CSE-505: Programming Languages

Lecture 3 — Operational Semantics

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2016

Where we are

- Done: OCaml tutorial, "IMP" syntax, structural induction
- Now: Operational semantics for our little "IMP" language
  - Most of what you need for Homework 1
  - (But Problem 4 requires proofs over semantics)

Review

IMP's abstract syntax is defined inductively:

$$
\begin{array}{rcl}
s & ::= & \textbf{skip} \mid x := e \mid s; s \mid \textbf{if } e\ s\ s \mid \textbf{while } e\ s \\
e & ::= & c \mid x \mid e + e \mid e * e \\
(c & \in & \{\ldots, -2, -1, 0, 1, 2, \ldots\}) \\
(x & \in & \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{z}_1, \mathrm{z}_2, \ldots, \ldots\})
\end{array}
$$

We haven't yet said what programs *mean*! (Syntax is boring)

Encode our "social understanding" about variables and control flow

Outline

- Semantics for expressions
  1. Informal idea; the need for *heaps*
  2. Definition of heaps
  3. The evaluation *judgment* (a relation form)
  4. The evaluation *inference rules* (the relation definition)
  5. Using inference rules
     - *Derivation trees* as interpreters
     - Or as *proofs* about expressions
  6. *Metatheory*: Proofs about the semantics
- Then semantics for statements
  - ...

Informal idea

Given e, what c does e evaluate to?

$$1 + 2 \qquad\qquad \mathrm{x} + 2$$

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It depends on the values of variables (of course)

Use a heap H for a total function from variables to constants

- Could use partial functions, but then $\exists\ H$ and e for which there is no c

We'll define a *relation* over triples of H, e, and c

- Will turn out to be *function* if we view H and e as inputs and c as output
- With our metalanguage, easier to define a relation and then prove it is a function (if, in fact, it is)

Heaps

$$H ::= \cdot \mid H, x \mapsto c$$

A lookup-function for heaps:

$$H(x) = \begin{cases} c & \text{if} \quad H = H', x \mapsto c \\ H'(x) & \text{if} \quad H = H', y \mapsto c' \text{ and } y \neq x \\ 0 & \text{if} \quad H = \cdot \end{cases}$$

- Last case avoids "errors" (makes function *total*)

"What heap to use" will arise in the semantics of statements

- For expression evaluation, "we are given an H"

The judgment

We will write: $\boxed{H \; ; e \Downarrow c}$

to mean, "e evaluates to c under heap H"

It is just a relation on triples of the form (H, e, c)

We just made up metasyntax $H \; ; e \Downarrow c$ to follow PL convention and to distinguish it from other relations

We can write: $., x \mapsto 3 \; ; x + y \Downarrow 3$, which will turn out to be *true*
(this triple will be in the relation we define)

Or: $., x \mapsto 3 \; ; x + y \Downarrow 6$, which will turn out to be *false*
(this triple will not be in the relation we define)

Inference rules

$$\frac{}{H \,;\, c \Downarrow c}\ \textsc{const} \qquad \frac{}{H \,;\, x \Downarrow H(x)}\ \textsc{var}$$

$$\frac{H \,;\, e_1 \Downarrow c_1 \qquad H \,;\, e_2 \Downarrow c_2}{H \,;\, e_1 + e_2 \Downarrow c_1 + c_2}\ \textsc{add} \qquad \frac{H \,;\, e_1 \Downarrow c_1 \qquad H \,;\, e_2 \Downarrow c_2}{H \,;\, e_1 * e_2 \Downarrow c_1 * c_2}\ \textsc{mult}$$

Top: *hypotheses*
Bottom: *conclusion* (read first)

By definition, if all hypotheses hold, then the conclusion holds

Each rule is a *schema* you "instantiate consistently"

- So rules "work" "for all" H, c, e_1, etc.
- But "each" e_1 has to be the "same" expression

Instantiating rules

Example instantiation:

$$\frac{\cdot, \mathrm{y} \mapsto 4 \,;\, 3 + \mathrm{y} \Downarrow 7 \qquad \cdot, \mathrm{y} \mapsto 4 \,;\, 5 \Downarrow 5}{\cdot, \mathrm{y} \mapsto 4 \,;\, (3 + \mathrm{y}) + 5 \Downarrow 12}$$

Instantiates:

$$\frac{H \,;\, e_1 \Downarrow c_1 \qquad H \,;\, e_2 \Downarrow c_2}{H \,;\, e_1 + e_2 \Downarrow c_1 + c_2}\ \textsc{add}$$

with
$H = \cdot, \mathrm{y} \mapsto 4$
$e_1 = (3 + \mathrm{y})$
$c_1 = 7$
$e_2 = 5$
$c_2 = 5$

Derivations

A *(complete) derivation* is a tree of instantiations with *axioms* at the leaves

Example:

$$\frac{\dfrac{\dfrac{}{\cdot, \mathrm{y}\mapsto 4 \,;\, 3 \Downarrow 3} \qquad \dfrac{}{\cdot, \mathrm{y}\mapsto 4 \,;\, \mathrm{y} \Downarrow 4}}{\cdot, \mathrm{y}\mapsto 4 \,;\, 3 + \mathrm{y} \Downarrow 7} \qquad \dfrac{}{\cdot, \mathrm{y}\mapsto 4 \,;\, 5 \Downarrow 5}}{\cdot, \mathrm{y}\mapsto 4 \,;\, (3 + \mathrm{y}) + 5 \Downarrow 12}$$

By definition, $H \,;\, e \Downarrow c$ if there exists a derivation with $H \,;\, e \Downarrow c$ at the root

Back to relations

So what relation do our inference rules define?

- Start with empty relation (no triples) R_0
- Let R_i be R_{i-1} union all $H \,;\, e \Downarrow c$ such that we can instantiate some inference rule to have conclusion $H \,;\, e \Downarrow c$ and all hypotheses in R_{i-1}
  - So R_i is all triples at the bottom of height-j complete derivations for $j \leq i$
- R_∞ is the relation we defined
  - All triples at the bottom of complete derivations

For the math folks: R_∞ is the smallest relation closed under the inference rules

What are these things?

We can view the inference rules as defining an *interpreter*

- Complete derivation shows recursive calls to the "evaluate expression" function
  - Recursive calls from conclusion to hypotheses
  - *Syntax-directed* means the interpreter need not "search"
- See OCaml code in Homework 1

Or we can view the inference rules as defining a *proof system*

- Complete derivation proves facts from other facts starting with axioms
  - Facts established from hypotheses to conclusions

Some theorems

- Progress: For all H and e, there exists a c such that $H\ ;\ e \Downarrow c$
- Determinacy: For all H and e, there is at most one c such that $H\ ;\ e \Downarrow c$

We rigged it that way...
what would division, undefined-variables, or `gettime()` do?

Proofs are by induction on the the structure (i.e., height) of the expression e

On to statements

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- If it terminates

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We could define $H_1 \; ; \; s \Downarrow H_2$

- Would be a partial function from H_1 and s to H_2
- Works fine; could be a homework problem

On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.

- If it terminates

We could define $H_1 \; ; \; s \Downarrow H_2$

- Would be a partial function from H_1 and s to H_2
- Works fine; could be a homework problem

Instead we'll define a "small-step" semantics and then "iterate" to "run the program"

$$\boxed{H_1 \; ; \; s_1 \to H_2 \; ; \; s_2}$$

Statement semantics

$$\boxed{H_1 \; ; \; s_1 \to H_2 \; ; \; s_2}$$

$$\text{ASSIGN}\quad \frac{H \; ; \; e \Downarrow c}{H \; ; \; x := e \to H, x \mapsto c \; ; \; \textsf{skip}}$$

$$\text{SEQ1}\quad \frac{}{H \; ; \; \textsf{skip}; s \to H \; ; \; s} \qquad \text{SEQ2}\quad \frac{H \; ; \; s_1 \to H' \; ; \; s_1'}{H \; ; \; s_1; s_2 \to H' \; ; \; s_1'; s_2}$$

$$\text{IF1}\quad \frac{H \; ; \; e \Downarrow c \qquad c > 0}{H \; ; \; \textsf{if}\ e\ s_1\ s_2 \to H \; ; \; s_1} \qquad \text{IF2}\quad \frac{H \; ; \; e \Downarrow c \qquad c \leq 0}{H \; ; \; \textsf{if}\ e\ s_1\ s_2 \to H \; ; \; s_2}$$

Statement semantics cont'd

What about **while** e s (do s and loop if $e > 0$)?

Statement semantics cont'd

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$$\frac{\text{WHILE}}{H\ ;\ \textbf{while}\ e\ s \to H\ ;\ \textbf{if}\ e\ (s; \textbf{while}\ e\ s)\ \textbf{skip}}$$

Many other equivalent definitions possible

Program semantics

Defined $H\ ;\ s \to H'\ ;\ s'$, but what does "s" mean/do?

Our machine iterates: $H_1;s_1 \to H_2;s_2 \to H_3;s_3 \ldots,$
with each step justified by a complete derivation using our single-step statement semantics

Let $H_1\ ;\ s_1 \ \to^n\ H_2\ ;\ s_2$ mean "becomes after n steps"

Let $H_1\ ;\ s_1 \to^* H_2\ ;\ s_2$ mean "becomes after 0 or more steps"

Pick a special "answer" variable $\texttt{ans}$

The program s produces c if $\cdot\ ;\ s \to^* H\ ;\ \textbf{skip}$ and $H(\texttt{ans}) = c$

Does every s produce a c?

Example program execution

$\texttt{x} := 3; (\texttt{y} := 1; \textbf{while}\ \texttt{x}\ (\texttt{y} := \texttt{y} * \texttt{x}; \texttt{x} := \texttt{x}{-}1))$

Let's write some of the state sequence. You can justify each step with a full derivation. Let $s = (\texttt{y} := \texttt{y} * \texttt{x}; \texttt{x} := \texttt{x}{-}1)$.

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$$\rightarrow \quad \cdot, \mathrm{x} \mapsto \mathbf{3}, \mathrm{y} \mapsto \mathbf{1}; \textbf{if } \mathrm{x}\ (s; \textbf{while } \mathrm{x}\ s)\ \textbf{skip}$$

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 & \cdot; \mathtt{x} := \mathbf{3}; \mathtt{y} := \mathbf{1}; \textsf{while}\ \mathtt{x}\ s \\
\to & \cdot, \mathtt{x} \mapsto \mathbf{3}; \textsf{skip}; \mathtt{y} := \mathbf{1}; \textsf{while}\ \mathtt{x}\ s \\
\to & \cdot, \mathtt{x} \mapsto \mathbf{3}; \mathtt{y} := \mathbf{1}; \textsf{while}\ \mathtt{x}\ s \\
\to^2 & \cdot, \mathtt{x} \mapsto \mathbf{3}, \mathtt{y} \mapsto \mathbf{1}; \textsf{while}\ \mathtt{x}\ s \\
\to & \cdot, \mathtt{x} \mapsto \mathbf{3}, \mathtt{y} \mapsto \mathbf{1}; \textsf{if}\ \mathtt{x}\ (s; \textsf{while}\ \mathtt{x}\ s)\ \textsf{skip} \\
\to & \cdot, \mathtt{x} \mapsto \mathbf{3}, \mathtt{y} \mapsto \mathbf{1}; \mathtt{y} := \mathtt{y} * \mathtt{x}; \mathtt{x} := \mathtt{x} - \mathbf{1}; \textsf{while}\ \mathtt{x}\ s
\end{array}
$$

Continued...

$$
\to^2 \quad \cdot, \mathtt{x} \mapsto \mathbf{3}, \mathtt{y} \mapsto \mathbf{1}, \mathtt{y} \mapsto \mathbf{3}; \mathtt{x} := \mathtt{x} - \mathbf{1}; \textsf{while}\ \mathtt{x}\ s
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Continued...

$$
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$\rightarrow^2 \quad \cdot, \mathrm{x} \mapsto \mathbf{3}, \mathrm{y} \mapsto \mathbf{1}, \mathrm{y} \mapsto \mathbf{3};\ \mathrm{x} := \mathrm{x}-\mathbf{1}; \textbf{while}\ \mathrm{x}\ s$

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Continued...

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$\ldots$

$\rightarrow \quad \ldots, \mathrm{y} \mapsto \mathbf{6}, \mathrm{x} \mapsto \mathbf{0};\ \textbf{skip}$

Where we are

Defined $H\ ;\ e \Downarrow c$ and $H\ ;\ s \rightarrow H'\ ;\ s'$ and extended the latter to give s a meaning

- The way we did expressions is "large-step operational semantics"
- The way we did statements is "small-step operational semantics"
- So now you have seen both

Definition by interpretation: program means what an interpreter (written in a metalanguage) says it means

- Interpreter represents a (very) abstract machine that runs code

Large-step does not distinguish errors and divergence

- But we defined IMP to have no errors
- And expressions never diverge

Establishing Properties

We can prove a property of a terminating program by "running" it

Example: Our last program terminates with x holding $\mathbf{0}$

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Example: **while** $\mathbf{1}$ **skip**

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Example: **while** $\mathbf{1}$ **skip**

By induction on n, but requires a *stronger induction hypothesis*

More General Proofs

We can prove properties of executing all programs (satisfying another property)

Example: If H and s have no negative constants and $H\ ;\ s \rightarrow^{*} H'\ ;\ s'$, then H' and s' have no negative constants.

Example: If for all H, we know s_1 and s_2 terminate, then for all H, we know $H;(s_1; s_2)$ terminates.