In class we sketched several proofs, but proof sketches invariably skip steps and have small errors. Here are the proofs more carefully laid out, as one might do on a homework assignment.

**Theorem:** $H ; e * 2 \downarrow c$ if and only if $H ; e + e \downarrow c$.

**Proof:** (Does not use induction)

- First assume $H ; e * 2 \downarrow c$ and show $H ; e + e \downarrow c$. Any derivation of $H ; e * 2 \downarrow c$ must end with the **MULT** rule, which means there must exist derivations of $H ; e \downarrow c'$ and $H ; 2 \downarrow 2$, and $c$ must be $2c'$. That is, there must be a derivation that looks like this:

  \[
  \vdots \\
  \frac{H ; e \downarrow c'}{H ; e * 2 \downarrow 2c'} \\
  \frac{H ; 2 \downarrow 2}{H ; e \downarrow c'}
  \]

  So given that there exists a derivation of $H ; e \downarrow c'$, we can use **ADD** to derive:

  \[
  \frac{H ; e \downarrow c'}{H ; e + e \downarrow c' + c'}
  \]

  Math provides $c' + c' = 2c'$, so the conclusion of this derivation is what we need.

- Now assume $H ; e + e \downarrow c$ and show $H ; e * 2 \downarrow c$. Any derivation of $H ; e + e \downarrow c$ must end with the **ADD** rule, which means there exists a derivation that looks like this (where $c = c_1 + c_2$):

  \[
  \vdots \\
  \frac{H ; e \downarrow c_1}{H ; e + e \downarrow c_1 + c_2} \\
  \frac{H ; e \downarrow c_2}{H ; e \downarrow c_1}
  \]

  In fact, we earlier proved determinacy (there is at most one $c$ such that $H ; e \downarrow c$), so the derivation must have this form (where $c = c_1 + c_1$):

  \[
  \vdots \\
  \frac{H ; e \downarrow c_1}{H ; e + e \downarrow c_1 + c_1} \\
  \frac{H ; e \downarrow c_1}{H ; e \downarrow c_1}
  \]

  So given that there exists a derivation of $H ; e \downarrow c_1$, we can use **MULT** to derive:

  \[
  \frac{H ; e \downarrow c_1}{H ; e * 2 \downarrow 2c_1}
  \]

  Math provides $c_1 + c_1 = 2c_1$, so the conclusion of this derivation is what we need.
Theorem: \( H ; C[e * 2] \Downarrow c \) if and only if \( H ; C[e + e] \Downarrow c \).

Proof: By induction on (the height of) the structure of \( C \):

- If the height is 0, then \( C \) is \([\cdot]\), so \( C[e * 2] = e * 2 \) and \( C[e + e] = e + e \). So the previous theorem is exactly what we need.

- If the height is greater than 0, then \( C \) has one of four forms:
  - If \( C \) is \( C' + e' \) for some \( C' \) and \( e' \), then \( C[e * 2] \) is \( C'[e * 2] + e' \) and \( C[e + e] \) is \( C'[e + e] + e' \). Since \( C' \) is shorter than \( C \), induction ensures that for any constant \( c' \), \( H ; C'[e * 2] \Downarrow c' \) if and only if \( H ; C'[e + e] \Downarrow c' \).

Assume \( H ; C'[e * 2] + e' \Downarrow c \) and show \( H ; C'[e + e] + e' \Downarrow c \): Any derivation of \( H ; C'[e * 2] + e' \Downarrow c \) must end with ADD, i.e., it looks like this (where \( c = c' + c'' \)):

\[
\vdots \\
H ; C'[e * 2] \Downarrow e' \quad H ; e' \Downarrow e'' \\
\hline
H ; C'[e + e] + e' \Downarrow c
\]

As argued above, the existence of a derivation of \( H ; C'[e * 2] \Downarrow c' \) ensures the existence of a derivation of \( H ; C'[e + e] \Downarrow c' \). So using ADD and the existence of a derivation of \( H ; e' \Downarrow e'' \), we can derive:

\[
\begin{align*}
H ; C'[e + e] \Downarrow c' & \quad H ; e' \Downarrow e'' \\
\hline
H ; C'[e + e] + e' \Downarrow c
\end{align*}
\]

Now assume \( H ; C'[e + e] + e' \Downarrow c \) and show \( H ; C'[e * 2] + e' \Downarrow c \): Any derivation of \( H ; C'[e + e] + e' \Downarrow c \) must end with ADD, i.e., it looks like this (where \( c = c' + c'' \)):

\[
\vdots \\
H ; C'[e + e] \Downarrow e' \quad H ; e' \Downarrow e'' \\
\hline
H ; C'[e + e] + e' \Downarrow c
\]

As argued above, the existence of a derivation of \( H ; C'[e + e] \Downarrow c' \) ensures the existence of a derivation of \( H ; C'[e * 2] \Downarrow c' \). So using ADD and the existence of a derivation of \( H ; e' \Downarrow e'' \), we can derive:

\[
\begin{align*}
H ; C'[e * 2] \Downarrow e' & \quad H ; e' \Downarrow e'' \\
\hline
H ; C'[e * 2] + e' \Downarrow c
\end{align*}
\]

- The other 3 cases are similar. (Try them out.)
Theorem: The two semantics below are equivalent, i.e., $H ; e \Downarrow c$ if and only if $H; e \rightarrow^* c$.

<table>
<thead>
<tr>
<th>CONST</th>
<th>VAR</th>
<th>ADD</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H; c \Downarrow c$</td>
<td>$H; x \Downarrow H(x)$</td>
<td>$H; e_1 \Downarrow c_1$ $H; e_2 \Downarrow c_2$ $H; e_1 + e_2 \Downarrow c_1 + c_2$</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>SVAR</th>
<th>SADD</th>
<th>SLEFT</th>
<th>SRIGHT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H; x \rightarrow H(x)$</td>
<td>$H; c_1 + e_2 \rightarrow c_1 + c_2$</td>
<td>$H; e_1 \rightarrow e'_1$ $H; e_1 + e_2 \rightarrow e'_1 + e_2$ $H; e_2 \rightarrow e'_2$ $H; e_1 + e_2 \rightarrow e'_1 + e'_2$</td>
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Proof: We prove the two directions separately.

First assume $H; e \Downarrow c$; show $\exists n. H; e \rightarrow^n c$. By induction on the height $h$ of derivation of $H; e \Downarrow c$:

- $h = 1$: Then the derivation must end with CONST or VAR. For CONST, $e$ is $c$ and trivially $H; e \rightarrow^0 c$. For VAR, $e$ is some $x$ where $H(x) = c$, so using SVAR, $H; e \rightarrow^1 c$.

- $h > 1$: Then the derivation must end with ADD, so $e$ is some $e_1 + e_2$ where $H; e_1 \Downarrow c_1$, $H; e_2 \Downarrow c_2$, and $c$ is $c_1 + c_2$. By induction $\exists n_1, n_2$. $H; e_1 \rightarrow^{n_1} c_1$ and $H; e_2 \rightarrow^{n_2} c_2$. Therefore, using the lemma below, $H; e_1 + e_2 \rightarrow^{n_1 + n_2} c_1 + c_2$, so ADD lets us derive $H; e_1 + e_2 \rightarrow^{n_1 + n_2 + 1} c$.

Lemma: If $H; e \rightarrow^n e'$, then $H; e_1 + e_2 \rightarrow^n e'_1 + e'_2$. By induction on $n$. If $n = 0$, the result is trivial because $e = e'$. If $n > 0$, then there exists some $e''$ such that $H; e \rightarrow^{n-1} e''$ and $H; e'' \rightarrow^1 e'$. So by induction $H; e_1 + e_2 \rightarrow^{n-1} c_1 + c_2$ and $H; e_2 \rightarrow^{n-1} c_2$. Using SRIGHT and SLEFT respectively, $H; e'' \rightarrow^1 e'$ ensures $H; e_1 + e'' \rightarrow^1 c_1 + e'$ and $H; e'' + e_2 \rightarrow^1 e' + e_2$. So with the inductive hypotheses, $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e_2 \rightarrow^n e' + e_2$.

Now assume $\exists n. H; e \rightarrow^n c$; show $H; e \Downarrow c$. By induction on $n$:

- $n = 0$: $e$ is $c$ and CONST lets us derive $H; c \Downarrow c$.

- $n > 0$: So $\exists e'$. $H; e \rightarrow e'$ and $H; e' \rightarrow^{n-1} c$. By induction $H; e' \Downarrow c$. So this lemma suffices: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$. Prove the lemma by induction on height $h$ of derivation of $H; e \rightarrow e'$:

  - $h = 1$: Then the derivation ends with SVAR or SADD. For SVAR, $e$ is some $x$ and $e' = H(x) = c$. So with VAR we can derive $H; x \Downarrow H(x)$, i.e., $H; e \Downarrow c$. For SADD, $e$ is some $c_1 + c_2$ and $e' = c = c_1 + c_2$. So with ADD, we can derive $H; c_1 + c_2 \Downarrow c_1 + c_2$, i.e., $H; e \Downarrow c$. (Note the $h = 1$ case may look a little weird because in fact in this case $n = 1$, i.e., $e'$ must be a constant.)

  - $h > 1$: Then the derivation ends with SLEFT or SRIGHT. For SLEFT, the assumed derivations end like this:

    $H; e_1 \rightarrow e'_1$ $H; e'_1 \Downarrow c_1$ $H; e_2 \Downarrow c_2$ $H; e'_1 + e_2 \Downarrow c_1 + c_2$ $H; e_1 + e_2 \rightarrow e'_1 + e_2$ $H; e_2 \rightarrow e'_2$ $H; e_1 \Downarrow c_1$ $H; e'_2 \Downarrow c_2$ $H; e_1 + e_2 \rightarrow e'_1 + e'_2$ $H; e_1 \Downarrow c_1$ $H; e_2 \Downarrow c_2$ $H; e_1 + e_2 \rightarrow e'_1 + e'_2$ $H; e_1 \Downarrow c_1$ $H; e_2 \Downarrow c_2$ $H; e_1 + e_2 \rightarrow e'_1 + e'_2$