Lattice-Theoretic Data Flow Analysis Framework

Goals:

- provide a single, formal model that describes all DFAs
- formalize notions of “safe”, “conservative”, “optimistic”
- place precise bounds on time complexity of DF analysis
- enable connecting analysis to underlying semantics for correctness proofs

Plan:

- define **domain** of program properties computed by DFA, organize domain elements as a **lattice**
- define flow functions, merge function over this domain using lattice operations
- benefit from lattice theory in attacking above issues

History: Kildall [POPL 73], Kam & Ullman [JACM 76]

Lattices

Define lattice $D = (S, \leq)$:

- S is a set of elements of the lattice
- $\leq$ is a binary relation over elements of S

Required properties of $\leq$:

- $\leq$ induces a **partial order** over S
  - reflexive, transitive, & anti-symmetric
- every pair of elements of S has
  a unique **greatest lower bound** (a.k.a. meet) and
  a unique **least upper bound** (a.k.a. join)

Height of D =
longest path through partial order from greatest to least

- infinite lattice can have finite height (but infinite width)

Top (T) = unique element of S that’s greatest, if exists

Bottom ($\perp$) = unique element of S that’s least, if exists

Lattice models in data flow analysis

Model information by elements of a lattice

- top = best case info
- bottom = worst case info
- merge function = g.l.b. (meet) on lattice elements
- if $a \leq b$, then a is a conservative approximation to b

(Opposite conventions used in semantics!)

Reaching definitions:

- elements:
- $\leq$:
  - meet:
  - top:
  - bottom:

Reaching constants:

- elements:
- $\leq$:
  - meet:
  - top:
  - bottom:

Tuples of lattices

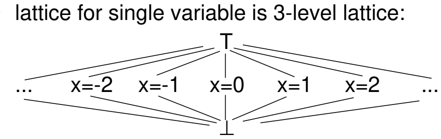
Often helpful to break down a complex lattice into a tuple of lattices, one per variable being analyzed

Formally: $D_T = \langle S_T, \leq_T \rangle \;=\; (D = \langle S, \leq \rangle)^N$

- $S_T = S_1 \times S_2 \times \ldots \times S_N$
  - element of tuple domain is a tuple of elements from each variable’s domain
  - i^{th} component of tuple is info about i^{th} variable
- $\langle \ldots, d_{1i}, \ldots \rangle \leq_T \langle \ldots, d_{2i}, \ldots \rangle \equiv d_{1i} \leq d_{2i}, \forall i$
  - i.e. pointwise ordering
- meet: pointwise meet
- top: tuple of tops
- bottom: tuple of bottoms
- height(D_T) = N * height(D)

E.g. reaching constants

- lattice for single variable is 3-level lattice:



- whole problem is tuple of individual lattices

Some typical lattice domains

Single-point lattice: just bottom

- trivial do-nothing analysis

Two-point lattice: top and bottom

- computes a boolean property
- a tuple of two-point lattices ⇔ a bit-vector

A lifted set: a set of incomparable values, plus top & bottom

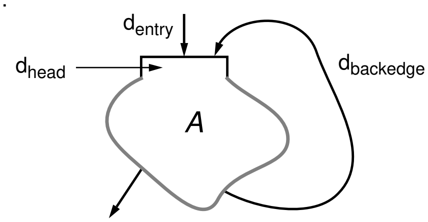
- e.g. reaching constants domain

Powerset lattice: set of all subsets of a set *S*, ordered somehow

- top & bottom = ∅ & *S*, or vice versa
- "a collecting analysis"
- isomorphic to tuple of booleans
  indicating membership in subset of elements of *S*

Analysis of loops in lattice model

Consider:



Assume A(d) computes info at backedge given d, info at head

Want solution to constraints:

$d_{head} = d_{entry} \cap d_{backedge}$

$d_{backedge} = A(d_{head})$

Let $F(d) = d_{entry} \cap A(d)$

Then want fixpoint of F:

$d_{head} = F(d_{head})$

Iterative analysis in lattice model

Iterative analysis computes fixpoint by iterative approximation:

$F^0 = d_{entry} \cap T$

$F^1 = d_{entry} \cap A(F^0) = F(F^0) = F(d_{entry})$

$F^2 = d_{entry} \cap A(F^1) = F(F^1) = F(F(F^0)) = F(F(d_{entry}))$

. . .

$F^k = d_{entry} \cap A(F^{k-1}) = F(F^{k-1}) = F(F(...(F(d_{entry}))...))$

until

$F^{k+1} = d_{entry} \cap A(F^k) = F(F^k) \quad = \quad F^k$

Is *k* finite?

If so, how big can it be?

Termination of iterative analysis

In general, *k* need not be finite

Sufficient conditions for finiteness:

- flow functions (e.g. *F*) are **monotonic**
- lattice is of finite height

A function *F* is monotonic iff:

$d_1 \le d_2 \;\Rightarrow\; F(d_1) \le F(d_2)$

- for application of DFA, this means that giving a flow function more conservative inputs leads to more conservative outputs, never more optimistic outputs

For monotonic *F* over domain *D*, the maximum number of times that *F* can be applied to itself, starting w/ any element of D, w/o reaching fixpoint, is height(*D*)-1

- start at top of D
- go down one level in lattice each application of *F*
- eventually must hit fixpoint or bottom
  (which is guaranteed to be a fixpoint), if D of finite height