CSE 505: Programming Languages

Lecture 15 — Parametric Polymorphism

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Last Time

Saw structural subtyping

- constraints over record fields
- propagate constraints to "bigger" types
- covariance, contravariance

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- propagate constraints to "bigger" types
- covariance, contravariance

Provided polymorphism over records with "enough" fields ...
but **at fixed types**.

What if code imposes *no constraints* on some types?

This Time: Parametric Polymorphism

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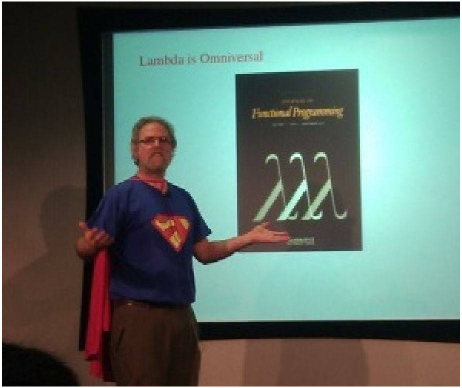
Before we figure out what that means, a word from a luminary:



Goal: Everybody Wins!

Understand what this interface means and why it matters:

```
empty   : forall a, mylist a
cons    : forall a, a -> mylist a -> mylist a
decons  : forall a, mylist a -> option (a * mylist a)
length  : forall a, mylist a -> nat
map     : forall a b, (a -> b) -> mylist a -> mylist b
```

From two perspectives:

1. Client: Code against this specification
2. Library: Implement this specification

Goal: Client Wins!

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   - Different lists with elements of different types
   - New reusable functions outside of library, e.g.:
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   - `a` and `b` held abstract from library
   - e.g., suppose `foo:  mylist a -> int`, then
     `foo [1;2;3]` totally equivalent to `foo [(5,4);(7,2);(9,2)]`

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   - Why? Still true if we have downcasts?
   - Proof left as exercise to the reader
   - In theory, means less (re-)integration testing

Goal: Library Wins!

1. Reusability — all the same reasons client likes it
2. Abstraction of `mylist` from clients
   - Clients can only assume interface, no implementation details
   - Free to change/optimize hidden details of `mylist a`
   - Clients typechecked knowing only:
     there exists some type constructor `mylist`
   - Unlike Java/C++ cannot downcast a `t mylist` to, e.g., a pair

Start Simple

The `mylist` interface has a lot going on:

1. Element types *held abstract* from library
2. List type (constructor) *held abstract* from client
3. Reuse of type variables constrains expressions over abstract types
4. Lists need some form of recursive type

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- First using a formal language with explicit type abstraction
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Note: Much more interesting than "not getting stuck"

Recipe for Extension

1. Add syntax
2. Add semantics
3. Add typing rules
4. Patch up type safety proof

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Summary of new things:

- Terms: Type abstraction and type application
- Types: Type variables and universal types
- Type contexts: what type variables are in scope

2. Add Semantics

What is this $\boldsymbol{\Lambda}$ (big lambda) thing? Informally:

1. $\boldsymbol{\Lambda\alpha.\ e}$: a value that takes some $\boldsymbol{\tau}$, plugs it in for $\boldsymbol{\alpha}$, then runs $\boldsymbol{e}$
   - type-check $\boldsymbol{e}$ knowing $\boldsymbol{\alpha}$ is *some* type, but not *which* type

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What is this $\forall$ (upside down "A") thing? Informally:

Types can use type variables α, β, etc., but only if they're *in scope* (just like term variables)

- Type-checking $\Delta; \Gamma \vdash e : \tau$ uses Δ to scope type vars in e
- universal type $\forall\alpha.\tau$, brings α into scope for τ

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Formal, small-step, CBV, left-to-right operational semantics:

- Recall: $\Lambda\alpha.\ e$ is a value

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Old: $\dfrac{e_1 \rightarrow e_1'}{e_1\ e_2 \rightarrow e_1'\ e_2}$ $\quad \dfrac{e_2 \rightarrow e_2'}{v\ e_2 \rightarrow v\ e_2'}$ $\quad \dfrac{}{(\lambda x{:}\tau.\ e)\ v \rightarrow e[v/x]}$

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Plus now have 3 different kinds of substitution, all defined in straightforward capture-avoiding way:

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Example

Example (using addition):

$$(\Lambda\alpha.\ \Lambda\beta.\ \lambda x:\alpha.\ \lambda f{:}\alpha \to \beta.\ f\ x)\ [\mathsf{int}]\ [\mathsf{int}]\ 3\ (\lambda y:\mathsf{int}.\ y + y)$$

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$$\to\ 6$$

3. Add Typing Rules

Need to be picky about "no free type variables"

- Typing judgment has the form $\Delta; \Gamma \vdash e : \tau$
  (whole program $\cdot; \cdot \vdash e : \tau$)
- Uses helper judgment $\Delta \vdash \tau$
  - "all *free* type variables in τ are in Δ"

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Rules are boring, but smart people found out the hard way that allowing free type variables is a pernicious source of language/compiler bugs.

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Old (with one technical change to prevent free type variables):

$$\frac{}{\Delta; \Gamma \vdash x : \Gamma(x)} \qquad \frac{}{\Delta; \Gamma \vdash c : \mathsf{int}}$$

$$\frac{\Delta; \Gamma, x{:}\tau_1 \vdash e : \tau_2 \qquad \Delta \vdash \tau_1}{\Delta; \Gamma \vdash \lambda x{:}\tau_1.\ e : \tau_1 \to \tau_2}$$

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New:

$$\frac{\Delta, \alpha;\Gamma \vdash e : \tau_1}{\Delta;\Gamma \vdash \Lambda\alpha.\ e : \forall\alpha.\tau_1}$$

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Ouch.

Just a syntax-directed derivation by instantiating the typing rules.
Still, machines are better suited to this stuff.

System F *(Tah Dah!)*

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$$

$$
\frac{e \to e'}{e\ e_2 \to e'\ e_2} \qquad \frac{e \to e'}{v\ e \to v\ e'} \qquad \frac{e \to e'}{e[\tau] \to e'[\tau]}
$$

$$
\frac{}{(\lambda x{:}\tau.\ e)\ v \to e[v/x]} \qquad \frac{}{(\Lambda\alpha.\ e)[\tau] \to e[\tau/\alpha]}
$$

$$
\frac{}{\Delta;\Gamma \vdash x : \Gamma(x)} \qquad \frac{}{\Delta;\Gamma \vdash c : \mathsf{int}}
$$

$$
\frac{\Delta;\Gamma, x{:}\tau_1 \vdash e : \tau_2 \qquad \Delta \vdash \tau_1}{\Delta;\Gamma \vdash \lambda x{:}\tau_1.\ e : \tau_1 \to \tau_2} \qquad \frac{\Delta, \alpha;\Gamma \vdash e : \tau_1}{\Delta;\Gamma \vdash \Lambda\alpha.\ e : \forall\alpha.\tau_1}
$$

$$
\frac{\Delta;\Gamma \vdash e_1 : \tau_2 \to \tau_1 \qquad \Delta;\Gamma \vdash e_2 : \tau_2}{\Delta;\Gamma \vdash e_1\ e_2 : \tau_1} \qquad \frac{\Delta;\Gamma \vdash e : \forall\alpha.\tau_1 \qquad \Delta \vdash \tau_2}{\Delta;\Gamma \vdash e[\tau_2] : \tau_1[\tau_2/\alpha]}
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- id $[\textbf{int}]$ has type $\textbf{int} \rightarrow \textbf{int}$
- id $[\textbf{int} * \textbf{int}]$ has type $(\textbf{int} * \textbf{int}) \rightarrow (\textbf{int} * \textbf{int})$

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In ML you can't do the last one! What?!

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- Could this be any more polymorphic?

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Note: Mutation breaks everything :(

What next?

Now that we have System F...

- What hath we wrought? Example of our mighty new powers.
- How/why ML is more restrictive and implicit.

Security from safety?

Example: A process e should not access files it did not open (fopen checks permissions)

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Require an untrusted process e to type-check as follows:
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This type ensures that a process won't "forge a file handle" and pass it to `fread`

So `fread` doesn't need to check (faster), file handles don't need to be encrypted (safer), etc.

Moral of Example

In STLC, type safety just meant not getting stuck

Type abstraction gives us new powers, e.g. secure interfaces!

Suppose we (the system library) implement file-handles as ints.
Then we instantiate α with **int**, but untrusted code *cannot tell*

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Memory safety is a necessary but insufficient condition for language-based *enforcement of strong abstractions*

Are types used at run-time?

We said polymorphism was about "many types for same term", but for clarity and easy checking, we changed:

- The syntax via $\Lambda\alpha.\ e$ and $e\ [\tau]$
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Claim: The operational semantics did not "really" change; types need not exist at run-time

More formally: *Erasing* all types from System F produces an equivalent program in the untyped lambda calculus

Strengthened induction hypothesis: If $e \rightarrow e_1$ in System F and $erase(e) \rightarrow e_2$ in untyped lambda-calculus, then $e_2 = erase(e_1)$

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"Erasure and evaluation commute"

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Erasure is easy to define:

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In pure System F, preserving evaluation order isn't crucial, but it is with fix, exceptions, mutation, etc.

Connection to reality... or at least ML

System F has been one of the most important theoretical PL models since the 1970s and inspires languages like ML.

But you have seen ML polymorphism and it looks different. In fact, it is an implicitly typed restriction of System F.

These two qualifications ((1) implicit, (2) restriction) are deeply related.

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- Let variables can be polymorphic only if `e1` is a "syntactic value"
  - A variable, constant, function definition, ...
  - Called the "value restriction" (relaxed partially in OCaml)

ML Restrictions: Why?

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- Type inference for ML is decidable and efficient in practice, though pathological programs of size $O(n)$ and run-time $O(n)$ can have types of size $O(2^{2^n})$
- The type inference algorithm is *unsound* in the presence of ML-style mutation, but value-restriction restores soundness
  - Based on *unification*

Recover Lost Ground

Extensions to the ML type system to be closer to System F:

- Usually require some type annotations
- Are judged by:
  - Soundness: Do programs still not get stuck?
  - Conservatism: Do all (or most) old ML programs still type-check?
  - Power: Does it accept many more useful programs?
  - Convenience: Are many new types still inferred?