems

Homework: Adding back mutation
Omitted: Type inference

Adding constants

Enrich the Lambda Calculus with integer constants:

- Not strictly necessary, but makes types seem more natural

\[ e ::= \lambda x. e \mid x \mid e \mid e \mid c \]
\[ v ::= \lambda x. e \mid c \]

No new operational-semantics rules since constants are values

We could add + and other primitives

- Then we would need new rules (e.g., 3 small-step for +)
- Alternately, parameterize “programs” by primitives:
  \[ \lambda \text{plus. } \lambda \text{times. } \ldots e \]
  - Like Pervasives in OCaml
  - A great way to keep language definitions small

Stuck

Key issue: can a program “get stuck” (reach a “bad” state)?

- Definition: \( e \) is stuck if \( e \) is not a value and there is no \( e' \) such that \( e \to e' \)
- Definition: \( e \) can get stuck if there exists an \( e' \) such that \( e \to^* e' \) and \( e' \) is stuck
  - In a deterministic language, \( e \) "gets stuck"

Most people don’t appreciate that stuckness depends on the operational semantics

- Inherent given the definitions above

What’s stuck?

Given our language, what are the set of stuck expressions?

- Note: Explicitly defining the stuck states is unusual

\[ e ::= \lambda x. e \mid x \mid e \mid e \mid c \]
\[ v ::= \lambda x. e \mid c \]

\[ (\lambda x. e) v \to e[v/x] \quad \frac{e_1 \to e_1'}{e_1 \to e_1'} \quad \frac{e_2 \to e_2'}{v e_2 \to v e_2'} \]

(Hint: The full set is recursively defined.)

\[ S ::= \]

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What's stuck?

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\[
\begin{align*}
e &::= \lambda x. e \mid x \mid e\ e \mid c \\
v &::= \lambda x. e \mid c \\
(\lambda x. e)\ v &\to e[v/x] & e_1\ e_2 &\to e_1'\ e_2 & e_2 &\to e_2' \\
(\lambda x. e)\ v &\to e[v/x] & e_1\ e_2 &\to e_1'\ e_2 & e_2 &\to e_2' \\
\end{align*}
\]

(Hint: The full set is recursively defined.)

\[
S ::= x \mid c\ v \mid S\ e \mid v\ S
\]

Note: Can have fewer stuck states if we add more rules

▶ Example: Javascript

▶ Example: 

▶ In unsafe languages, stuck states can set the computer on fire

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Soundness and Completeness

A type system is a judgment for classifying programs

▶ “accepts” a program if some complete derivation gives it a type, else “rejects”

A sound type system never accepts a program that can get stuck

▶ No false negatives

A complete type system never rejects a program that can’t get stuck

▶ No false positives

It is typically undecidable whether a stuck state can be reachable

▶ Corollary: If we want an algorithm for deciding if a type system accepts a program, then the type system cannot be sound and complete

▶ We’ll choose soundness, try to reduce false positives in practice

Wrong Attempt

\[
\begin{align*}
\tau &::= \text{int} \mid \text{fn} \\
\vdash e : \tau \\
\vdash \lambda x. e : \text{fn} &\quad \vdash c : \text{int} &\quad \vdash e_1\ e_2 : \text{int}
\end{align*}
\]
Wrong Attempt

\[ \tau ::= \text{int} | \text{fn} \]
\[ \vdash e : \tau \]
\[ \vdash \lambda x. e : \text{fn} \]
\[ \vdash e_1 : \text{fn} \]
\[ \vdash e_2 : \text{int} \]
\[ \vdash e_1 e_2 : \text{int} \]

1. NO: can get stuck, e.g., \((\lambda x. y) 3\)
2. NO: too restrictive, e.g., \((\lambda x. x 3) (\lambda y. y)\)
3. NO: types not preserved, e.g., \((\lambda x. \lambda y. y) 3\)

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Getting it right

1. Need to type-check function bodies, which have free variables
2. Need to classify functions using argument and result types

For (1): \(\Gamma ::= \cdot | \Gamma, x : \tau \) and \(\Gamma \vdash e : \tau\)

- Require whole program to type-check under empty context \(\cdot\)

For (2): \(\tau ::= \text{int} | \tau \rightarrow \tau\)

- An infinite number of types:
  \(\text{int} \rightarrow \text{int}, (\text{int} \rightarrow \text{int}) \rightarrow \text{int}, \text{int} \rightarrow (\text{int} \rightarrow \text{int})\)

Concrete syntax note: \(\rightarrow\) is right-associative, so \(\tau_1 \rightarrow \tau_2 \rightarrow \tau_3\) is \(\tau_1 \rightarrow (\tau_2 \rightarrow \tau_3)\)

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STLC Type System

\[ \tau ::= \text{int} | \tau \rightarrow \tau \]
\[ \Gamma ::= \cdot | \Gamma, x : \tau \]
\[ \Gamma \vdash e : \tau \]
\[ \Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2 \]

Where did \(\tau_1\) come from?

- Our rule “inferred” or “guessed” it

To be syntax directed, change \(\lambda x. e\) to \(\lambda x : \tau. e\) and use that \(\tau\)

Can think of “adding \(x\)” as shadowing or requiring \(x \notin \text{Dom}(\Gamma)\)

- Systematic renaming (\(\alpha\)-conversion) ensures \(x \notin \text{Dom}(\Gamma)\) is not a problem

The \textit{function-introduction} rule is the interesting one...
A closer look

\[ \Gamma, x : \tau_1 \vdash e : \tau_2 \]
\[ \Gamma \vdash \lambda x. e : \tau_1 \rightarrow \tau_2 \]

Is our type system too restrictive?
- That’s a matter of opinion
- But it does reject programs that don’t get stuck

Example: \((\lambda x. (x (\lambda y. y)) (x 3)) \lambda z. z\)
- Does not get stuck: Evaluates to 3
- Does not type-check:
  - There is no \(\tau_1, \tau_2\) such that \(x : \tau_1 \vdash (x (\lambda y. y)) (x 3) : \tau_2\)
    because you have to pick one type for \(x\)

How does STLC measure up?

So far, STLC is sound:
- As language dictators, we decided \(c v\) and undefined variables were “bad” meaning neither values nor reducible
- Our type system is a conservative checker that an expression will never get stuck

But STLC is far too restrictive:
- In practice, just too often that it prevents safe and natural code reuse
- More fundamentally, it’s not even Turing-complete
  - Turns out all (well-typed) programs terminate
  - A good-to-know and useful property, but inappropriate for a general-purpose PL
  - That’s okay: We will add more constructs and typing rules

Always restrictive

Whether or not a program “gets stuck” is undecidable:
- If \(e\) has no constants or free variables, then \(e (3 4)\) or \(e x\) gets stuck if and only if \(e\) terminates (cf. the halting problem)

Old conclusion: “Strong types for weak minds”
- Need a back door (unchecked casts)

Modern conclusion: Unsafe constructs almost never worth the risk
- Make “false positives” (rejecting safe program) rare enough
  - Have compile-time resources for “fancy” type systems
- Make workarounds for false positives convenient enough

Type Soundness

We will take a syntactic (operational) approach to soundness/safety
- The popular way since the early 1990s

Theorem (Type Safety): If \(\cdot \vdash e : \tau\) then \(e\) diverges or \(e \rightarrow^n v\)
for an \(n\) and \(v\) such that \(\cdot \vdash v : \tau\)
- That is, if \(\cdot \vdash e : \tau\), then \(e\) cannot get stuck

Proof: Next lecture