

# CSE 505: Programming Languages

## Lecture 8 — Reduction Strategies; Substitution

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## Review

$\lambda$ -calculus syntax:

$$\begin{aligned} e & ::= \lambda x. e \mid x \mid e e \\ v & ::= \lambda x. e \end{aligned}$$

Call-By-Value Left-To-Right Small-Step Operational Semantics:

$$\boxed{e \rightarrow e'}$$

$$\frac{}{(\lambda x. e) v \rightarrow e[v/x]} \quad \frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e_2 \rightarrow e'_2}{v e_2 \rightarrow v e'_2}$$

Previously wrote the first rule as follows:

$$\frac{e[v/x] = e'}{(\lambda x. e) v \rightarrow e'}$$

- ▶ The more concise axiom is more common
- ▶ But the more verbose version fits better with how we will formally define substitution at the end of this lecture

## Other Reduction “Strategies”

Suppose we allowed any substitution to take place in any order:

$$\boxed{e \rightarrow e'}$$

$$\frac{}{(\lambda x. e) e' \rightarrow e[e'/x]} \quad \frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e_2 \rightarrow e'_2}{e_1 e_2 \rightarrow e_1 e'_2}$$
$$\frac{e \rightarrow e'}{\lambda x. e \rightarrow \lambda x. e'}$$

Programming languages do not typically do this, but it has uses:

- ▶ Optimize/pessimize/partially evaluate programs
- ▶ Prove programs equivalent by reducing them to the same term

## Church-Rosser

The order in which you reduce is a “strategy”

Non-obvious fact — “Confluence” or “Church-Rosser”:

In this pure calculus,

If  $e \rightarrow^* e_1$  and  $e \rightarrow^* e_2$ ,  
then there exists an  $e_3$  such that  $e_1 \rightarrow^* e_3$  and  $e_2 \rightarrow^* e_3$

“No strategy gets painted into a corner”

- ▶ Useful: No rewriting via the full-reduction rules prevents you from getting an answer (Wow!)

Any *rewriting system* with this property is said to,  
“have the Church-Rosser property”

## Equivalence via rewriting

We can add two more rewriting rules:

- ▶ Replace  $\lambda x. e$  with  $\lambda y. e'$  where  $e'$  is  $e$  with “free”  $x$  replaced with  $y$  (assuming  $y$  not already used in  $e$ )

$$\overline{\lambda x. e \rightarrow \lambda y. e[y/x]}$$

- ▶ Replace  $\lambda x. e x$  with  $e$  if  $x$  does not occur “free” in  $e$

$$\frac{x \text{ is not free in } e}{\lambda x. e x \rightarrow e}$$

Analogies: `if e then true else false`

`List.map (fun x -> f x) lst`

But beware side-effects/non-termination under call-by-value

## No more rules to add

Now consider the system with:

- ▶ The 4 rules on slide 3
- ▶ The 2 rules on slide 5
- ▶ Rules can also run backwards (rewrite right-side to left-side)

Amazing: Under the natural denotational semantics (basically treat lambdas as functions),  $e$  and  $e'$  denote the same thing if and only if this rewriting system can show  $e \rightarrow^* e'$

- ▶ So the rules are *sound*, meaning they respect the semantics
- ▶ So the rules are *complete*, meaning there is no need to add any more rules in order to show some equivalence they can't

But program equivalence in a Turing-complete PL is undecidable

- ▶ So there is no perfect (always terminates, always correctly says yes or no) rewriting strategy for equivalence

## Some other common semantics

We have seen “full reduction” and left-to-right CBV

- ▶ (OCaml is unspecified order, but actually right-to-left)

Claim: Without assignment, I/O, exceptions, . . . , you cannot distinguish left-to-right CBV from right-to-left CBV

- ▶ How would you prove this equivalence? (Hint: Lecture 6)

Another option: call-by-name (CBN) — even “smaller” than CBV!

$$\boxed{e \rightarrow e'}$$

$$\frac{}{(\lambda x. e) e' \rightarrow e[e'/x]}$$

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2}$$

Diverges strictly less often than CBV, e.g.,  $(\lambda y. \lambda z. z) e$

Can be faster (fewer steps), but not usually (reuse args)

## More on evaluation order

In “purely functional” code, evaluation order matters “only” for performance and termination

Example: Imagine CBV for conditionals!

```
let rec f n = if n=0 then 1 else n*(f (n-1))
```

Call-by-need or “lazy evaluation”:

- ▶ Evaluate the argument the first time it’s used and *memoize the result*
  - ▶ Useful idiom for programmers too

Best of both worlds?

- ▶ For purely functional code, total equivalence with CBN and asymptotically no slower than CBV. (Note: *asymptotic!*)
- ▶ But hard to reason about side-effects

## More on Call-By-Need

This course will mostly assume Call-By-Value

Haskell uses Call-By-Need

Example:

```
four = length (9:(8+5):17:42:[])  
eight = four + four  
main = do { putStrLn (show eight) }
```

Example:

```
ones = 1 : ones  
nats_from x = x : (nats_from (x + 1))
```

## Formalism not done yet

Need to define substitution (used in our function-call rule)

- ▶ Shockingly subtle

Informally:  $e[e'/x]$  “replaces occurrences of  $x$  in  $e$  with  $e'$ ”

Examples:

$$x[(\lambda y. y)/x] = \lambda y. y$$

$$(\lambda y. y x)[(\lambda z. z)/x] = \lambda y. y \lambda z. z$$

$$(x x)[(\lambda x. x x)/x] = (\lambda x. x x)(\lambda x. x x)$$

## Substitution gone wrong

Attempt #1:

$$\boxed{e_1[e_2/x] = e_3}$$

$$\frac{}{x[e/x] = e} \quad \frac{y \neq x}{y[e/x] = y} \quad \frac{e_1[e/x] = e'_1}{(\lambda y. e_1)[e/x] = \lambda y. e'_1}$$

$$\frac{e_1[e/x] = e'_1 \quad e_2[e/x] = e'_2}{(e_1 e_2)[e/x] = e'_1 e'_2}$$

Recursively replace every  $x$  leaf with  $e$

The rule for substituting into (nested) functions is wrong: If the function's argument binds the same variable (shadowing), we should not change the function's body

Example program:  $(\lambda x. \lambda x. x) 42$

## Substitution gone wrong: Attempt #2

$$e_1[e_2/x] = e_3$$

$$\frac{}{x[e/x] = e}$$

$$\frac{y \neq x}{y[e/x] = y}$$

$$\frac{e_1[e/x] = e'_1 \quad y \neq x}{(\lambda y. e_1)[e/x] = \lambda y. e'_1}$$

$$\frac{}{(\lambda x. e_1)[e/x] = \lambda x. e_1} \quad \frac{e_1[e/x] = e'_1 \quad e_2[e/x] = e'_2}{(e_1 e_2)[e/x] = e'_1 e'_2}$$

Recursively replace every  $x$  leaf with  $e$  but respect shadowing

Substituting into (nested) functions is still wrong: If  $e$  uses an outer  $y$ , then substitution *captures*  $y$  (actual technical name)

- ▶ Example program capturing  $y$ :  
 $(\lambda x. \lambda y. x) (\lambda z. y) \rightarrow \lambda y. (\lambda z. y)$ 
  - ▶ Different(!) from:  $(\lambda a. \lambda b. a) (\lambda z. y) \rightarrow \lambda b. (\lambda z. y)$
- ▶ Capture won't happen under CBV/CBN if our source program has *no free variables*, but can happen under full reduction

## Attempt #3

First define the “free variables of an expression”  $FV(e)$ :

$$\begin{aligned}FV(x) &= \{x\} \\FV(e_1 e_2) &= FV(e_1) \cup FV(e_2) \\FV(\lambda x. e) &= FV(e) - \{x\}\end{aligned}$$

$$e_1[e_2/x] = e_3$$

$$\frac{}{x[e/x] = e} \quad \frac{y \neq x}{y[e/x] = y} \quad \frac{e_1[e/x] = e'_1 \quad y \neq x \quad y \notin FV(e)}{(\lambda y. e_1)[e/x] = \lambda y. e'_1}$$

$$\frac{}{(\lambda x. e_1)[e/x] = \lambda x. e_1} \quad \frac{e_1[e/x] = e'_1 \quad e_2[e/x] = e'_2}{(e_1 e_2)[e/x] = e'_1 e'_2}$$

But this is a *partial* definition

- ▶ Could get stuck if there is no substitution

# Implicit Renaming

- ▶ A *partial* definition because of the *syntactic accident* that  $y$  was used as a binder
  - ▶ Choice of local names should be irrelevant/invisible
- ▶ So we allow *implicit systematic renaming* of a binding and all its bound occurrences
- ▶ So via renaming the rule with  $y \neq x$  can *always* apply and we can remove the rule where  $x$  is shadowed
- ▶ In general, we *never* distinguish terms that differ only in the names of variables (A key language-design principle!)
- ▶ So now even “different syntax trees” can be the “same term”
  - ▶ Treat particular choice of variable as a concrete-syntax thing

## Correct Substitution

Assume *implicit* systematic renaming of a binding and all its bound occurrences

- ▶ Lets one rule match any substitution into a function

And these rules:

$$\boxed{e_1[e_2/x] = e_3}$$

$$\frac{}{x[e/x] = e} \quad \frac{y \neq x}{y[e/x] = y} \quad \frac{e_1[e/x] = e'_1 \quad e_2[e/x] = e'_2}{(e_1 \ e_2)[e/x] = e'_1 \ e'_2}$$

$$\frac{e_1[e/x] = e'_1 \quad y \neq x \quad y \notin FV(e)}{(\lambda y. e_1)[e/x] = \lambda y. e'_1}$$

## More explicit approach

While everyone in PL:

- ▶ Understands the capture problem
- ▶ Avoids it via implicit systematic renaming

you may find that unsatisfying, especially if you have to implement substitution and full reduction in a meta-language that doesn't have implicit renaming

This more explicit version also works

$$\frac{z \neq x \quad z \notin FV(e_1) \quad z \notin FV(e) \quad e_1[z/y] = e'_1 \quad e'_1[e/x] = e''_1}{(\lambda y. e_1)[e/x] = \lambda z. e''_1}$$

- ▶ You have to find an appropriate  $z$ , but one always exists and `__$compilerGenerated` appended to a global counter works

## Some jargon

If you want to study/read PL research, some jargon for things we have studied is helpful...

- ▶ Implicit systematic renaming is  $\alpha$ -conversion. If renaming in  $e_1$  can produce  $e_2$ , then  $e_1$  and  $e_2$  are  $\alpha$ -equivalent.
  - ▶  $\alpha$ -equivalence is an equivalence relation
- ▶ Replacing  $(\lambda x. e_1) e_2$  with  $e_1[e_2/x]$ , i.e., doing a function call, is a  $\beta$ -reduction
  - ▶ (The reverse step is meaning-preserving, but unusual)
- ▶ Replacing  $\lambda x. e x$  with  $e$  is an  $\eta$ -reduction or  $\eta$ -contraction (since it's always smaller)
- ▶ Replacing  $e$  with  $e$  with  $\lambda x. e x$  is an  $\eta$ -expansion
  - ▶ It can delay evaluation of  $e$  under CBV
  - ▶ It is sometimes necessary in languages (e.g., OCaml does not treat constructors as first-class functions)