

CSE505 Graduate Programming Languages: Lecture 6 Supplement

In class we sketched several proofs, but proof sketches invariably skip steps and have small errors. Here are the proofs more carefully laid out, as one might do on a homework assignment.

Theorem: $H ; e * 2 \Downarrow c$ if and only if $H ; e + e \Downarrow c$.

Proof: (Does not use induction)

- First assume $H ; e * 2 \Downarrow c$ and show $H ; e + e \Downarrow c$. Any derivation of $H ; e * 2 \Downarrow c$ must end with the MULT rule, which means there must exist derivations of $H ; e \Downarrow c'$ and $H ; 2 \Downarrow 2$, and c must be $2c'$. That is, there must be a derivation that looks like this:

$$\frac{\begin{array}{c} \vdots \\ \hline H ; e \Downarrow c' \end{array} \quad \frac{\begin{array}{c} \vdots \\ \hline H ; 2 \Downarrow 2 \end{array}}{\hline H ; e * 2 \Downarrow 2c'}}$$

So given that there exists a derivation of $H ; e \Downarrow c'$, we can use ADD to derive:

$$\frac{H ; e \Downarrow c' \quad H ; e \Downarrow c'}{\hline H ; e + e \Downarrow c' + c'}}$$

Math provides $c' + c' = 2c'$, so the conclusion of this derivation is what we need.

- Now assume $H ; e + e \Downarrow c$ and show $H ; e * 2 \Downarrow c$. Any derivation of $H ; e + e \Downarrow c$ must end with the ADD rule, which means there exists a derivation that looks like this (where $c = c_1 + c_2$):

$$\frac{\begin{array}{c} \vdots \\ \hline H ; e \Downarrow c_1 \end{array} \quad \frac{\begin{array}{c} \vdots \\ \hline H ; e \Downarrow c_2 \end{array}}{\hline H ; e + e \Downarrow c_1 + c_2}}$$

In fact, we earlier proved determinacy (there is at most one c such that $H ; e \Downarrow c$), so the derivation must have this form (where $c = c_1 + c_1$):

$$\frac{\begin{array}{c} \vdots \\ \hline H ; e \Downarrow c_1 \end{array} \quad \frac{\begin{array}{c} \vdots \\ \hline H ; e \Downarrow c_1 \end{array}}{\hline H ; e + e \Downarrow c_1 + c_1}}$$

So given that there exists a derivation of $H ; e \Downarrow c_1$, we can use MULT to derive:

$$\frac{H ; e \Downarrow c_1 \quad \frac{\begin{array}{c} \vdots \\ \hline H ; 2 \Downarrow 2 \end{array}}{\hline}}{\hline H ; e * 2 \Downarrow 2c_1}}$$

Math provides $c_1 + c_1 = 2c_1$, so the conclusion of this derivation is what we need.

$$C ::= [\cdot] \mid C + e \mid e + C \mid C * e \mid e * C$$

Formal definition of “filling the hole”:

$$\begin{aligned} ([\cdot])[e] &= e \\ (C + e_1)[e] &= C[e] + e_1 \\ (e_1 + C)[e] &= e_1 + C[e] \\ (C * e_1)[e] &= C[e] * e_1 \\ (e_1 * C)[e] &= e_1 * C[e] \end{aligned}$$

Theorem: $H ; C[e * 2] \Downarrow c$ if and only if $H ; C[e + e] \Downarrow c$.

Proof: By induction on (the height of) the structure of C :

- If the height is 0, then C is $[\cdot]$, so $C[e * 2] = e * 2$ and $C[e + e] = e + e$. So the previous theorem is exactly what we need.
- If the height is greater than 0, then C has one of four forms:
 - If C is $C' + e'$ for some C' and e' , then $C[e * 2]$ is $C'[e * 2] + e'$ and $C[e + e]$ is $C'[e + e] + e'$. Since C' is shorter than C , induction ensures that for any constant c' , $H ; C'[e * 2] \Downarrow c'$ if and only if $H ; C'[e + e] \Downarrow c'$.

Assume $H ; C'[e * 2] + e' \Downarrow c$ and show $H ; C'[e + e] + e' \Downarrow c$: Any derivation of $H ; C'[e * 2] + e' \Downarrow c$ must end with ADD, i.e., it looks like this (where $c = c' + c''$):

$$\frac{\begin{array}{c} \vdots \\ H ; C'[e * 2] \Downarrow c' \end{array} \quad \begin{array}{c} \vdots \\ H ; e' \Downarrow c'' \end{array}}{H ; C'[e * 2] + e' \Downarrow c}$$

As argued above, the existence of a derivation of $H ; C'[e * 2] \Downarrow c'$ ensures the existence of a derivation of $H ; C'[e + e] \Downarrow c'$. So using ADD and the existence of a derivation of $H ; e' \Downarrow c''$, we can derive:

$$\frac{H ; C'[e + e] \Downarrow c' \quad H ; e' \Downarrow c''}{H ; C'[e + e] + e' \Downarrow c}$$

Now assume $H ; C'[e + e] + e' \Downarrow c$ and show $H ; C'[e * 2] + e' \Downarrow c$: Any derivation of $H ; C'[e + e] + e' \Downarrow c$ must end with ADD, i.e., it looks like this (where $c = c' + c''$):

$$\frac{\begin{array}{c} \vdots \\ H ; C'[e + e] \Downarrow c' \end{array} \quad \begin{array}{c} \vdots \\ H ; e' \Downarrow c'' \end{array}}{H ; C'[e + e] + e' \Downarrow c}$$

As argued above, the existence of a derivation of $H ; C'[e + e] \Downarrow c'$ ensures the existence of a derivation of $H ; C'[e * 2] \Downarrow c'$. So using ADD and the existence of a derivation of $H ; e' \Downarrow c''$, we can derive:

$$\frac{H ; C'[e * 2] \Downarrow c' \quad H ; e' \Downarrow c''}{H ; C'[e * 2] + e' \Downarrow c}$$

- The other 3 cases are similar. (Try them out.)

Theorem: The two semantics below are equivalent, i.e., $H ; e \Downarrow c$ if and only if $H ; e \rightarrow^* c$.

$$\begin{array}{c}
\text{CONST} \\
\frac{}{H ; c \Downarrow c} \\
\\
\text{VAR} \\
\frac{}{H ; x \Downarrow H(x)} \\
\\
\text{ADD} \\
\frac{H ; e_1 \Downarrow c_1 \quad H ; e_2 \Downarrow c_2}{H ; e_1 + e_2 \Downarrow c_1 + c_2} \\
\\
\text{SVAR} \\
\frac{}{H ; x \rightarrow H(x)} \\
\\
\text{SADD} \\
\frac{}{H ; c_1 + c_2 \rightarrow c_1 + c_2} \\
\\
\text{SLEFT} \\
\frac{H ; e_1 \rightarrow e'_1}{H ; e_1 + e_2 \rightarrow e'_1 + e_2} \\
\\
\text{SRIGHT} \\
\frac{H ; e_2 \rightarrow e'_2}{H ; e_1 + e_2 \rightarrow e_1 + e'_2}
\end{array}$$

Proof: We prove the two directions separately.

First assume $H ; e \Downarrow c$; show $\exists n. H ; e \rightarrow^n c$. By induction on the height h of derivation of $H ; e \Downarrow c$:

- $h = 1$: Then the derivation must end with CONST or VAR. For CONST, e is c and trivially $H ; e \rightarrow^0 c$. For VAR, e is some x where $H(x) = c$, so using SVAR, $H ; e \rightarrow^1 c$.
- $h > 1$: Then the derivation must end with ADD, so e is some $e_1 + e_2$ where $H ; e_1 \Downarrow c_1$, $H ; e_2 \Downarrow c_2$, and c is $c_1 + c_2$. By induction $\exists n_1, n_2. H ; e_1 \rightarrow^{n_1} c_1$ and $H ; e_2 \rightarrow^{n_2} c_2$. Therefore, using the lemma below, $H ; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H ; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$, so ADD lets us derive $H ; e_1 + e_2 \rightarrow^{n_1 + n_2 + 1} c$.

Lemma: If $H ; e \rightarrow^n e'$, then $H ; e_1 + e \rightarrow^n e_1 + e'$ and $H ; e + e_2 \rightarrow^n e' + e_2$.

Proof: By induction on n . If $n = 0$, the result is trivial because $e = e'$. If $n > 0$, then there exists some e'' such that $H ; e \rightarrow^{n-1} e''$ and $H ; e'' \rightarrow^1 e'$. So by induction $H ; e_1 + e \rightarrow^{n-1} e_1 + e''$ and $H ; e + e_2 \rightarrow^{n-1} e'' + e_2$. Using SRIGHT and SLEFT respectively, $H ; e'' \rightarrow^1 e'$ ensures $H ; e_1 + e'' \rightarrow^1 e_1 + e'$ and $H ; e'' + e_2 \rightarrow^1 e' + e_2$. So with the inductive hypotheses, $H ; e_1 + e \rightarrow^n e_1 + e'$ and $H ; e + e_2 \rightarrow^n e' + e_2$.

Now assume $\exists n. H ; e \rightarrow^n c$; show $H ; e \Downarrow c$. By induction on n :

- $n = 0$: e is c and CONST lets us derive $H ; c \Downarrow c$.
- $n > 0$: So $\exists e'. H ; e \rightarrow e'$ and $H ; e' \rightarrow^{n-1} c$. By induction $H ; e' \Downarrow c$. So this lemma suffices: If $H ; e \rightarrow e'$ and $H ; e' \Downarrow c$, then $H ; e \Downarrow c$. Prove the lemma by induction on height h of derivation of $H ; e \rightarrow e'$:
 - $h = 1$: Then the derivation ends with SVAR or SADD. For SVAR, e is some x and $e' = H(x) = c$. So with VAR we can derive $H ; x \Downarrow H(x)$, i.e., $H ; e \Downarrow c$. For SADD, e is some $c_1 + c_2$ and $e' = c = c_1 + c_2$. So with ADD, we can derive $H ; c_1 + c_2 \Downarrow c_1 + c_2$, i.e., $H ; e \Downarrow c$. (Note the $h = 1$ case may look a little weird because in fact in this case $n = 1$, i.e., e' must be a constant.)
 - $h > 1$: Then the derivation ends with SLEFT or SRIGHT. For SLEFT, the assumed derivations end like this:

$$\frac{H ; e_1 \rightarrow e'_1}{H ; e_1 + e_2 \rightarrow e'_1 + e_2} \qquad \frac{H ; e'_1 \Downarrow c_1 \quad H ; e_2 \Downarrow c_2}{H ; e'_1 + e_2 \Downarrow c_1 + c_2}$$

Using $H ; e_1 \rightarrow e'_1$, $H ; e'_1 \Downarrow c_1$, and the induction hypothesis, $H ; e_1 \Downarrow c_1$. Using this fact, $H ; e_2 \Downarrow c_2$, and ADD, we can derive $H ; e_1 + e_2 \Downarrow c_1 + c_2$.

For SRIGHT, the assumed derivations end like this:

$$\frac{H ; e_2 \rightarrow e'_2}{H ; e_1 + e_2 \rightarrow e_1 + e'_2} \qquad \frac{H ; e_1 \Downarrow c_1 \quad H ; e'_2 \Downarrow c_2}{H ; e_1 + e'_2 \Downarrow c_1 + c_2}$$

Using $H ; e_2 \rightarrow e'_2$, $H ; e'_2 \Downarrow c_2$, and the induction hypothesis, $H ; e_2 \Downarrow c_2$. Using this fact, $H ; e_1 \Downarrow c_1$, and ADD, we can derive $H ; e_1 + e_2 \Downarrow c_1 + c_2$.