Where we are

- Done: OCaml tutorial, “IMP” syntax, structural induction
- Now: Operational semantics for our little “IMP” language
  - Most of what you need for Homework 1
  - (But Problem 4 requires proofs over semantics)
IMP’s abstract syntax is defined inductively:

\[
\begin{align*}
    s & ::= \text{skip} \mid x := e \mid s; s \mid \text{if } e \text{ s s } \mid \text{while } e \text{ s} \\
    e & ::= c \mid x \mid e + e \mid e * e \\
     & (c \in \{\ldots, -2, -1, 0, 1, 2, \ldots \}) \\
     & (x \in \{x_1, x_2, \ldots, y_1, y_2, \ldots, z_1, z_2, \ldots, \ldots \})
\end{align*}
\]

We haven’t yet said what programs \textit{mean}! (Syntax is boring)

Encode our “social understanding” about variables and control flow
Outline

- Semantics for expressions
  1. Informal idea; the need for heaps
  2. Definition of heaps
  3. The evaluation judgment (a relation form)
  4. The evaluation inference rules (the relation definition)
  5. Using inference rules
    - Derivation trees as interpreters
    - Or as proofs about expressions
  6. Metatheory: Proofs about the semantics

- Then semantics for statements
  - ...
Informal idea

Given $e$, what $c$ does $e$ evaluate to?

$$1 + 2 \quad \quad x + 2$$
Informal idea

Given \( e \), what \( c \) does \( e \) evaluate to?

\[
1 + 2 \quad \quad \quad x + 2
\]

It depends on the values of variables (of course)

Use a heap \( H \) for a total function from variables to constants
  - Could use partial functions, but then \( \exists H \) and \( e \) for which there is no \( c \)

We’ll define a relation over triples of \( H, e, \) and \( c \)
  - Will turn out to be function if we view \( H \) and \( e \) as inputs and \( c \) as output
  - With our metalanguage, easier to define a relation and then prove it is a function (if, in fact, it is)
Heaps

\[ H ::= \cdot \mid H, x \mapsto c \]

A lookup-function for heaps:

\[
H(x) = \begin{cases} 
  c & \text{if } H = H', x \mapsto c \\
  H'(x) & \text{if } H = H', y \mapsto c' \text{ and } y \neq x \\
  0 & \text{if } H = \cdot .
\end{cases}
\]

- Last case avoids “errors” (makes function total)

“What heap to use” will arise in the semantics of statements

- For expression evaluation, “we are given an H”
The judgment

We will write: \( H ; e \downarrow c \)

to mean, “\( e \) evaluates to \( c \) under heap \( H \)”

It is just a relation on triples of the form \((H, e, c)\)

We just made up metasyntax \( H ; e \downarrow c \) to follow PL convention and to distinguish it from other relations

We can write: \(. , x \mapsto 3 ; x + y \downarrow 3\), which will turn out to be true

(this triple will be in the relation we define)

Or: \(. , x \mapsto 3 ; x + y \downarrow 6\), which will turn out to be false

(this triple will not be in the relation we define)
Inference rules

**CONST**

\[
H ; c \\downarrow c
\]

**VAR**

\[
H ; x \\downarrow H(x)
\]

**ADD**

\[
\frac{H ; e_1 \\downarrow c_1 \quad H ; e_2 \\downarrow c_2}{H ; e_1 + e_2 \\downarrow c_1 + c_2}
\]

**MULT**

\[
\frac{H ; e_1 \\downarrow c_1 \quad H ; e_2 \\downarrow c_2}{H ; e_1 * e_2 \\downarrow c_1 * c_2}
\]

Top: *hypotheses*
Bottom: *conclusion* (read first)

By definition, if all hypotheses hold, then the conclusion holds

Each rule is a *schema* you “instantiate consistently”

- So rules “work” “for all” \( H, c, e_1, \) etc.
- But “each” \( e_1 \) has to be the “same” expression
Instantiating rules

Example instantiation:

\[ \cdot, y \mapsto 4 ; 3 + y \Downarrow 7 \quad \cdot, y \mapsto 4 ; 5 \Downarrow 5 \]

\[ \cdot, y \mapsto 4 ; (3 + y) + 5 \Downarrow 12 \]

Instantiates:

\[
\begin{align*}
\text{ADD} \\
H ; e_1 \Downarrow c_1 & \quad H ; e_2 \Downarrow c_2 \\
\hline
H ; e_1 + e_2 \Downarrow c_1 + c_2
\end{align*}
\]

with

\[ H = \cdot, y \mapsto 4 \]
\[ e_1 = (3 + y) \]
\[ c_1 = 7 \]
\[ e_2 = 5 \]
\[ c_2 = 5 \]
Derivations

A \textit{(complete) derivation} is a tree of instantiations with \textit{axioms} at the leaves.

Example:

\[
\begin{align*}
\cdot, y \mapsto 4 & ; 3 \downarrow 3 \\
\cdot, y \mapsto 4 & ; 3 + y \downarrow 7 \\
\cdot, y \mapsto 4 & ; (3 + y) + 5 \downarrow 12
\end{align*}
\]

By definition, \( H ; e \downarrow c \) if there exists a derivation with \( H ; e \downarrow c \) at the root.
Back to relations

So what relation do our inference rules define?

- Start with empty relation (no triples) $R_0$

- Let $R_i$ be $R_{i-1}$ union all $H; e \downarrow c$ such that we can instantiate some inference rule to have conclusion $H; e \downarrow c$ and all hypotheses in $R_{i-1}$
  - So $R_i$ is all triples at the bottom of height-$j$ complete derivations for $j \leq i$

- $R_\infty$ is the relation we defined
  - All triples at the bottom of complete derivations

For the math folks: $R_\infty$ is the smallest relation closed under the inference rules
What are these things?

We can view the inference rules as defining an *interpreter*

- Complete derivation shows recursive calls to the “evaluate expression” function
  - Recursive calls from conclusion to hypotheses
  - *Syntax-directed* means the interpreter need not “search”

- See OCaml code in Homework 1

Or we can view the inference rules as defining a *proof system*

- Complete derivation proves facts from other facts starting with axioms
  - Facts established from hypotheses to conclusions
Some theorems

▶ Progress: For all $H$ and $e$, there exists a $c$ such that $H ; e \downarrow c$

▶ Determinacy: For all $H$ and $e$, there is at most one $c$ such that $H ; e \downarrow c$

We rigged it that way...
what would division, undefined-variables, or gettime() do?

Proofs are by induction on the the structure (i.e., height) of the expression $e$
On to statements

A statement does not produce a constant
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A statement does not produce a constant

It produces a new, possibly-different heap.

- If it terminates
On to statements

A statement does not produce a constant
It produces a new, possibly-different heap.
  ▶ If it terminates

We could define $H_1; s \downarrow H_2$
  ▶ Would be a partial function from $H_1$ and $s$ to $H_2$
  ▶ Works fine; could be a homework problem
On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.
  ▶ If it terminates

We could define $H_1 \Downarrow s \downarrow H_2$
  ▶ Would be a partial function from $H_1$ and $s$ to $H_2$
  ▶ Works fine; could be a homework problem

Instead we’ll define a “small-step” semantics and then “iterate” to “run the program”

\[ H_1 \downarrow s_1 \rightarrow H_2 \downarrow s_2 \]
Statement semantics

$$H_1 ; s_1 \rightarrow H_2 ; s_2$$

**ASSIGN**

$$\quad H \ ; \ e \Downarrow c \quad H \ ; \ x := e \rightarrow H, x \mapsto c \ ; \ \text{skip}$$

**SEQ1**

$$\frac{}{H \ ; \ \text{skip}; s \rightarrow H ; s}$$

**SEQ2**

$$\frac{H \ ; \ s_1 \rightarrow H' ; s_1'}{H \ ; \ s_1; s_2 \rightarrow H' ; s_1'; s_2}$$

**IF1**

$$\frac{H \ ; \ e \Downarrow c \quad c > 0}{H \ ; \ \text{if} \ e \ s_1 \ s_2 \rightarrow H ; s_1}$$

**IF2**

$$\frac{H \ ; \ e \Downarrow c \quad c \leq 0}{H \ ; \ \text{if} \ e \ s_1 \ s_2 \rightarrow H ; s_2}$$
What about \textbf{while} $e$ \textbf{s} (do \textbf{s} and loop if $e > 0$)?
What about `while e s` (do `s` and loop if `e > 0`)?

\[
\text{WHILE} \\
H ; \text{while } e \gets H ; \text{if } e \ (s ; \text{while } e \ s) \ \text{skip}
\]

Many other equivalent definitions possible
Program semantics

Defined $H ; s \rightarrow H' ; s'$, but what does “s” mean/do?

Our machine iterates: $H_1 ; s_1 \rightarrow H_2 ; s_2 \rightarrow H_3 ; s_3 \ldots$, 
with each step justified by a complete derivation using our single-step statement semantics

Let $H_1 ; s_1 \rightarrow^n H_2 ; s_2$ mean “becomes after n steps”

Let $H_1 ; s_1 \rightarrow^* H_2 ; s_2$ mean “becomes after 0 or more steps”

Pick a special “answer” variable ans

The program s produces c if · ; s \rightarrow^* H ; skip and $H(\text{ans}) = c$

Does every s produce a c?
Example program execution

\[
\text{x := 3; (y := 1; while x (y := y * x; x := x - 1))}
\]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y * x; x := x-1) \).
Example program execution

```
x := 3; (y := 1; while x (y := y * x; x := x−1))
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Let’s write some of the state sequence. You can justify each step with a full derivation. Let $s = (y := y * x; x := x−1)$.

```
.; x := 3; y := 1; while x s
```
Example program execution

\[ x := 3; (y := 1; \textbf{while } x (y := y \times x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \times x; x := x - 1) \).

\[
\cdot; x := 3; y := 1; \textbf{while } x \ s \\
\rightarrow \cdot, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while } x \ s
\]
Example program execution

\[ x := 3; (y := 1; \textbf{while} x \ (y := y \times x; x := x - 1)) \]

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\[
\begin{align*}
\cdot; x := 3; y := 1; \textbf{while} & \ x \ s \\
\rightarrow & \ , x \mapsto 3; \textbf{skip}; y := 1; \textbf{while} \ x \ s \\
\rightarrow & \ , x \mapsto 3; y := 1; \textbf{while} \ x \ s
\end{align*}
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\]
\[
\rightarrow \cdot, x \mapsto 3; \textbf{skip}; \ y := 1; \textbf{while} \ x \ s
\]
\[
\rightarrow \cdot, x \mapsto 3; \ y := 1; \textbf{while} \ x \ s
\]
\[
\rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1; \textbf{while} \ x \ s
\]
Example program execution

\[ x := 3; (y := 1; \textbf{while} x (y := y \ast x; x := x - 1)) \]

Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \ast x; x := x - 1) \).

\[ \cdot; x := 3; y := 1; \textbf{while} x s \]

\[ \rightarrow \ ·, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while} x s \]

\[ \rightarrow \ ·, x \mapsto 3; y := 1; \textbf{while} x s \]

\[ \rightarrow^2 \ ·, x \mapsto 3, y \mapsto 1; \textbf{while} x s \]

\[ \rightarrow \ ·, x \mapsto 3, y \mapsto 1; \textbf{if} x (s; \textbf{while} x s) \textbf{skip} \]
Example program execution

\[
x := 3; (y := 1; \textbf{while } x \ (y := y \times x; x := x - 1))
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Let’s write some of the state sequence. You can justify each step with a full derivation. Let \( s = (y := y \times x; x := x - 1) \).

\[
\cdot; x := 3; y := 1; \textbf{while } x \ s
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\[
\rightarrow \cdot, x \mapsto 3; \textbf{skip}; y := 1; \textbf{while } x \ s
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\[
\rightarrow \cdot, x \mapsto 3; y := 1; \textbf{while } x \ s
\]

\[
\rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1; \textbf{while } x \ s
\]

\[
\rightarrow \cdot, x \mapsto 3, y \mapsto 1; \textbf{if } x \ (s; \textbf{while } x \ s) \ \textbf{skip}
\]

\[
\rightarrow \cdot, x \mapsto 3, y \mapsto 1; y := y \times x; x := x - 1; \textbf{while } x \ s
\]
Continued...

\[ \rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; x := x-1; \textbf{while} \ x \ s \]
Continued...

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; x := x-1; \textbf{while} \ x \ s \]

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \textbf{while} \ x \ s \]
Continued...

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; \ x := x-1; \ \textbf{while} \ x \ s \]

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \ \textbf{while} \ x \ s \]

\[ \rightarrow \quad \ldots, y \mapsto 3, x \mapsto 2; \ \textbf{if} \ x \ (s; \ \textbf{while} \ x \ s) \ \textbf{skip} \]
Continued...

\[\rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; x := x - 1; \textbf{while} \ x \ s\]

\[\rightarrow^2 \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \textbf{while} \ x \ s\]

\[\rightarrow \ldots, y \mapsto 3, x \mapsto 2; \textbf{if} \ x (s; \textbf{while} \ x \ s) \textbf{skip}\]

\[\ldots\]
Continued...

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3; x := x - 1; \textbf{while} x \ s \]

\[ \rightarrow^2 \quad \cdot, x \mapsto 3, y \mapsto 1, y \mapsto 3, x \mapsto 2; \textbf{while} x \ s \]

\[ \rightarrow \quad \ldots, y \mapsto 3, x \mapsto 2; \textbf{if} \quad \textbf{(} s; \textbf{while} x \ s \textbf{)} \textbf{skip} \]

\[ \ldots \]

\[ \rightarrow \quad \ldots, y \mapsto 6, x \mapsto 0; \textbf{skip} \]
Where we are

Defined $H; e \downarrow c$ and $H; s \rightarrow H'; s'$ and extended the latter to give $s$ a meaning

- The way we did expressions is “large-step operational semantics”
- The way we did statements is “small-step operational semantics”
- So now you have seen both

Definition by interpretation: program means what an interpreter (written in a metalanguage) says it means

- Interpreter represents a (very) abstract machine that runs code

Large-step does not distinguish errors and divergence

- But we defined IMP to have no errors
- And expressions never diverge
Establishing Properties

We can prove a property of a terminating program by “running” it.

Example: Our last program terminates with $x$ holding 0.
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We can prove a property of a terminating program by “running” it

Example: Our last program terminates with \( x \) holding 0

We can prove a program diverges, i.e., for all \( H \) and \( n \),
\[ \cdot ; s \rightarrow^n H ; \text{skip} \]
cannot be derived

Example: \texttt{while 1 skip}
Establishing Properties

We can prove a property of a terminating program by “running” it.

Example: Our last program terminates with $x$ holding 0.

We can prove a program diverges, i.e., for all $H$ and $n$,

$\cdot ; s \rightarrow^n H ; \text{skip}$ cannot be derived.

Example: while 1 skip

By induction on $n$, but requires a stronger induction hypothesis.
More General Proofs

We can prove properties of executing all programs (satisfying another property)

Example: If $H$ and $s$ have no negative constants and $H ; s \rightarrow^* H' ; s'$, then $H'$ and $s'$ have no negative constants.

Example: If for all $H$, we know $s_1$ and $s_2$ terminate, then for all $H$, we know $H ; (s_1 ; s_2)$ terminates.