Theory Homework 2

CSE 493S/599S: Advanced Machine Learning
Instructor: Sewoong Oh
Due: Thursday, May 21st at 11:59pm

The goal of this homework is to help you better understand the ideas from theoretical machine learning we have covered in class.

Notes:

- You will be assigned a subset of the problems for each homework. Please submit the homework to Gradescope and link each page of your work to the corresponding problem.
- Please typeset your work using LaTeX.
- List every person with whom you discussed any problem in any depth, and every reference (outside of our course slides, lectures, and textbook) that you used.
- You may spend an arbitrary amount of time discussing and working out a solution with your listed collaborators, but **do not take notes, photos, or other artifacts of your collaboration**. Erase the board you were working on, and once you're alone, write up your answers yourself.
- The homework problems have been carefully chosen for their pedagogical value and hence might be similar or identical to those given out in similar courses at UW or other schools. Using any pre-existing solutions from these sources, from the Web or other textbooks constitutes a violation of the academic integrity expected of you and is strictly prohibited.

Version history:

V1 Initial version.

1 VC-dimension of axis aligned rectangles (taught in the 4th theory lecture)

Let $\mathcal{H}^d_{\text{rec}}$ be the class of axis aligned rectangles in $\mathbb{R}^d$. Prove that $\text{VCdim}(\mathcal{H}^d_{\text{rec}}) = 2d$.

[20 points]

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2 Infinite VC-dimension with one parameter (taught in the 4th theory lecture)

It is often the case that the VC-dimension of a hypothesis class equals (or can be bounded above by) the number of parameters one needs to set in order to define each hypothesis in the class. For instance, if $\mathcal{H}$ is the class of axis aligned rectangles in $\mathbb{R}^d$, then $\text{VCdim}(\mathcal{H}) = 2d$, which is equal to the number of parameters used to define a rectangle in $\mathbb{R}^d$. Here is an example that shows that this is not always the case. We will see that a hypothesis class might be very complex and even not learnable, although it has a small number of parameters.

Consider the domain $\mathcal{X} = \mathbb{R}$, and the hypothesis class

$$\mathcal{H} = \{x \mapsto \lceil \sin(\theta x) \rceil : \theta \in \mathbb{R}\}$$

(here, we take $\lceil -1 \rceil = 0$). Prove that $\text{VCdim}(\mathcal{H}) = \infty$.

Hint: There is more than one way to prove the required result. One option is by applying the following lemma: If $0.x_1x_2x_3\ldots$, is the binary expansion of $x \in (0, 1)$, then for any natural number m, $\lceil \sin(2^m \pi x) \rceil = (1 - x_m)$, provided that $\exists k \geq m$ s.t. $x_k = 1$.

[30 points]
