

Lecture 4: Neural Networks and Backpropagation

Administrative: Assignment 1

Due 4/16 11:59pm

- K-Nearest Neighbor
- Linear classifiers: SVM, Softmax

Administrative: EdStem

Please make sure to check and read all pinned EdStem posts.

Administrative: Fridays

This Friday 9:30-10:30am and again 12:30-1:30pm

Project Design & Backprop

Come to office hours to talk about your ideas

Administrative: Course Project

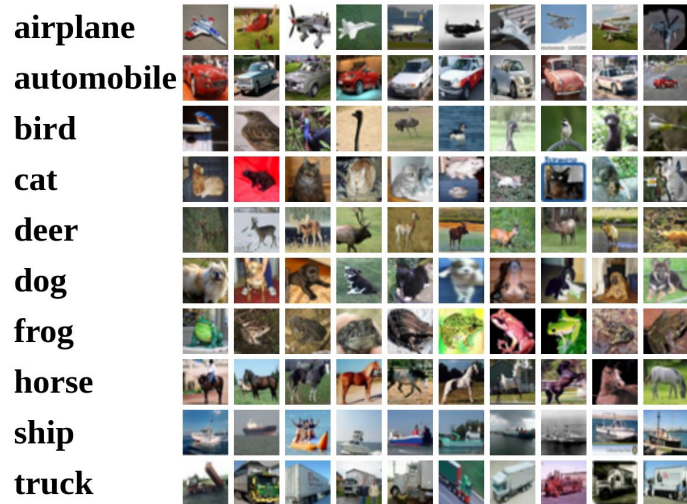
Project proposal due 4/29 11:59pm

“Is X a valid project for 493G1?”

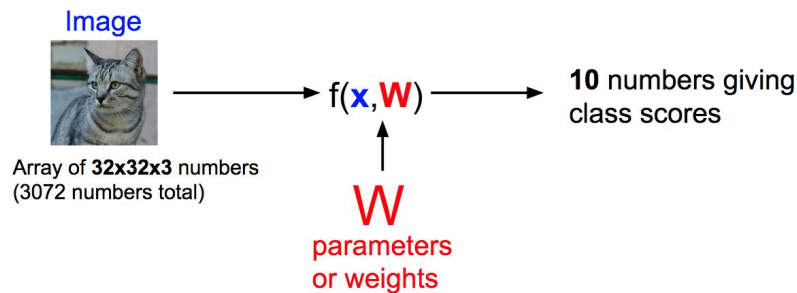
- Anything related to deep learning or computer vision
- Maximum of 3 students per team
- Make a EdStem private post or come to TA Office Hours

More info on the website

Recap: from last time

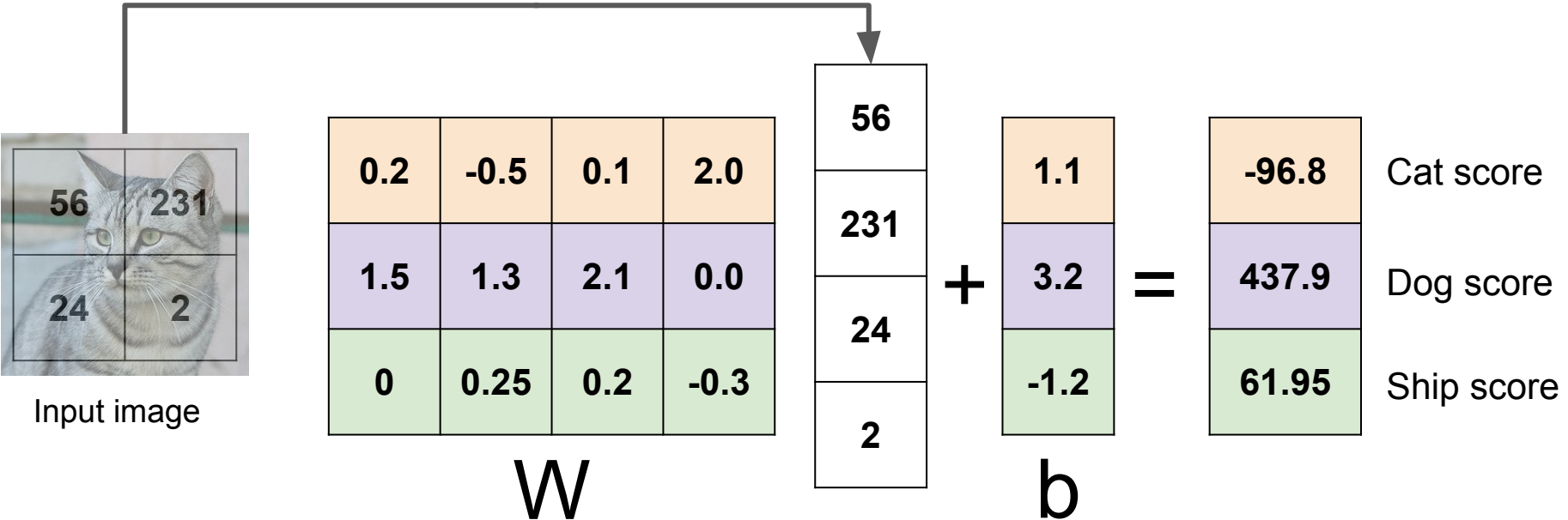


$$f(x, W) = Wx + b$$

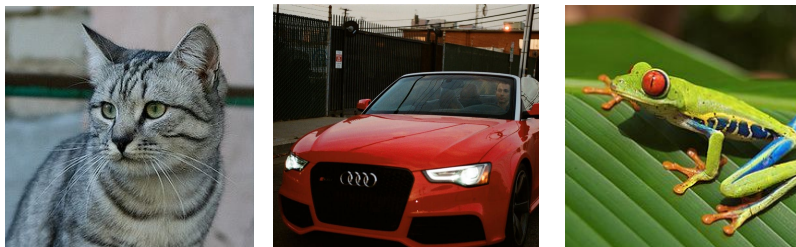


Algebraic viewpoint: Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Flatten tensors into a vector



Suppose: 3 training examples, 3 classes.
 With some W the scores $f(x, W) = Wx$ are:



cat	3.2	1.3	2.2
car	5.1	4.9	2.5
frog	-1.7	2.0	-3.1

Losses: **2.9**

Multiclass SVM loss:

Given an example (x_i, y_i)
 where x_i is the image and
 where y_i is the (integer) label,

and using the shorthand for the
 scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \max(0, 5.1 - 3.2 + 1) + \max(0, -1.7 - 3.2 + 1)$$

Regularization

λ = regularization strength
(hyperparameter)

$$L(W) = \underbrace{\frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i)}_{\text{Data loss}} + \underbrace{\lambda R(W)}_{\text{Regularization}}$$

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Why regularize?

- Express preferences over weights
- Make the model *simple* so it works on test data
- Improve optimization by adding curvature

Softmax classifier

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**



cat	3.2
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax
Function

cat	3.2
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$
 Softmax Function

Probabilities
must be ≥ 0

cat	3.2	exp	24.5
car	5.1	→	164.0
frog	-1.7		0.18

unnormalized
probabilities

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

Probabilities
must be ≥ 0

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax
Function

Probabilities
must sum to 1

cat	3.2
car	5.1
frog	-1.7

exp

24.5
164.0
0.18

normalize

0.13
0.87
0.00

unnormalized
probabilities

probabilities

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

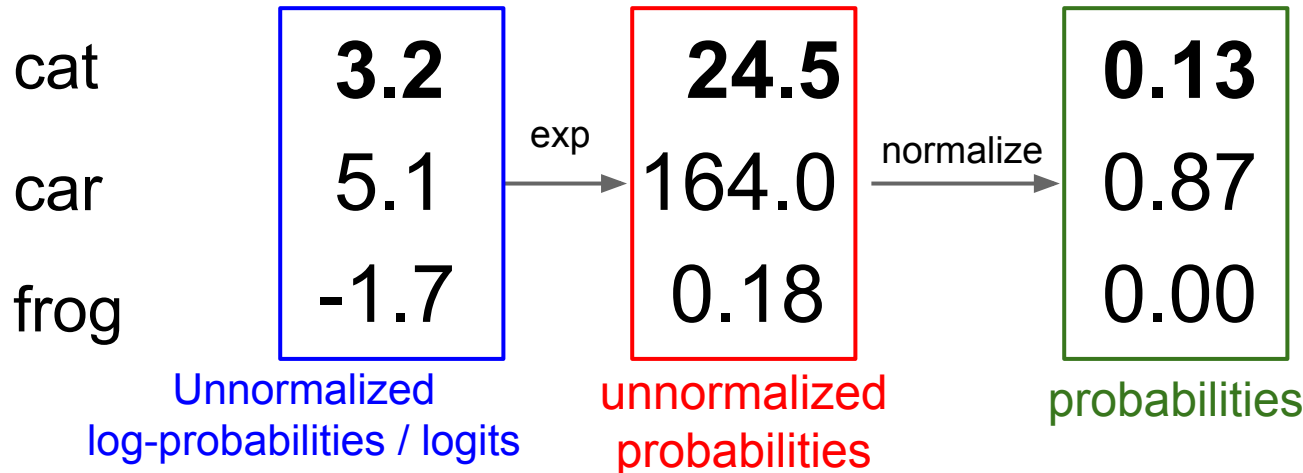
$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax
Function

Probabilities
must be ≥ 0

Probabilities
must sum to 1



Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \quad \text{Softmax Function}$$

Probabilities must be ≥ 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

cat
car
frog

3.2
5.1
-1.7

Unnormalized log-probabilities / logits

exp

24.5
164.0
0.18

unnormalized probabilities

normalize

0.13
0.87
0.00

probabilities

$$\rightarrow L_i = -\log(0.13) = 2.04$$

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$
 Softmax Function

Probabilities
must be ≥ 0

Probabilities
must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

cat
car
frog

3.2
5.1
-1.7

Unnormalized
log-probabilities / logits

exp

24.5
164.0
0.18

unnormalized
probabilities

normalize

0.13
0.87
0.00

probabilities

$$\rightarrow L_i = -\log(0.13) = 2.04$$

Maximum Likelihood Estimation
Choose weights to maximize the likelihood of the observed data

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

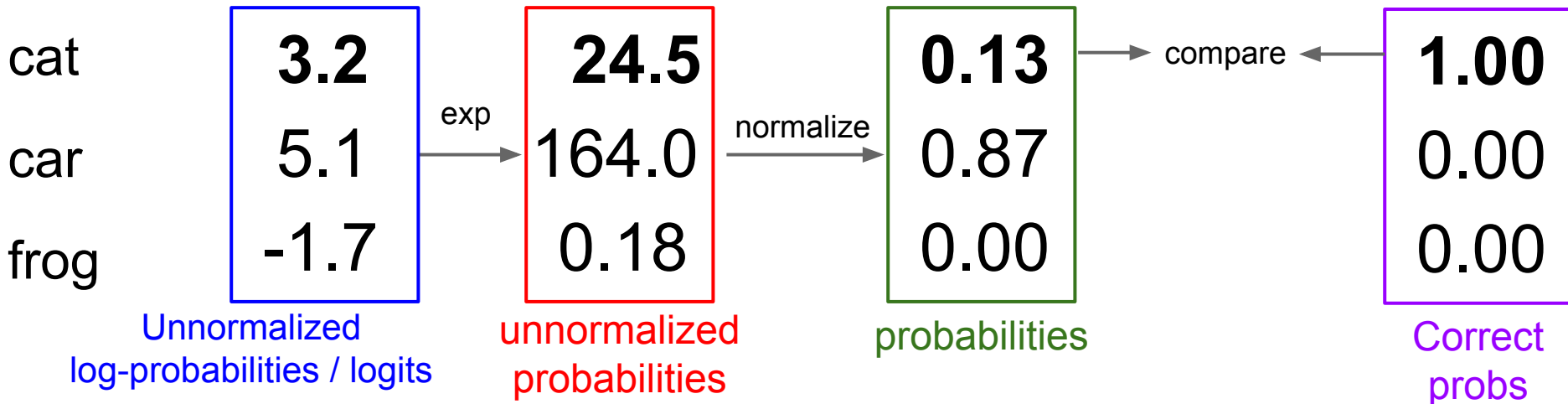
$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax
Function

Probabilities
must be ≥ 0

Probabilities
must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$



Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax Function

Probabilities must be ≥ 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

cat
car
frog

3.2
5.1
-1.7

Unnormalized log-probabilities / logits

exp

24.5
164.0
0.18

unnormalized probabilities

normalize

0.13
0.87
0.00

probabilities

compare

Kullback-Leibler divergence

$$D_{KL}(P||Q) = \sum_y P(y) \log \frac{P(y)}{Q(y)}$$

1.00
0.00
0.00

Correct probs

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

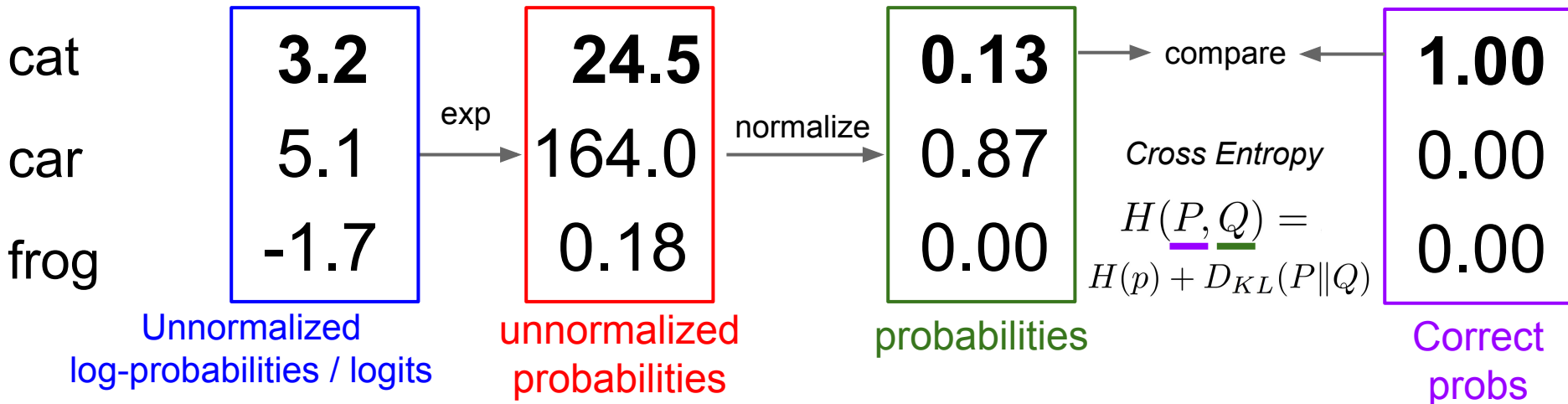
$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

Softmax Function

Probabilities must be ≥ 0

Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$



Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \quad \text{Softmax Function}$$

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

cat	3.2
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

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$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

cat **3.2**

car **5.1**

frog **-1.7**

Q1: What is the min/max possible softmax loss L_i ?

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \quad \text{Softmax Function}$$

Maximize probability of correct class

Putting it all together:

$$L_i = -\log P(Y = y_i | X = x_i)$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

cat **3.2**

car **5.1**

frog **-1.7**

Q1: What is the min/max possible softmax loss L_i ?

Q2: At initialization all s_j will be approximately equal; what is the softmax loss L_i , assuming C classes?

Softmax Classifier (Multinomial Logistic Regression)



Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}} \quad \text{Softmax Function}$$

Maximize probability of correct class

Putting it all together:

$$L_i = -\log P(Y = y_i | X = x_i)$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

cat **3.2**

car **5.1**

frog **-1.7**

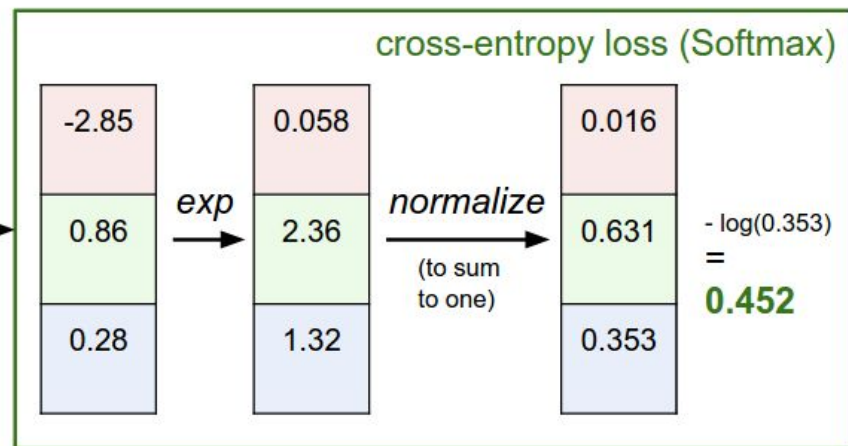
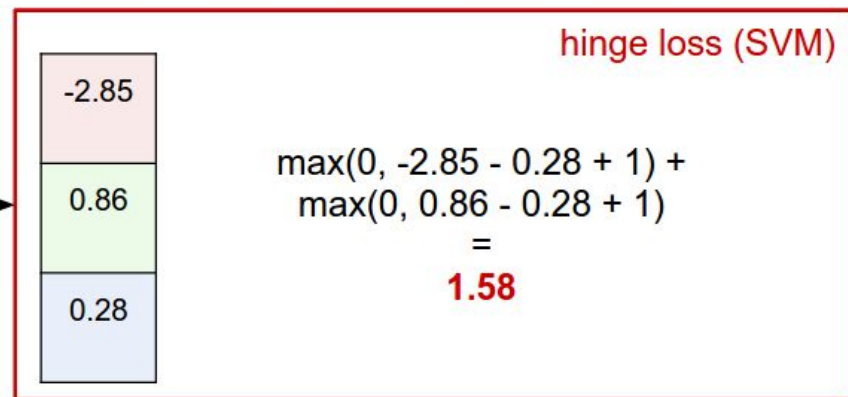
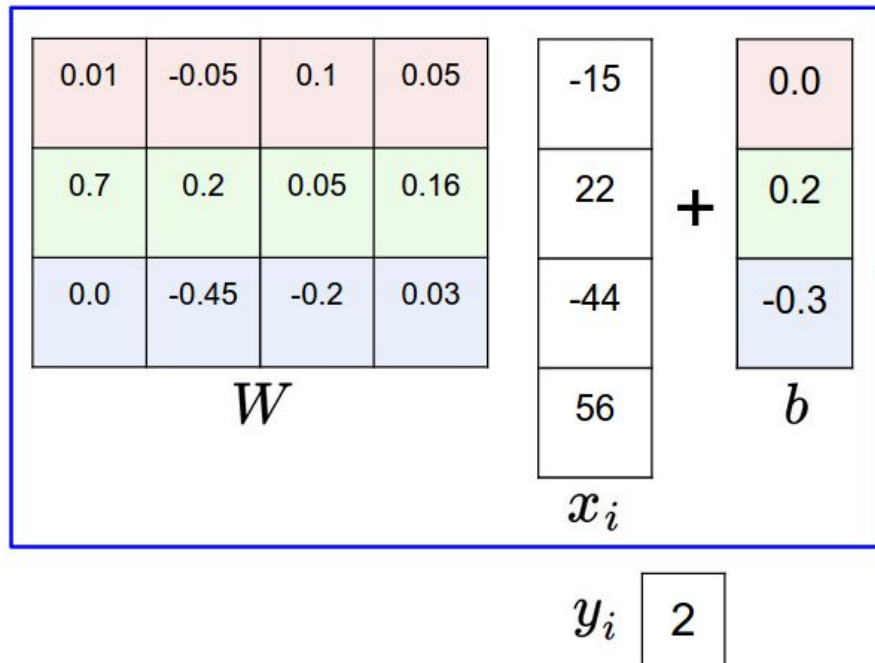
Q2: At initialization all s will be approximately equal; what is the loss?

A: $-\log(1/C) = \log(C)$,

If $C = 10$, then $L_i = \log(10) \approx 2.3$

Softmax vs. SVM

matrix multiply + bias offset



Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $y_i = 0$

Q: What is the **SVM loss**?

Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[10, -2, 3]

[10, 9, 9]

[10, -100, -100]

and $y_i = 0$

Q: What is the **SVM loss**?

Q: Is the **Softmax** loss zero for any of them?

Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

assume scores:

[20, -2, 3]

[20, 9, 9]

[20, -100, -100]

and $y_i = 0$

Q: What is the **SVM loss**?

Q: Is the **Softmax** loss zero for any of them?

I doubled the correct class score from 10 -> 20?

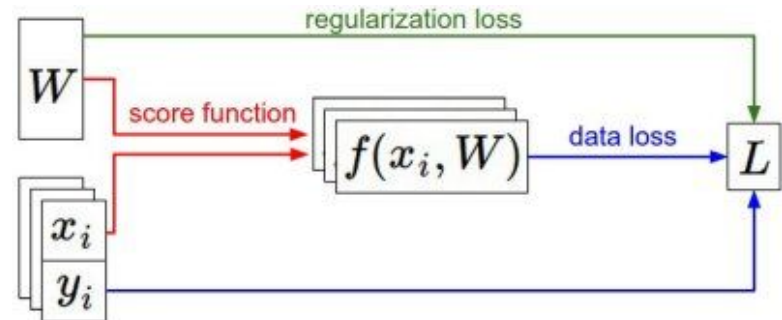
Recap

- We have some dataset of (x,y)
- We have a **score function**: $s = f(x; W) \stackrel{\text{e.g.}}{=} Wx$
- We have a **loss function**:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \quad \text{Softmax}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W) \quad \text{Full loss}$$



Recap

- We have some dataset of (x,y)
- We have a **score function**:
- We have a **loss function**:

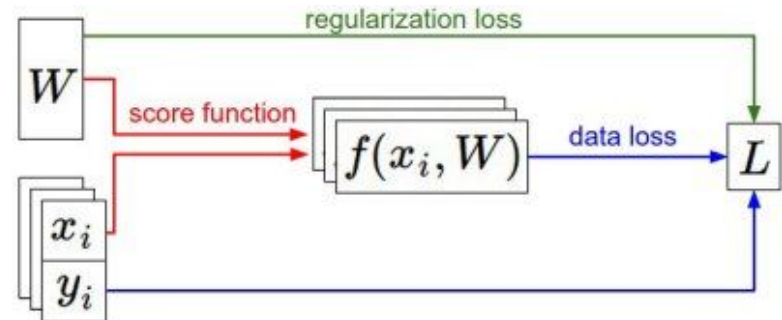
How do we find the best W ?

$$s = f(x; W) \stackrel{\text{e.g.}}{=} Wx$$

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right) \quad \text{Softmax}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W) \quad \text{Full loss}$$



Recap: loss functions

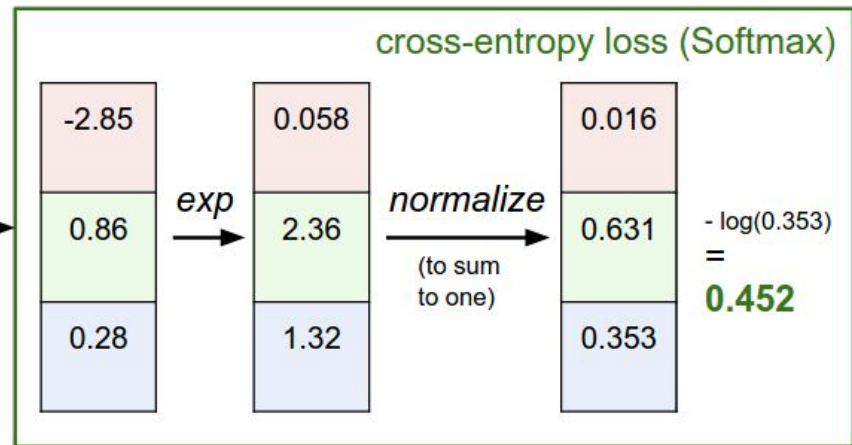
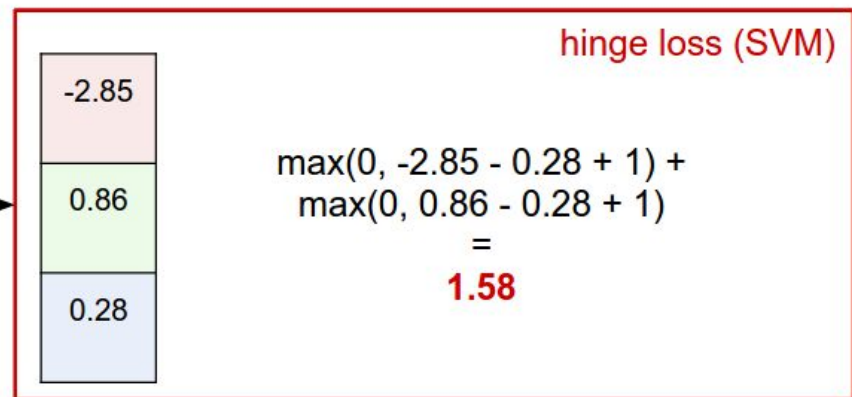
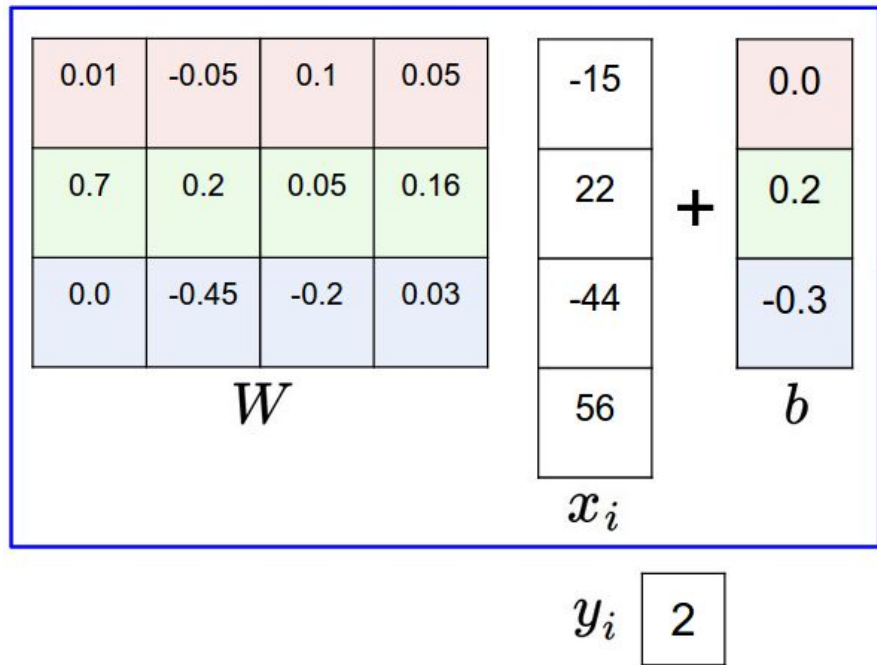
$$s = f(x; W) = Wx \quad \text{Linear score function}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM loss (or softmax)}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda \sum_k W_k^2 \quad \text{data loss + regularization}$$

Recap: Softmax vs. SVM

matrix multiply + bias offset



Optimization



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[Walking man image](#) is [CC0 1.0](#) public domain

Strategy #1: A first very bad idea solution: Random search

```
# assume X_train is the data where each column is an example (e.g. 3073 x 50,000)
# assume Y_train are the labels (e.g. 1D array of 50,000)
# assume the function L evaluates the loss function

bestloss = float("inf") # Python assigns the highest possible float value
for num in xrange(1000):
    W = np.random.randn(10, 3073) * 0.0001 # generate random parameters
    loss = L(X_train, Y_train, W) # get the loss over the entire training set
    if loss < bestloss: # keep track of the best solution
        bestloss = loss
        bestW = W
    print 'in attempt %d the loss was %f, best %f' % (num, loss, bestloss)

# prints:
# in attempt 0 the loss was 9.401632, best 9.401632
# in attempt 1 the loss was 8.959668, best 8.959668
# in attempt 2 the loss was 9.044034, best 8.959668
# in attempt 3 the loss was 9.278948, best 8.959668
# in attempt 4 the loss was 8.857370, best 8.857370
# in attempt 5 the loss was 8.943151, best 8.857370
# in attempt 6 the loss was 8.605604, best 8.605604
# ... (truncated: continues for 1000 lines)
```

Lets see how well this works on the test set...

```
# Assume X_test is [3073 x 10000], Y_test [10000 x 1]  
scores = Wbest.dot(Xte_cols) # 10 x 10000, the class scores for all test examples  
# find the index with max score in each column (the predicted class)  
Yte_predict = np.argmax(scores, axis = 0)  
# and calculate accuracy (fraction of predictions that are correct)  
np.mean(Yte_predict == Yte)  
# returns 0.1555
```

15.5% accuracy! not bad!
(SOTA is ~99.7%)

Strategy #2: Follow the slope



Strategy #2: Follow the slope

In 1-dimension, the derivative of a function:

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

In multiple dimensions, the **gradient** is the vector of (partial derivatives) along each dimension

The slope in any direction is the **dot product** of the direction with the gradient
The direction of steepest descent is the **negative gradient**

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

gradient dW:

[?,
?,
?,
?,
?,
?,
?,
?,
?,...]

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (first dim):

[0.34 + **0.0001**,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25322

gradient dW:

[?,
?,
?,
?,
?,
?,
?,
?,
?,...]

current **W**:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (first dim):

[0.34 + **0.0001**,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25322

gradient **dW**:

[-2.5,
?,
?,

$$(1.25322 - 1.25347)/0.0001 = -2.5$$

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

?,
?,...]

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (second dim):

[0.34,
-1.11 + **0.0001**,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25353

gradient dW:

[-2.5,
?,
?,
?,
?,
?,
?,
?,
?,...]

current **W**:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (second dim):

[0.34,
-1.11 + **0.0001**,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25353

gradient **dW**:

[-2.5,
0.6,
?,
?,

$$(1.25353 - 1.25347)/0.0001 = 0.6$$

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

?,...]

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (third dim):

[0.34,
-1.11,
0.78 + **0.0001**,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

gradient dW:

[-2.5,
0.6,
?,
?,
?,
?,
?,
?,
?,...]

current **W**:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (third dim):

[0.34,
-1.11,
0.78 + **0.0001**,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

gradient **dW**:

[-2.5,
0.6,
0,
?,
?

$(1.25347 - 1.25347)/0.0001 = 0$

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

?, ...]

current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

W + h (third dim):

[0.34,
-1.11,
0.78 + **0.0001**,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

gradient dW:

[-2.5,
0.6,
0,
?,
?

Numeric Gradient

- Slow! Need to loop over all dimensions
- Approximate

?,...]

This is silly. The loss is just a function of W :

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \sum_k W_k^2$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$s = f(x; W) = Wx$$

want $\nabla_W L$

This is silly. The loss is just a function of W :

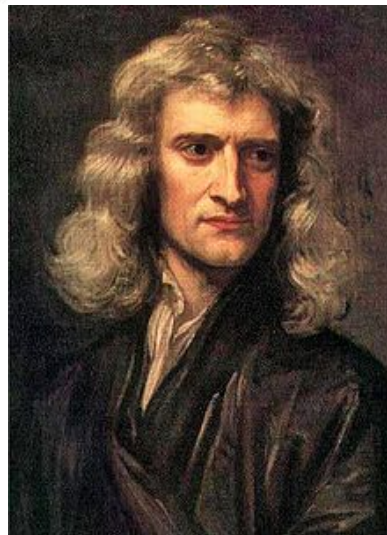
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$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$s = f(x; W) = Wx$$

want $\nabla_W L$

Use calculus to compute an
analytic gradient



[This image](#) is in the public domain



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current W:

[0.34,
-1.11,
0.78,
0.12,
0.55,
2.81,
-3.1,
-1.5,
0.33,...]

loss 1.25347

$dW = \dots$
(some function
data and W)



gradient dW:

[-2.5,
0.6,
0,
0.2,
0.7,
-0.5,
1.1,
1.3,
-2.1,...]

In summary:

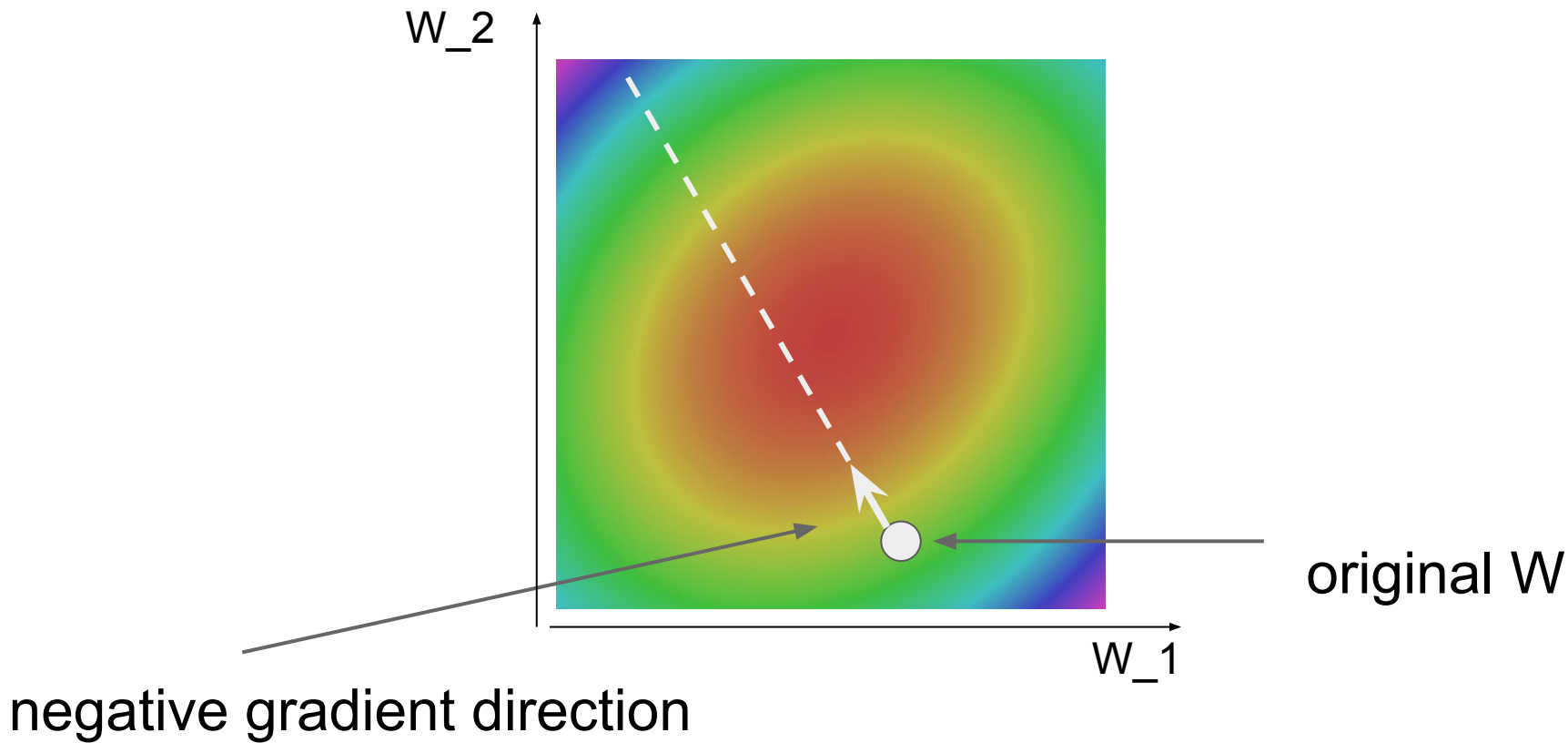
- Numerical gradient: approximate, slow, easy to write
- Analytic gradient: exact, fast, error-prone

=>

In practice: Always use analytic gradient, but check implementation with numerical gradient. This is called a **gradient check**.

Gradient Descent

```
# Vanilla Gradient Descent  
  
while True:  
    weights_grad = evaluate_gradient(loss_fun, data, weights)  
    weights += - step_size * weights_grad # perform parameter update
```



Stochastic Gradient Descent (SGD)

$$L(W) = \frac{1}{N} \sum_{i=1}^N L_i(x_i, y_i, W) + \lambda R(W)$$

$$\nabla_W L(W) = \frac{1}{N} \sum_{i=1}^N \nabla_W L_i(x_i, y_i, W) + \lambda \nabla_W R(W)$$

Full sum expensive
when N is large!

Approximate sum
using a **minibatch** of
examples
32 / 64 / 128 common

```
# Vanilla Minibatch Gradient Descent
```

```
while True:
```

```
    data_batch = sample_training_data(data, 256) # sample 256 examples
```

```
    weights_grad = evaluate_gradient(loss_fun, data_batch, weights)
```

```
    weights += - step_size * weights_grad # perform parameter update
```

Calculating the analytical gradient requires calculus!

$$s = f(x; W) = Wx \quad \text{Linear score function}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM loss (or softmax)}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda \sum_k W_k^2 \quad \text{data loss + regularization}$$

How to find the best W ?

$$\nabla_W L$$

Before we discuss how to calculate gradients analytically,

let's introduce **neural networks**

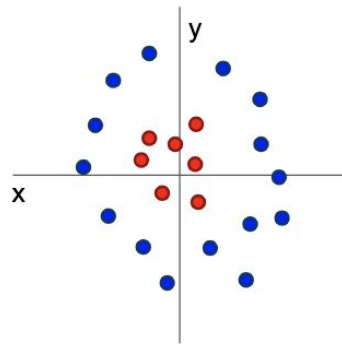
Problem: Linear Classifiers are not very powerful

Visual Viewpoint



Linear classifiers learn
one template per class

Geometric Viewpoint



Linear classifiers
can only draw linear
decision boundaries

Pixel Features



Class
scores

$$f(x) = Wx$$



Image Features

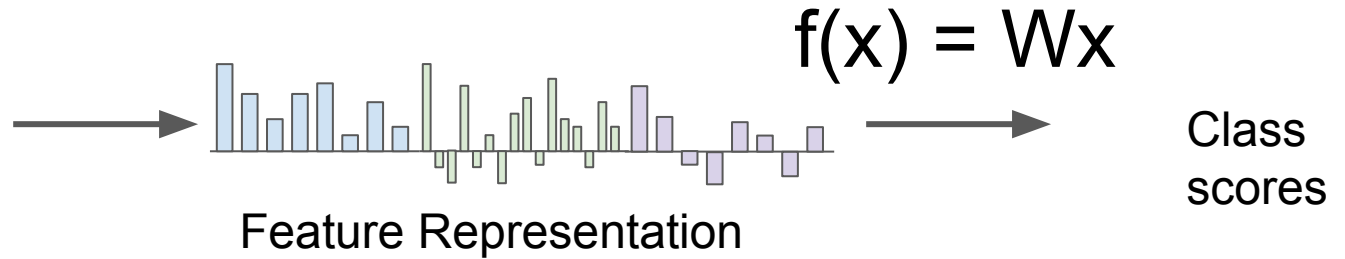
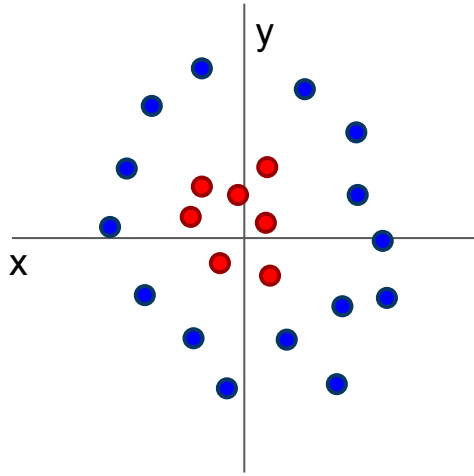
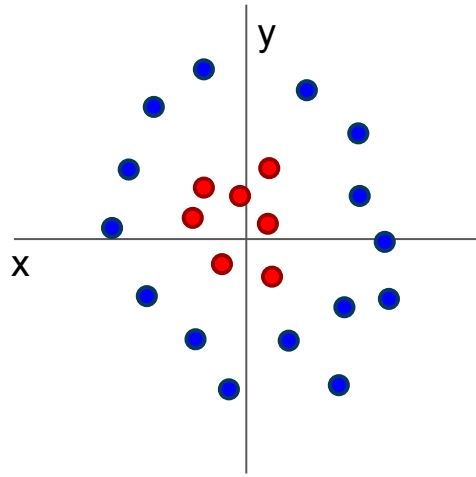


Image Features: Motivation



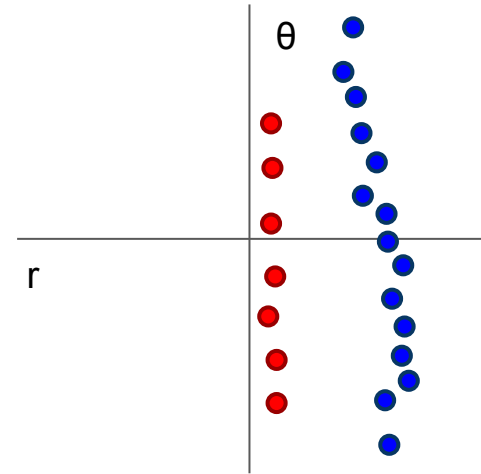
Cannot separate red
and blue points with
linear classifier

Features become linearly separable through a non-linear transformation



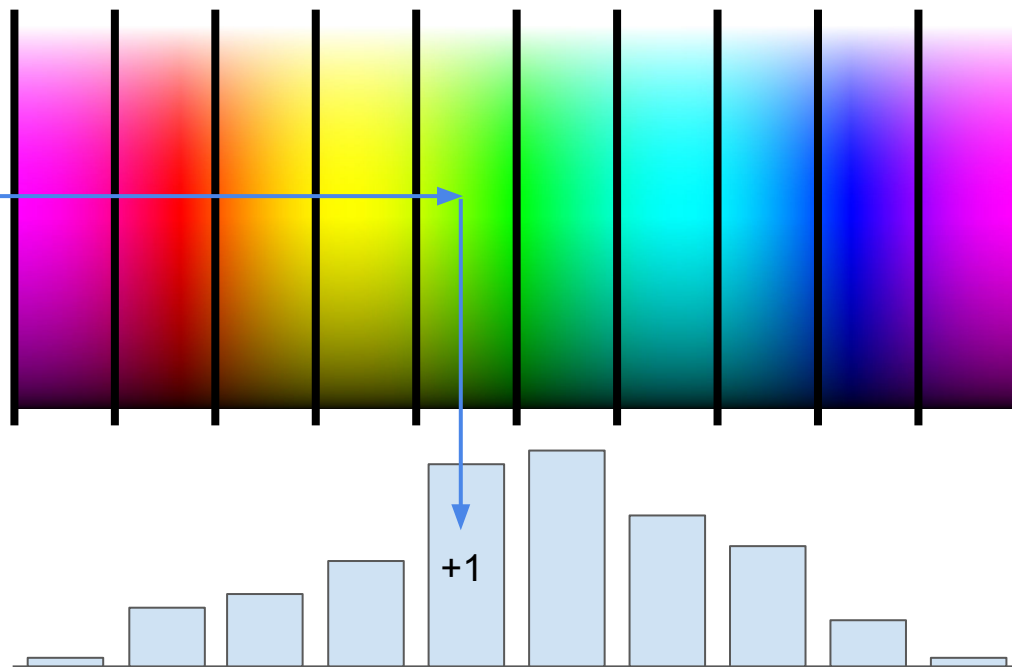
Cannot separate red and blue points with linear classifier

$$f(x, y) = (r(x, y), \theta(x, y))$$



After applying feature transform, points can be separated by linear classifier

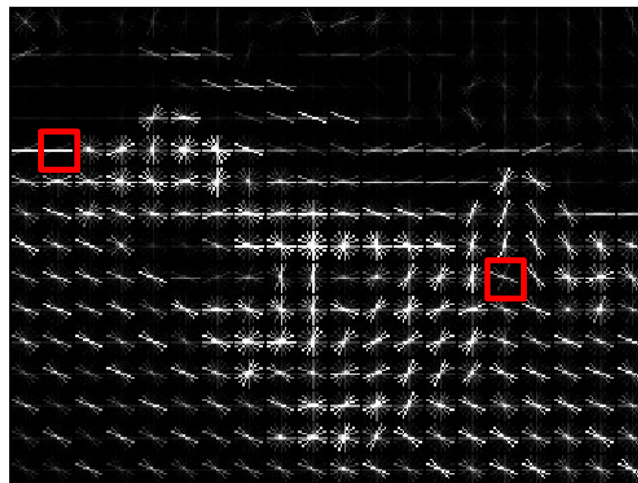
Example: Color Histogram



Example: Histogram of Oriented Gradients (HoG)



Divide image into 8x8 pixel regions
Within each region quantize edge
direction into 9 bins



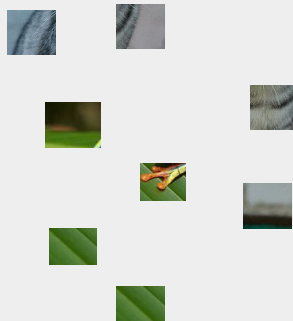
Example: 320x240 image gets divided
into 40x30 bins; in each bin there are
9 numbers so feature vector has
 $30 \times 40 \times 9 = 10,800$ numbers

Example: Bag of Words

Step 1: Build codebook



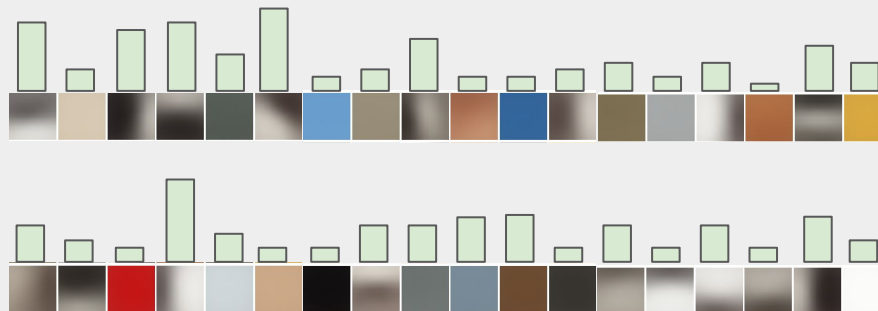
Extract random patches



Cluster patches to form "codebook" of "visual words"



Step 2: Encode images



Fei-Fei and Perona, "A bayesian hierarchical model for learning natural scene categories", CVPR 2005

Combine many different features if unsure which features are better

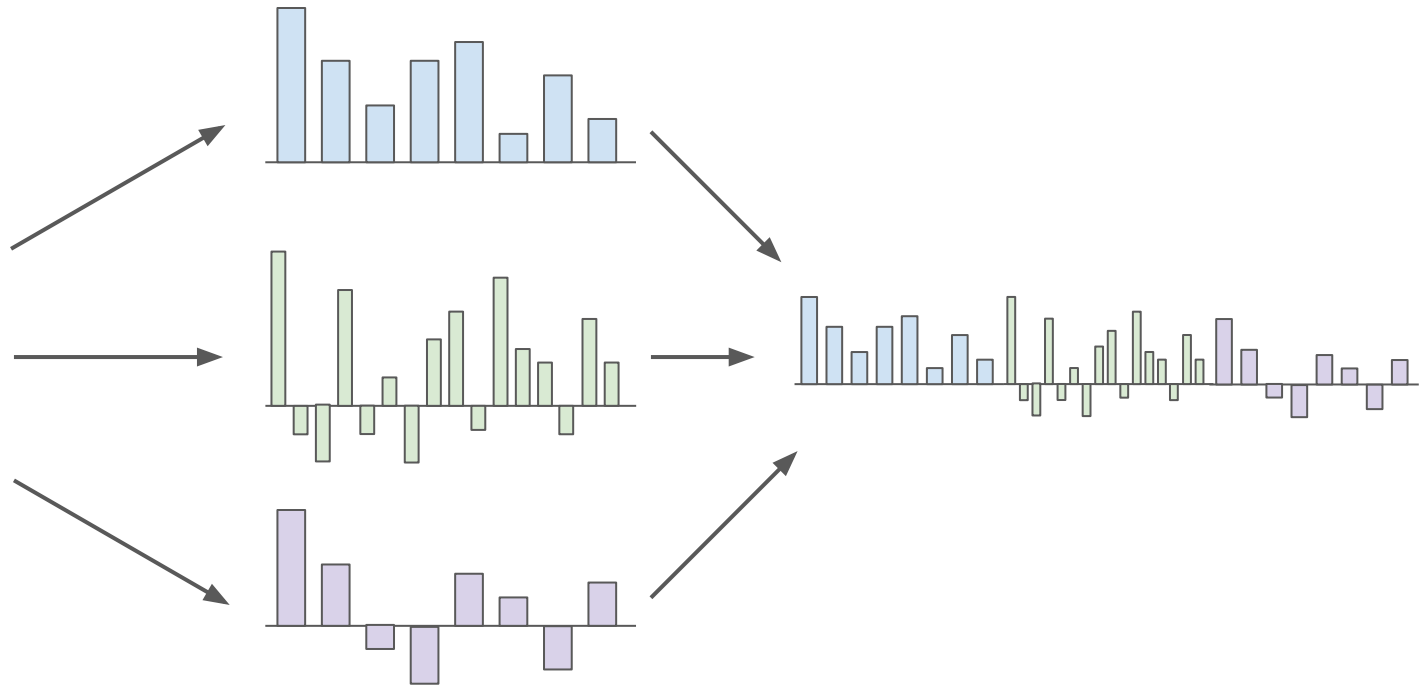
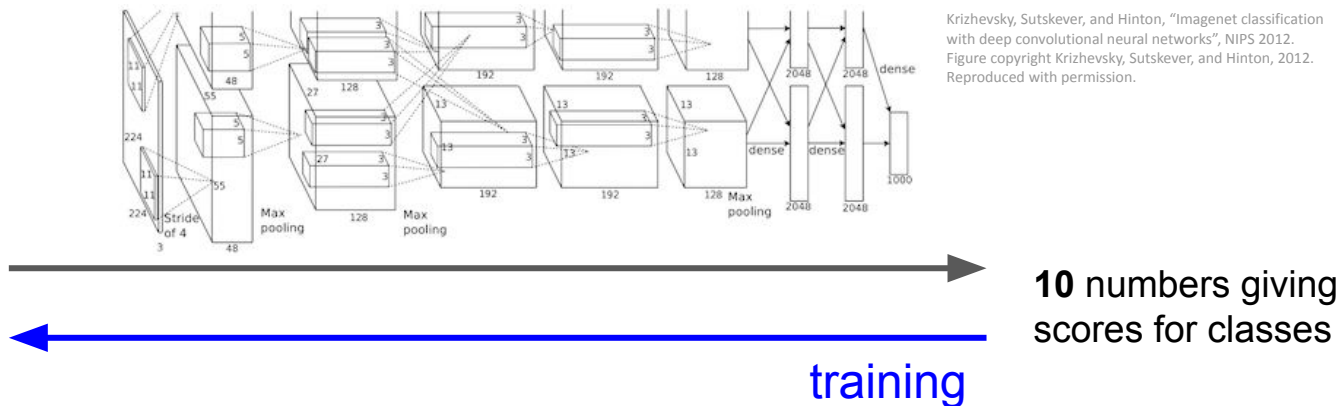
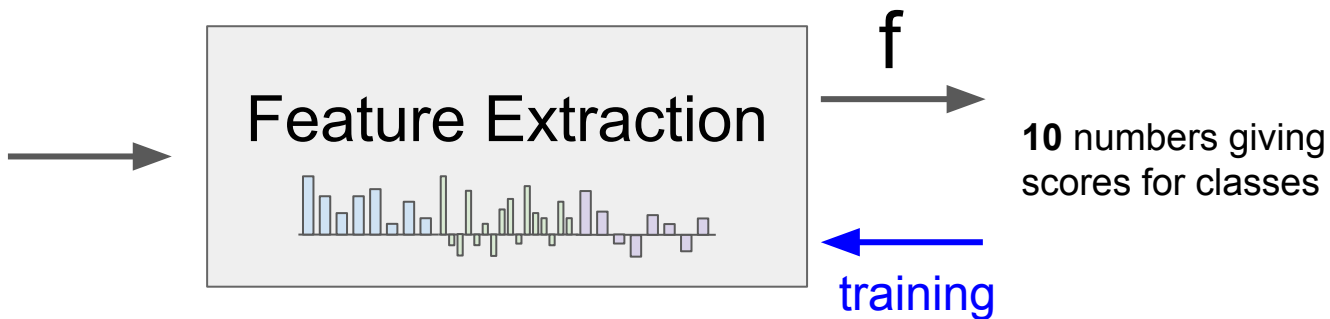
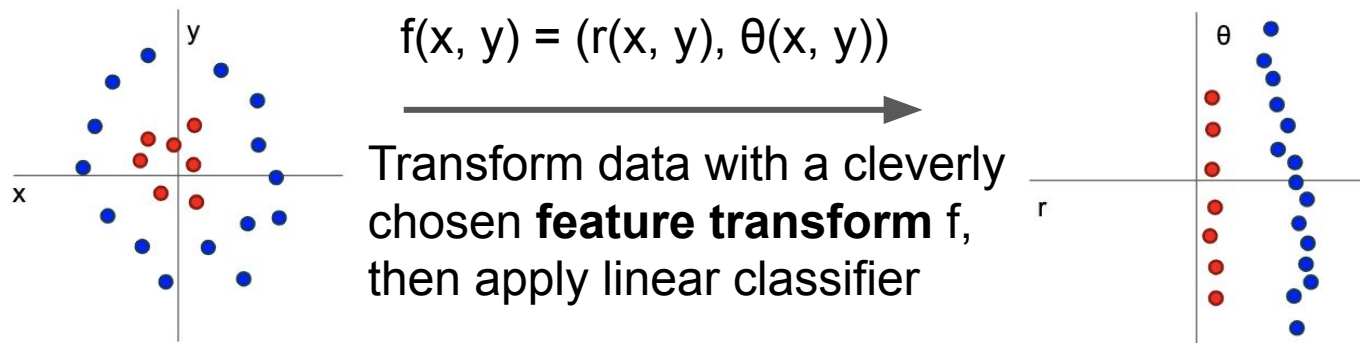


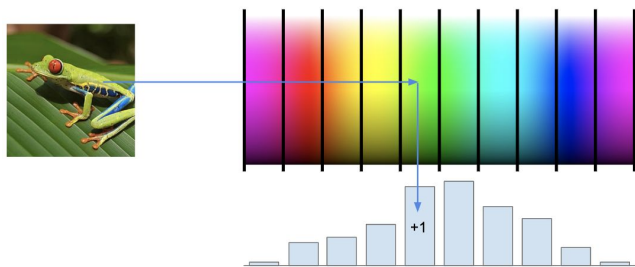
Image features vs neural networks



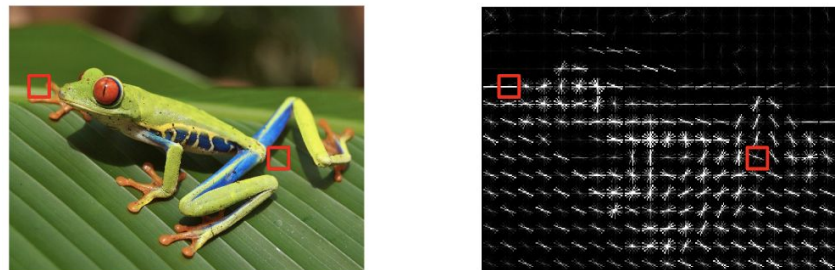
One Solution: Non-linear feature transformation



Color Histogram



Histogram of Oriented Gradients (HoG)



Neural Networks

Neural networks: the original linear classifier

(**Before**) Linear score function: $f = Wx$

$$x \in \mathbb{R}^D, W \in \mathbb{R}^{C \times D}$$

Neural networks: 2 layers

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

(In practice we will usually add a learnable bias at each layer as well)

Neural networks: also called fully connected network

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

“Neural Network” is a very broad term; these are more accurately called “fully-connected networks” or sometimes “multi-layer perceptrons” (**MLP**)

(In practice we will usually add a learnable bias at each layer as well)

Neural networks: 3 layers

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network
or 3-layer Neural Network $f = W_2 \max(0, W_1 x)$

$$f = W_3 \max(0, W_2 \max(0, W_1 x))$$

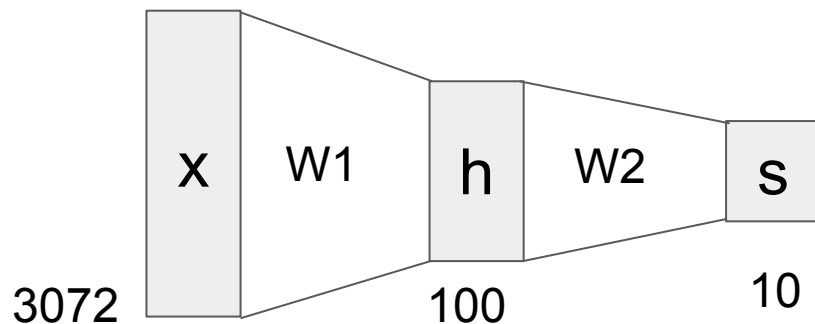
$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H_1 \times D}, W_2 \in \mathbb{R}^{H_2 \times H_1}, W_3 \in \mathbb{R}^{C \times H_2}$$

(In practice we will usually add a learnable bias at each layer as well)

Neural networks: hierarchical computation

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

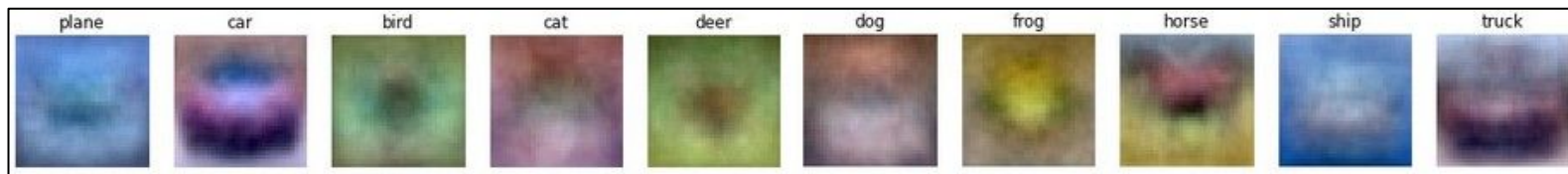
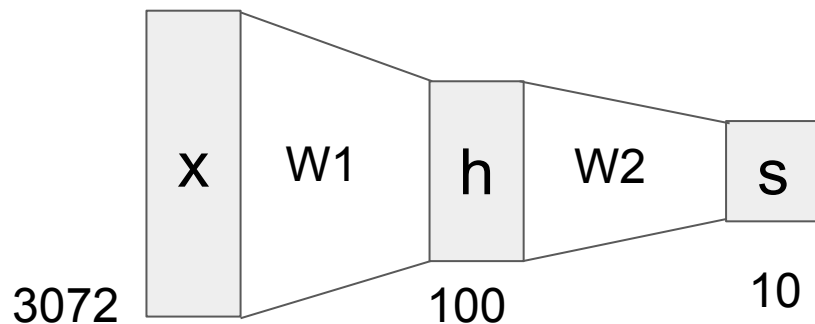


$$x \in \mathbb{R}^D, W_1 \in \mathbb{R}^{H \times D}, W_2 \in \mathbb{R}^{C \times H}$$

Neural networks: learning 100s of templates

(**Before**) Linear score function: $f = Wx$

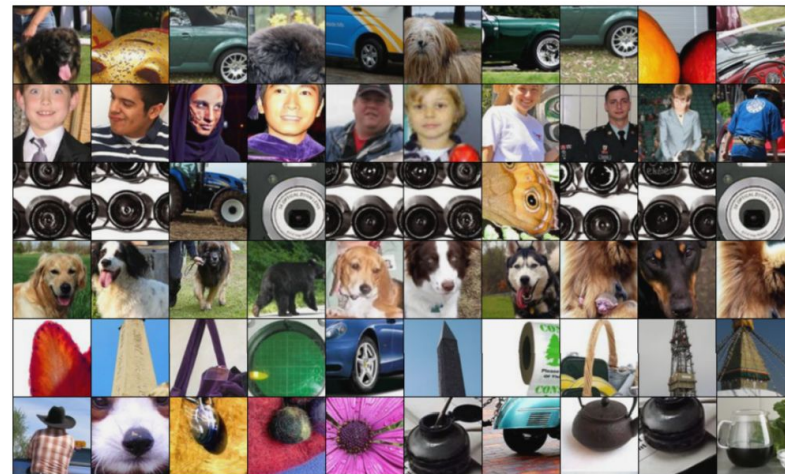
(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$



Learn 100 templates instead of 10.

Share templates between classes

Examples of templates from real neural networks



Springenberg et al, "Striving for Simplicity: The All Convolutional Net", ICLR Workshop 2015
Figure copyright Jost Tobias Springenberg, Alexey Dosovitskiy, Thomas Brox, Martin Riedmiller, 2015; reproduced with permission.

Neural networks: why is max operator important?

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

The function $\max(0, z)$ is called the **activation function**.

Q: What if we try to build a neural network without one?

$$f = W_2 W_1 x$$

Neural networks: why is max operator important?

(**Before**) Linear score function: $f = Wx$

(**Now**) 2-layer Neural Network $f = W_2 \max(0, W_1 x)$

The function $\max(0, z)$ is called the **activation function**.

Q: What if we try to build a neural network without one?

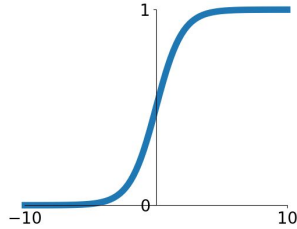
$$f = W_2 W_1 x \quad W_3 = W_2 W_1 \in \mathbb{R}^{C \times H}, f = W_3 x$$

A: We end up with a linear classifier again!

Activation functions

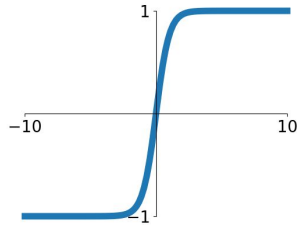
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



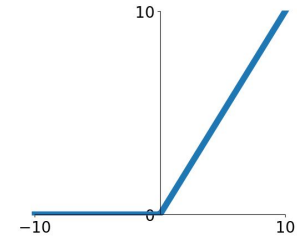
tanh

$$\tanh(x)$$



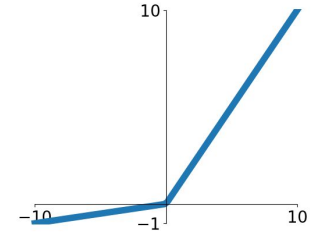
ReLU

$$\max(0, x)$$



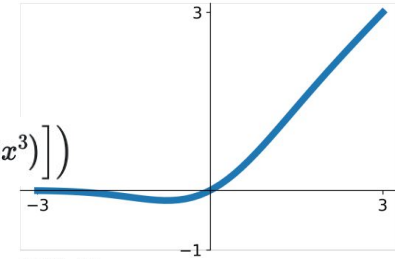
Leaky ReLU

$$\max(0.1x, x)$$



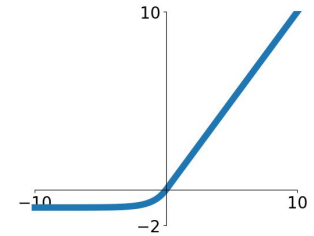
GeLU

$$0.5x \left(1 + \tanh \left[\sqrt{2/\pi} (x + 0.044715x^3) \right] \right)$$



ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$

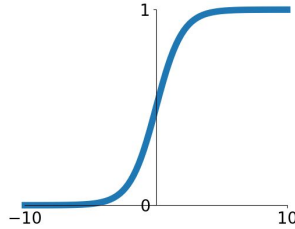


Activation functions

ReLU is a good default choice for most problems

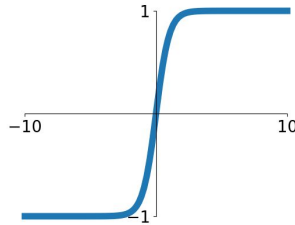
Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



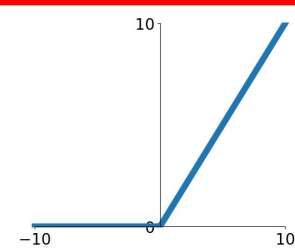
tanh

$$\tanh(x)$$



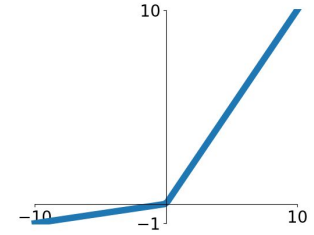
ReLU

$$\max(0, x)$$



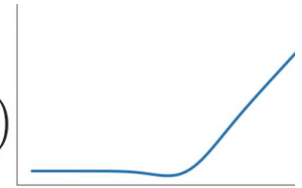
Leaky ReLU

$$\max(0.1x, x)$$



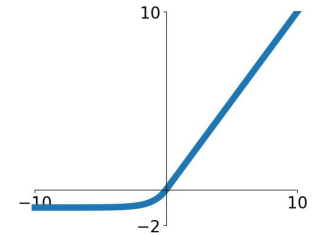
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ELU

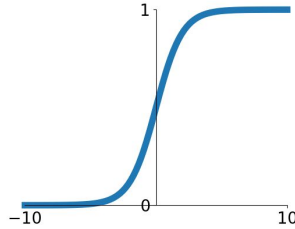
$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



Activation functions

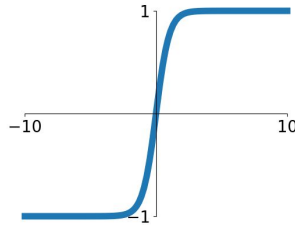
Sigmoid

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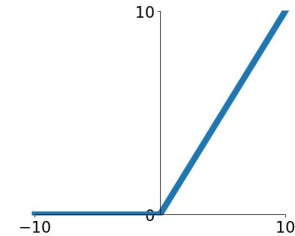
tanh

$$\tanh(x)$$



ReLU

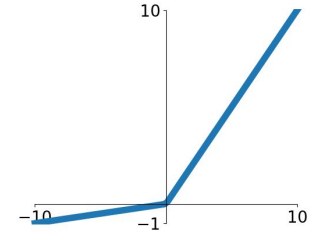
$$\max(0, x)$$



GeLU is the default choice
for transformers

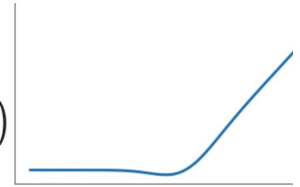
Leaky ReLU

$$\max(0.1x, x)$$



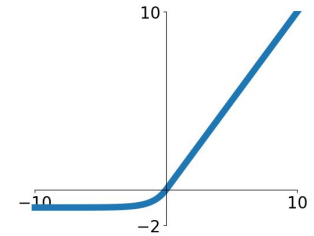
GeLU

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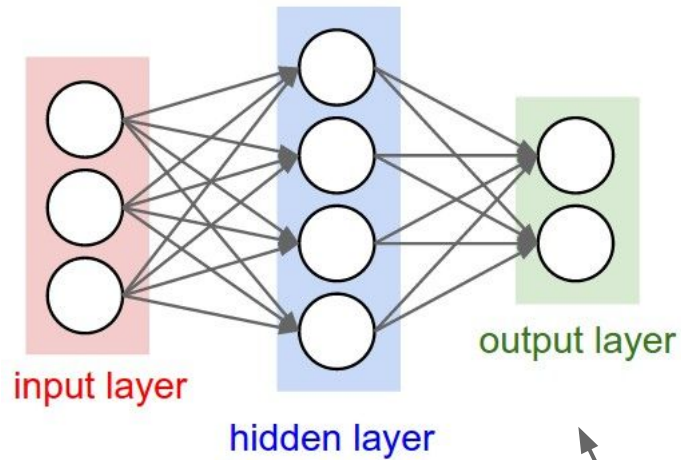


ELU

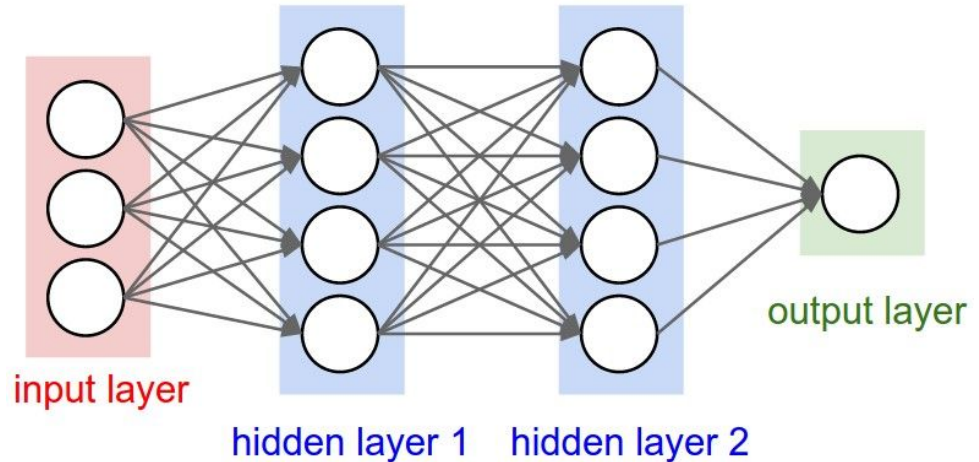
$$\begin{cases} x & x \geq 0 \\ \alpha(e^x - 1) & x < 0 \end{cases}$$



Neural networks: Architectures



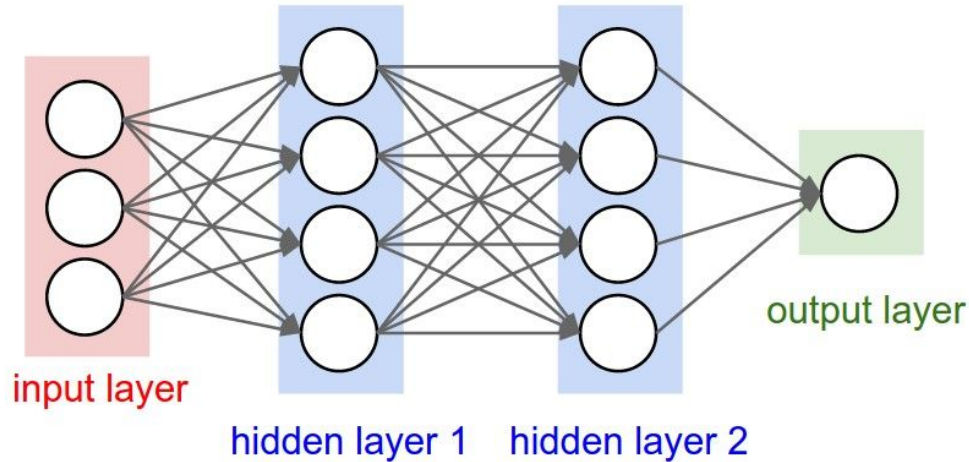
"2-layer Neural Net", or
"1-hidden-layer Neural Net"



"3-layer Neural Net", or
"2-hidden-layer Neural Net"

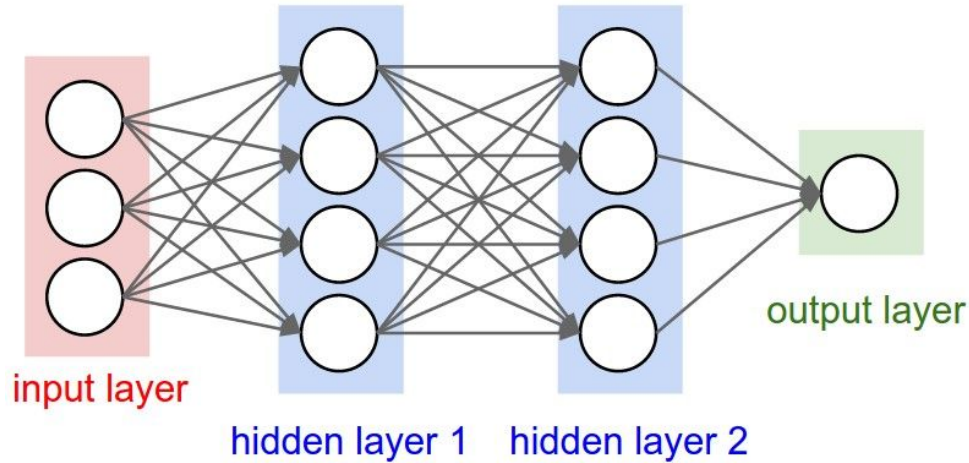
"Fully-connected" layers

Example feed-forward computation of a neural network



```
# forward-pass of a 3-layer neural network:  
f = lambda x: 1.0/(1.0 + np.exp(-x)) # activation function (use sigmoid)  
x = np.random.randn(3, 1) # random input vector of three numbers (3x1)  
h1 = f(np.dot(W1, x) + b1) # calculate first hidden layer activations (4x1)  
h2 = f(np.dot(W2, h1) + b2) # calculate second hidden layer activations (4x1)  
out = np.dot(W3, h2) + b3 # output neuron (1x1)
```

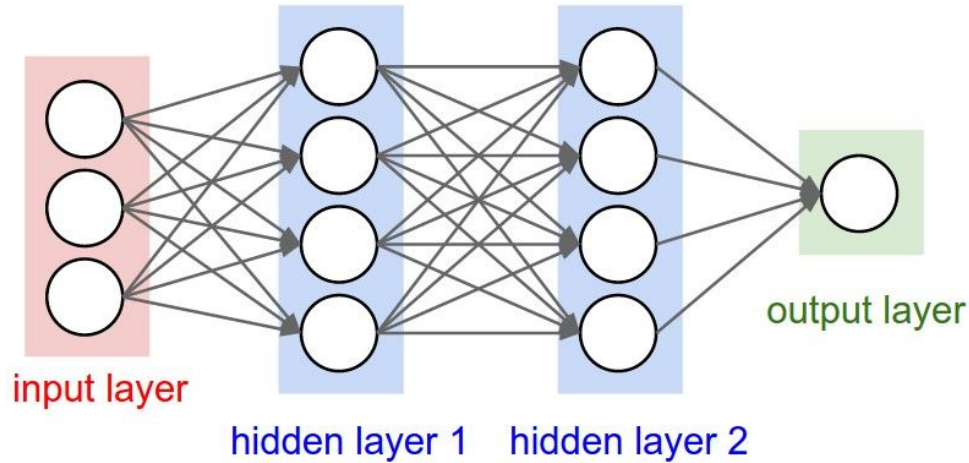
Example feed-forward computation of a neural network



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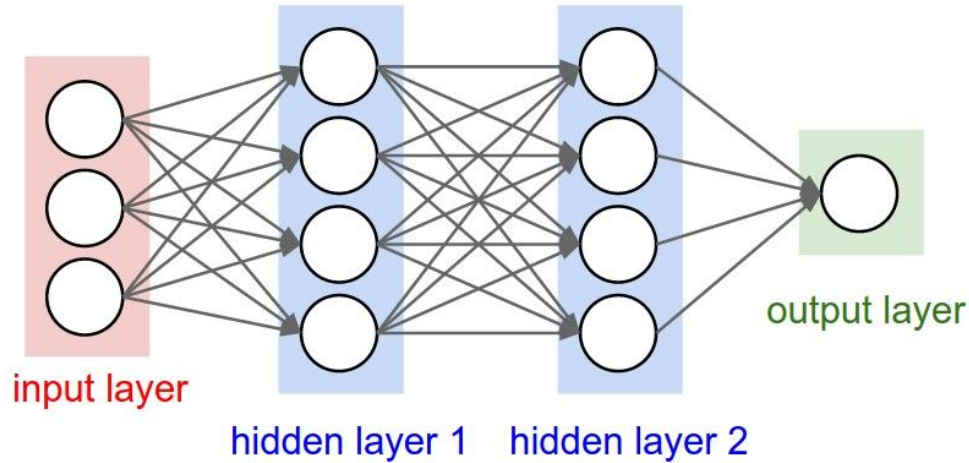
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Example feed-forward computation of a neural network



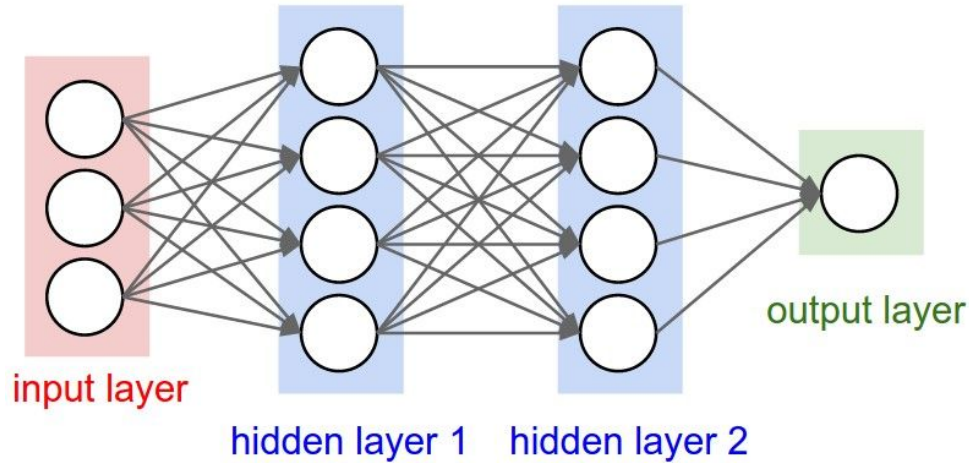
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Example feed-forward computation of a neural network



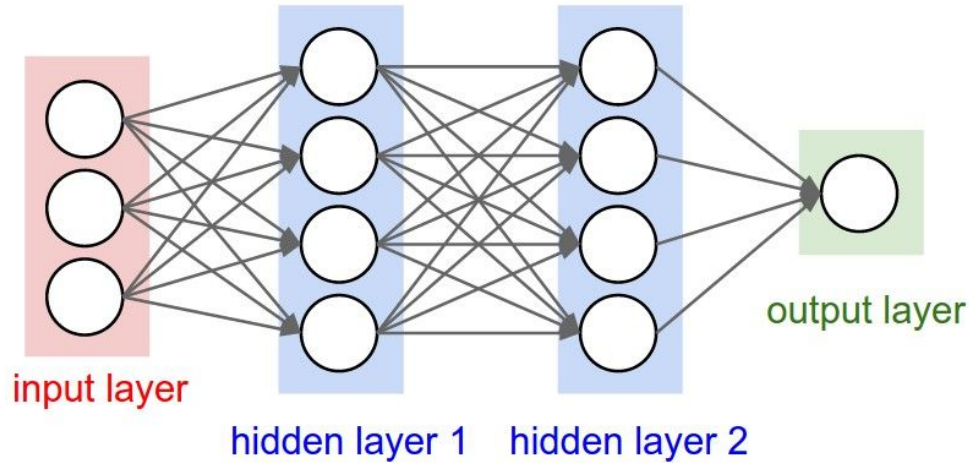
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Example feed-forward computation of a neural network



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Example feed-forward computation of a neural network



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out = np.dot(W3, h2) + b3 # output neuron (1x1)
```

Full implementation of training a 2-layer Neural Network needs ~20 lines:

```
1 import numpy as np
2 from numpy.random import randn
3
4 N, D_in, H, D_out = 64, 1000, 100, 10
5 x, y = randn(N, D_in), randn(N, D_out)
6 w1, w2 = randn(D_in, H), randn(H, D_out)
7
8 for t in range(2000):
9     h = 1 / (1 + np.exp(-x.dot(w1)))
10    y_pred = h.dot(w2)
11    loss = np.square(y_pred - y).sum()
12    print(t, loss)
13
14    grad_y_pred = 2.0 * (y_pred - y)
15    grad_w2 = h.T.dot(grad_y_pred)
16    grad_h = grad_y_pred.dot(w2.T)
17    grad_w1 = x.T.dot(grad_h * h * (1 - h))
18
19    w1 -= 1e-4 * grad_w1
20    w2 -= 1e-4 * grad_w2
```

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15    grad_w2 = h.T.dot(grad_y_pred)
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17    grad_w1 = x.T.dot(grad_h * h * (1 - h))
18
19    w1 -= 1e-4 * grad_w1
20    w2 -= 1e-4 * grad_w2
```

Define the network

Full implementation of training a 2-layer Neural Network needs ~20 lines:

```
1 import numpy as np
2 from numpy.random import randn
3
4 N, D_in, H, D_out = 64, 1000, 100, 10
5 x, y = randn(N, D_in), randn(N, D_out)
6 w1, w2 = randn(D_in, H), randn(H, D_out)
```

Define the network

```
7
8 for t in range(2000):
9     h = 1 / (1 + np.exp(-x.dot(w1)))
10    y_pred = h.dot(w2)
11    loss = np.square(y_pred - y).sum()
12    print(t, loss)
13
14    grad_y_pred = 2.0 * (y_pred - y)
15    grad_w2 = h.T.dot(grad_y_pred)
16    grad_h = grad_y_pred.dot(w2.T)
17    grad_w1 = x.T.dot(grad_h * h * (1 - h))
18
19    w1 -= 1e-4 * grad_w1
20    w2 -= 1e-4 * grad_w2
```

Forward pass

Full implementation of training a 2-layer Neural Network needs ~20 lines:

```
1 import numpy as np
2 from numpy.random import randn
3
4 N, D_in, H, D_out = 64, 1000, 100, 10
5 x, y = randn(N, D_in), randn(N, D_out)
6 w1, w2 = randn(D_in, H), randn(H, D_out)
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Define the network

Forward pass

Calculate the analytical gradients

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```

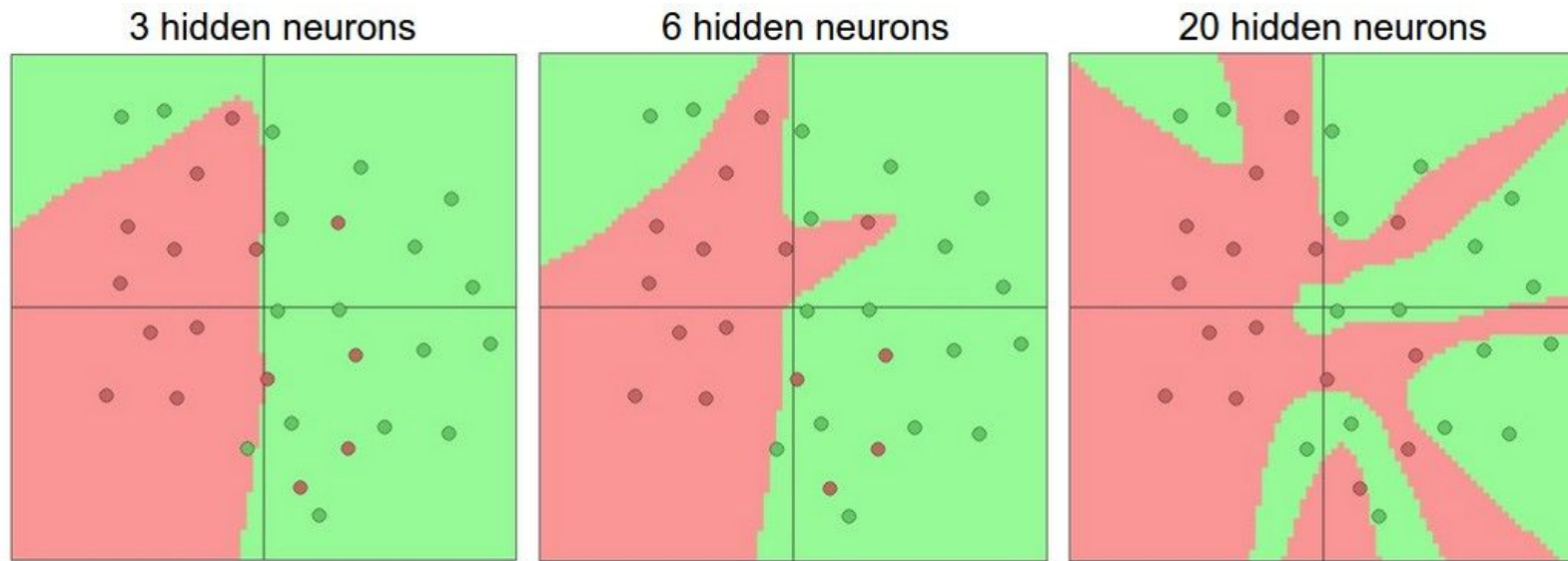
Define the network

Forward pass

Calculate the analytical gradients

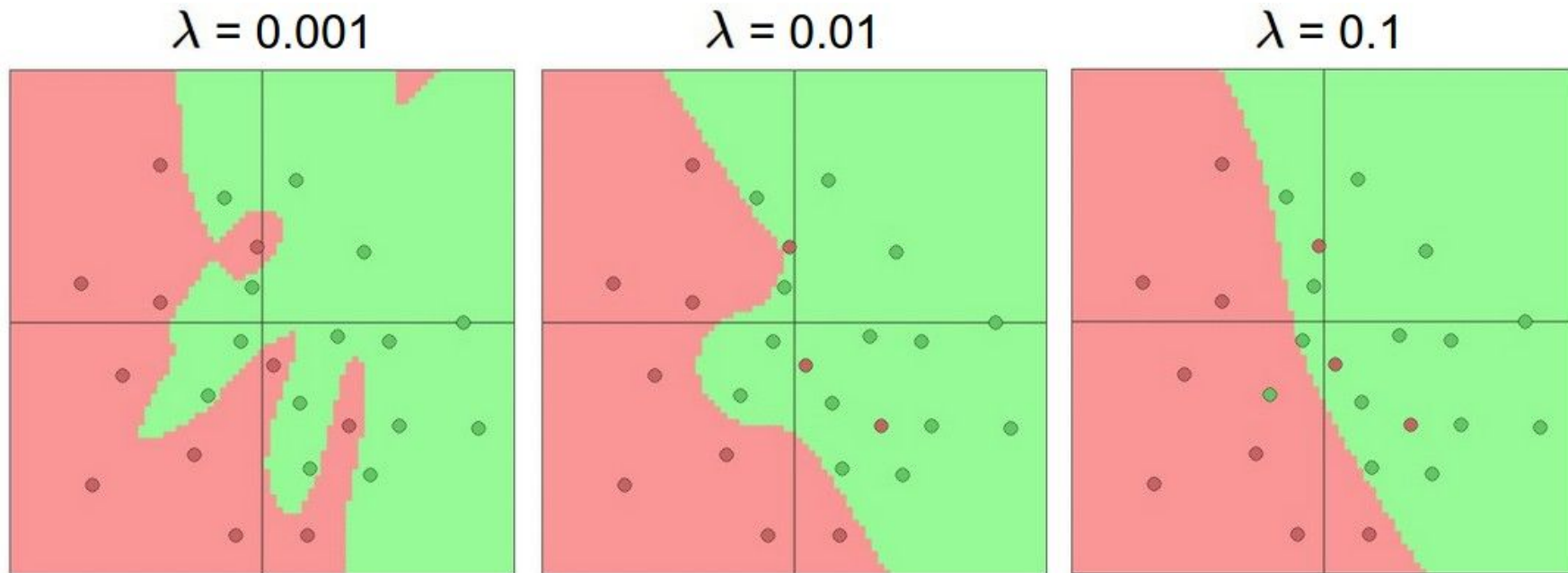
Gradient descent

Setting the number of layers and their sizes



more neurons = more capacity

Do not use size of neural network as a regularizer. Use stronger regularization instead:

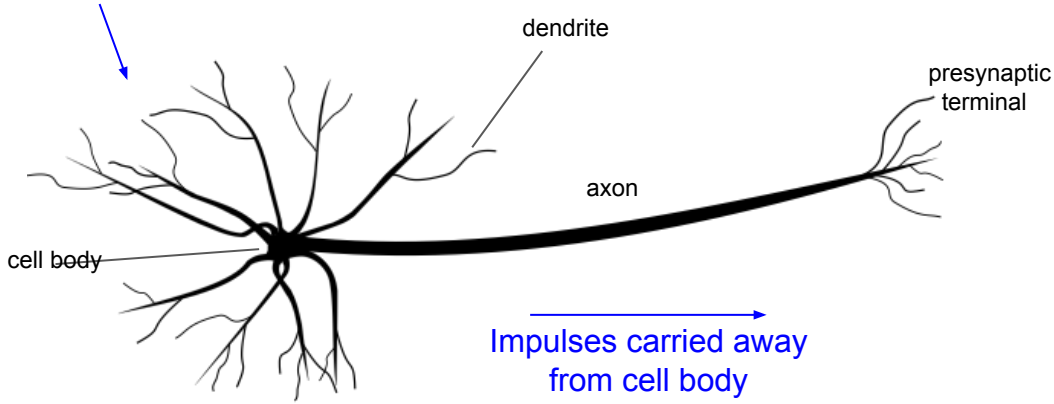


$$L(W) = \frac{1}{N} \sum_{i=1}^N L_i(f(x_i, W), y_i) + \lambda R(W)$$



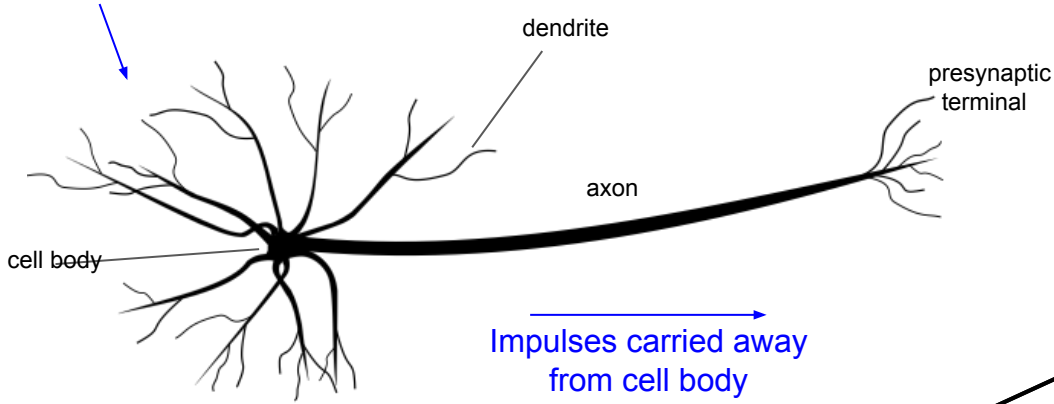
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Impulses carried toward cell body

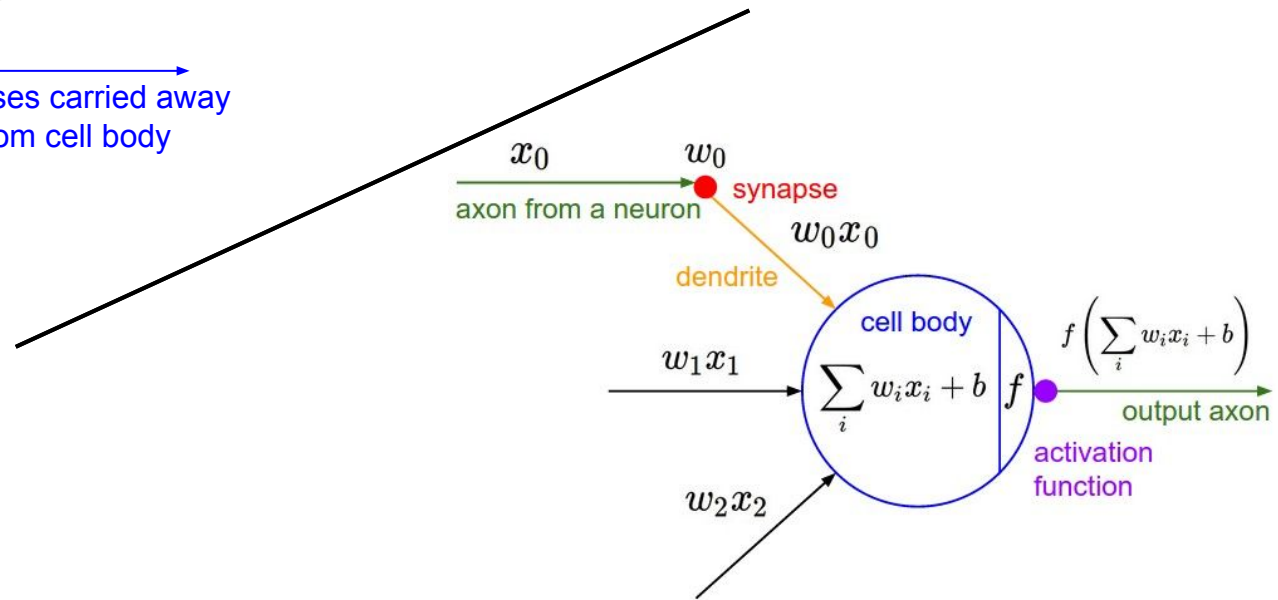


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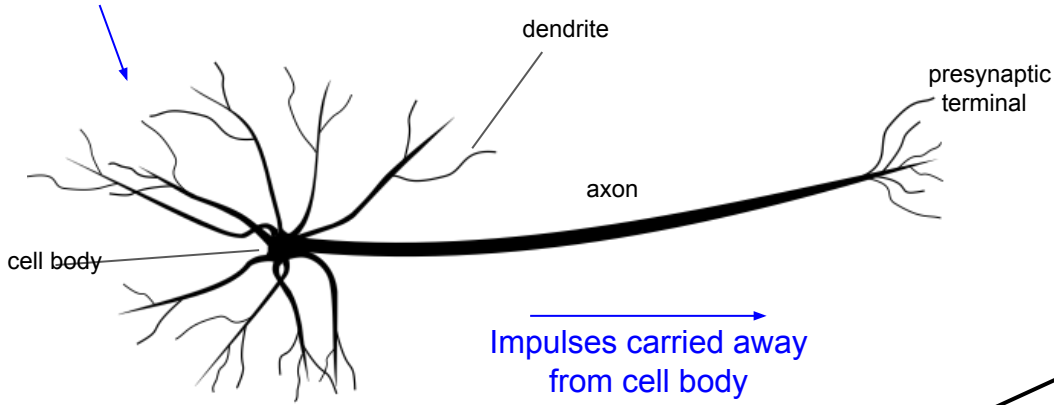
Impulses carried toward cell body



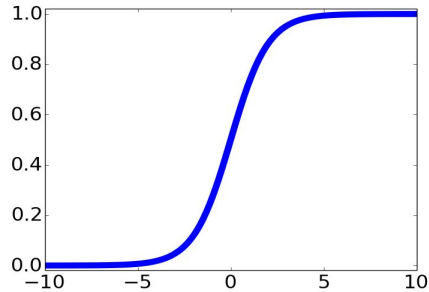
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Impulses carried toward cell body

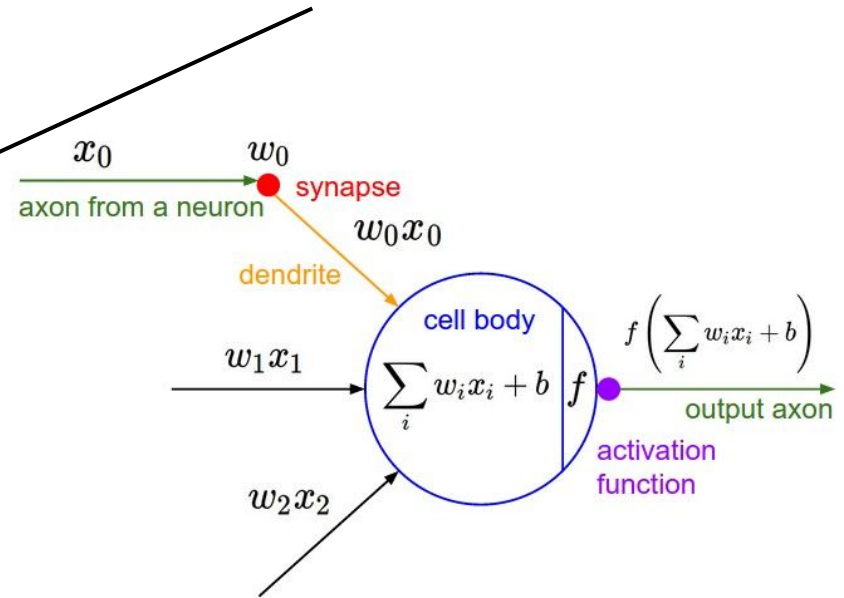


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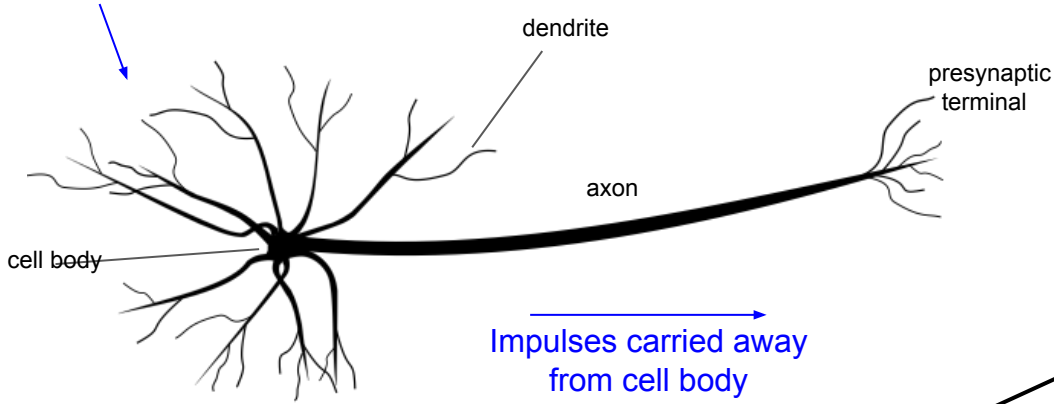


sigmoid activation function

$$\frac{1}{1 + e^{-x}}$$

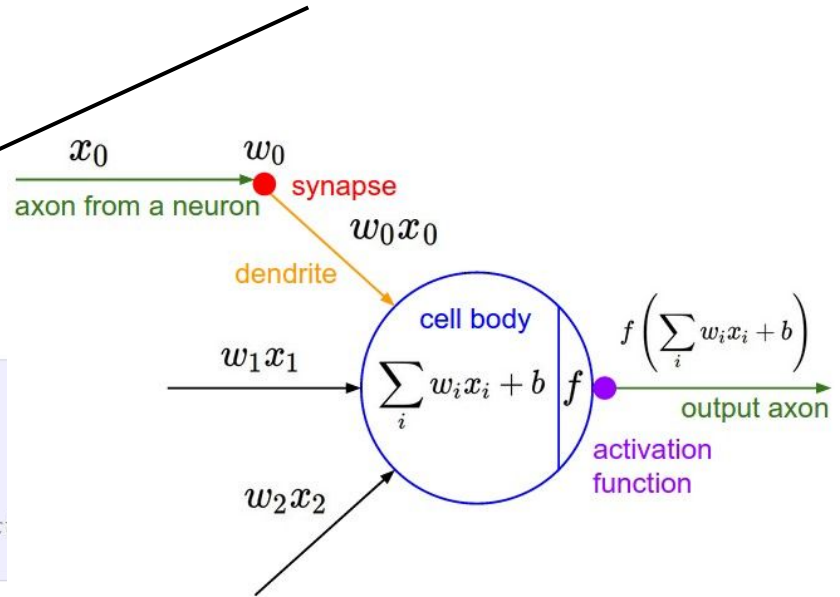


Impulses carried toward cell body

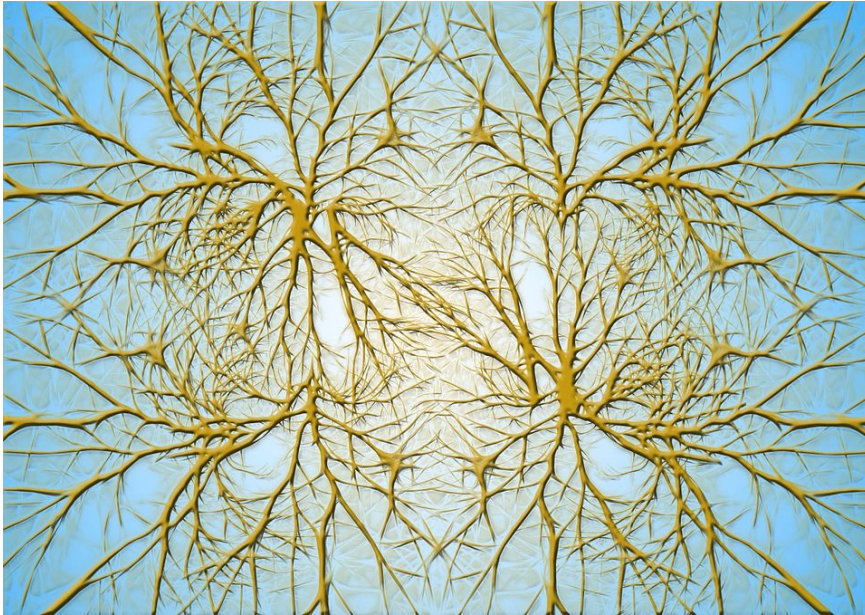


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```
class Neuron:  
    # ...  
    def neuron_tick(inputs):  
        """ assume inputs and weights are 1-D numpy arrays and bias is a number """  
        cell_body_sum = np.sum(inputs * self.weights) + self.bias  
        firing_rate = 1.0 / (1.0 + math.exp(-cell_body_sum)) # sigmoid activation func  
        return firing_rate
```

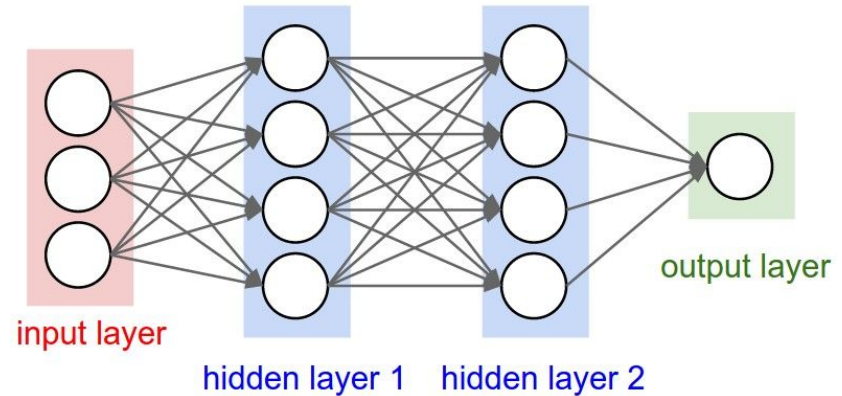


Biological Neurons: Complex connectivity patterns

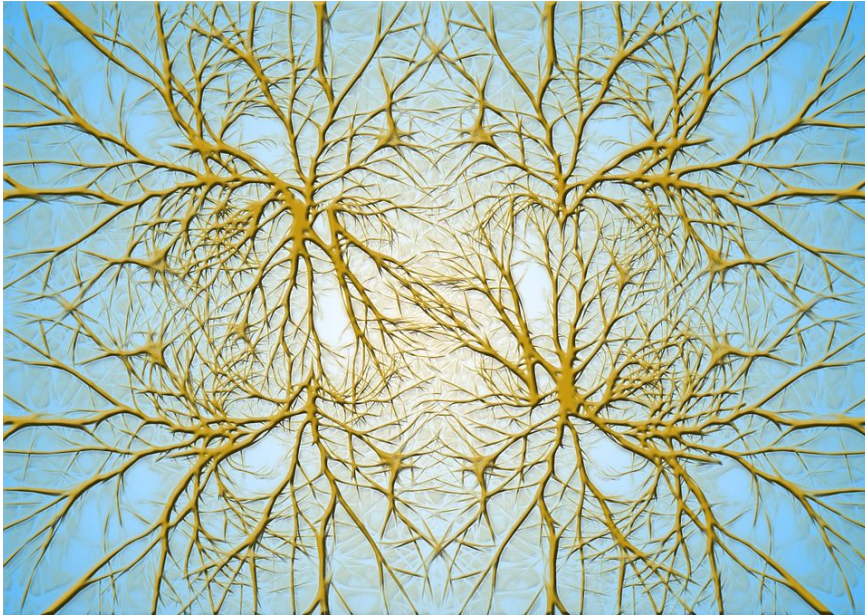


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Neurons in a neural network: Organized into regular layers for computational efficiency

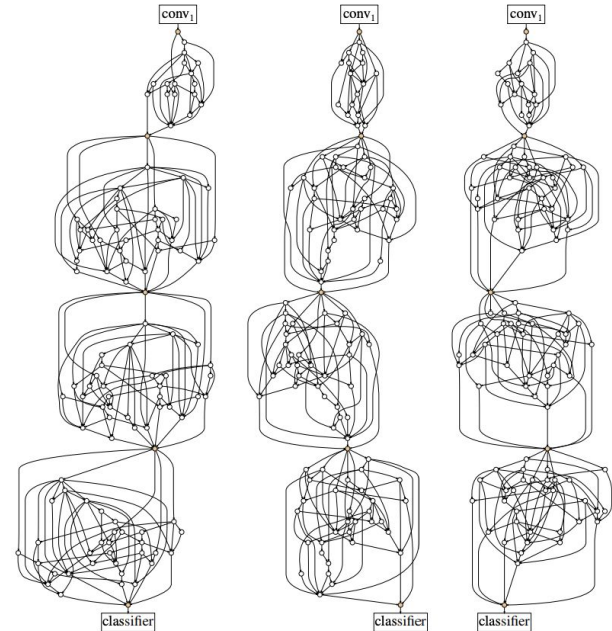


Biological Neurons: Complex connectivity patterns



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But neural networks with random connections can work too!



Xie et al, "Exploring Randomly Wired Neural Networks for Image Recognition", arXiv 2019

Be very careful with your brain analogies!

Biological Neurons:

- Many different types
- Dendrites can perform complex non-linear computations
- Synapses are not a single weight but a complex non-linear dynamical system

[Dendritic Computation. London and Hausser]

Now let's calculate the analytical gradients

Plugging in neural networks with loss functions

$$s = f(x; W_1, W_2) = W_2 \max(0, W_1 x) \quad \text{Nonlinear score function}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM Loss on predictions}$$

$$R(W) = \sum_k W_k^2 \quad \text{Regularization}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda R(W_1) + \lambda R(W_2) \quad \text{Total loss: data loss + regularization}$$

Problem: How to compute gradients?

$$s = f(x; W_1, W_2) = W_2 \max(0, W_1 x) \quad \text{Nonlinear score function}$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1) \quad \text{SVM Loss on predictions}$$

$$R(W) = \sum_k W_k^2 \quad \text{Regularization}$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda R(W_1) + \lambda R(W_2) \quad \text{Total loss: data loss + regularization}$$

If we can compute $\frac{\partial L}{\partial W_1}$, $\frac{\partial L}{\partial W_2}$ then we can learn W_1 and W_2

(Bad) Idea: Derive $\nabla_W L$ on paper

$$s = f(x; W) = Wx$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$= \sum_{j \neq y_i} \max(0, W_{j,:} \cdot x + W_{y_i,:} \cdot x + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + \lambda \sum_k W_k^2$$

$$= \frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, W_{j,:} \cdot x + W_{y_i,:} \cdot x + 1) + \lambda \sum_k W_k^2$$

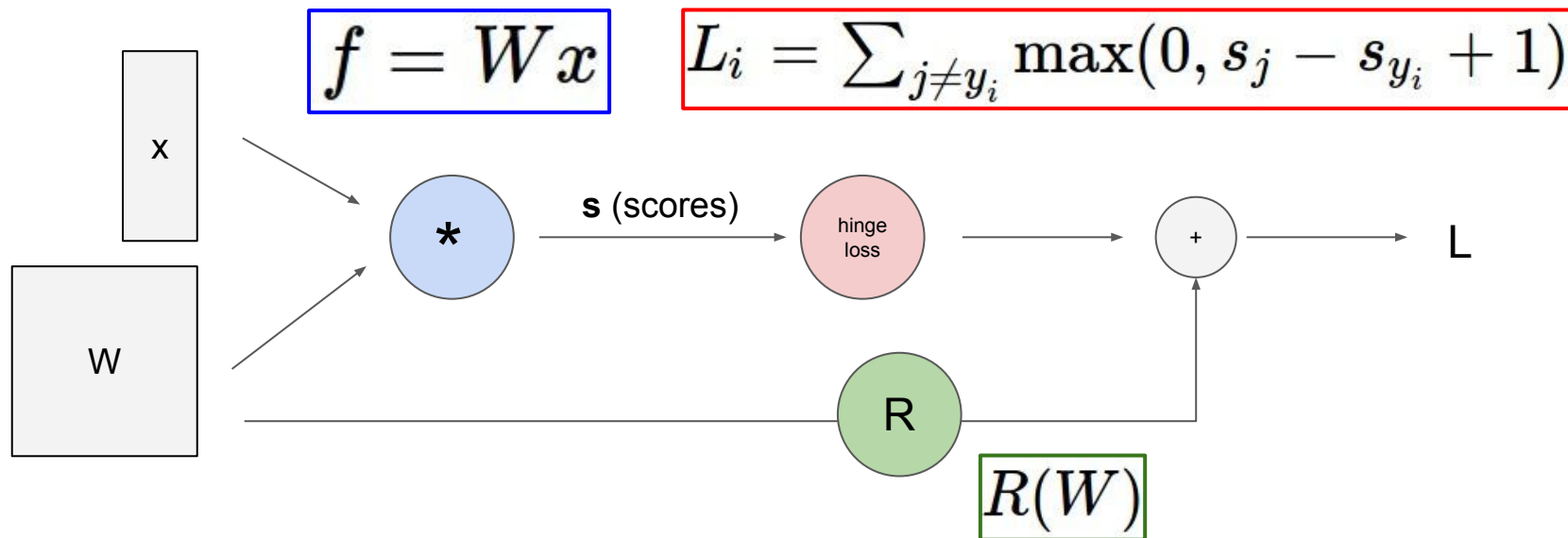
$$\nabla_W L = \nabla_W \left(\frac{1}{N} \sum_{i=1}^N \sum_{j \neq y_i} \max(0, W_{j,:} \cdot x + W_{y_i,:} \cdot x + 1) + \lambda \sum_k W_k^2 \right)$$

Problem: Very tedious: Lots of matrix calculus, need lots of paper

Problem: What if we want to change loss? E.g. use softmax instead of SVM? Need to re-derive from scratch = (

Problem: Not feasible for very complex models!

Better Idea: Computational graphs + Backpropagation



Convolutional network (AlexNet)

input image

weights

loss

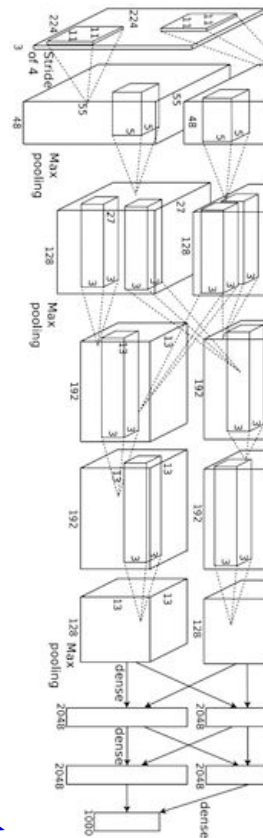


Figure copyright Alex Krizhevsky, Ilya Sutskever, and Geoffrey Hinton, 2012. Reproduced with permission.

Really complex neural networks!!

input image

loss

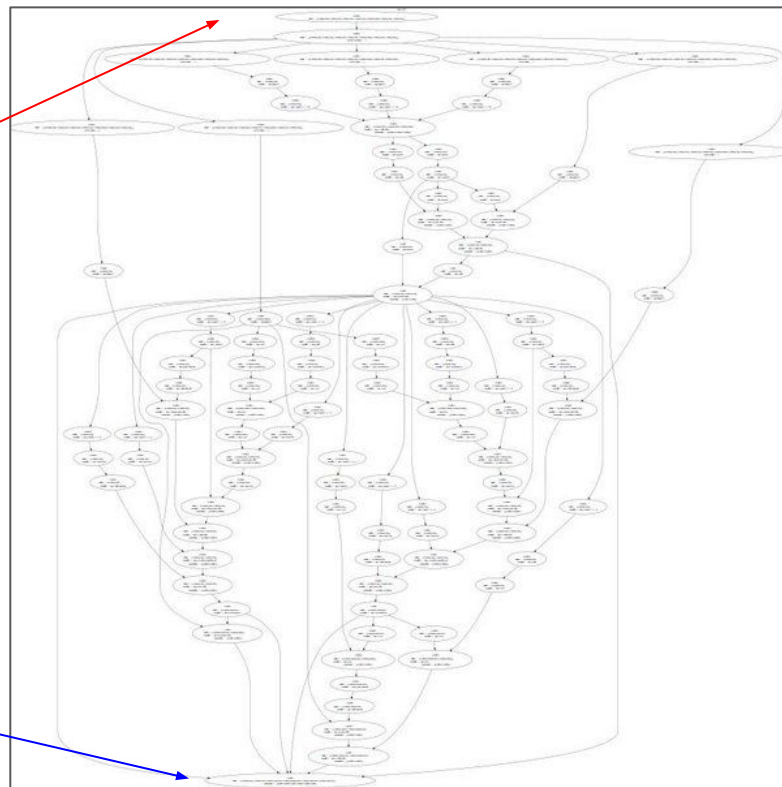


Figure reproduced with permission from a [Twitter post](#) by Andrej Karpathy.

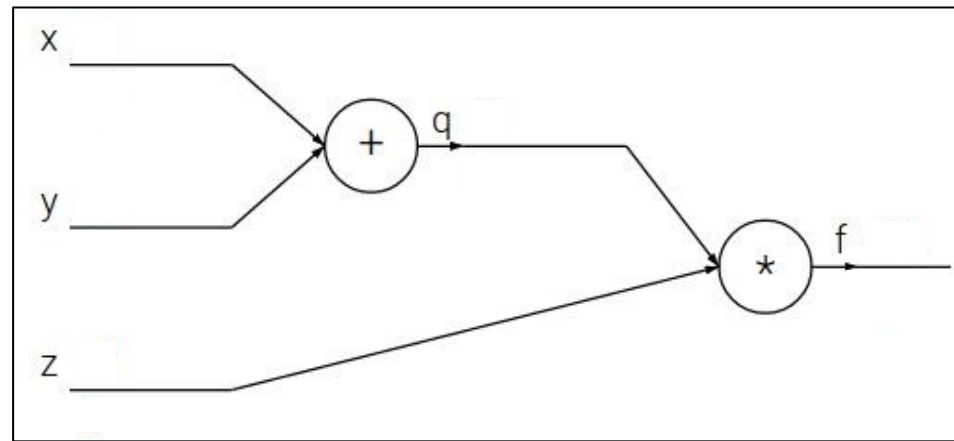
Solution: Backpropagation

Backpropagation: a simple example

$$f(x, y, z) = (x + y)z$$

Backpropagation: a simple example

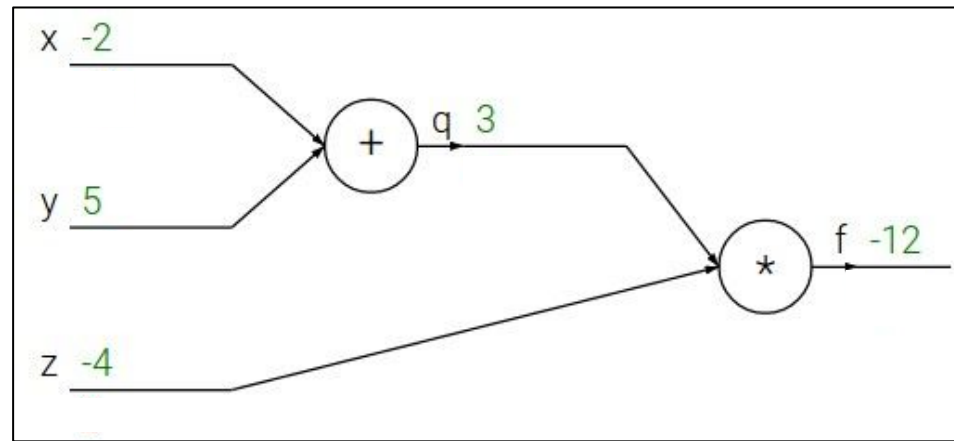
$$f(x, y, z) = (x + y)z$$



Backpropagation: a simple example

$$f(x, y, z) = (x + y)z$$

e.g. $x = -2, y = 5, z = -4$

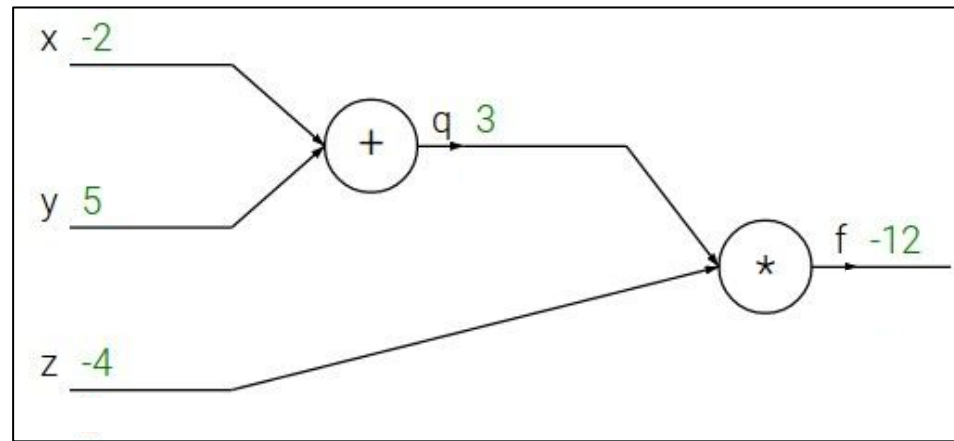


Backpropagation: a simple example

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e.g. $x = -2, y = 5, z = -4$

$$q = x + y \quad \frac{\partial q}{\partial x} = 1, \frac{\partial q}{\partial y} = 1$$



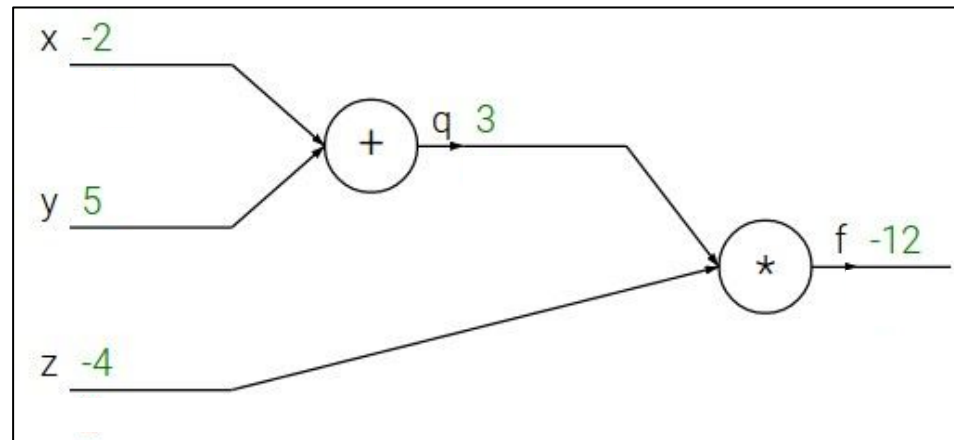
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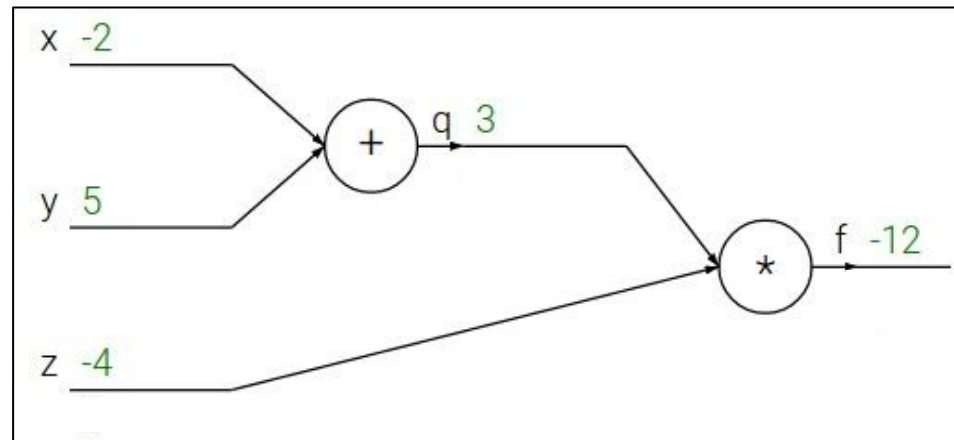
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Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



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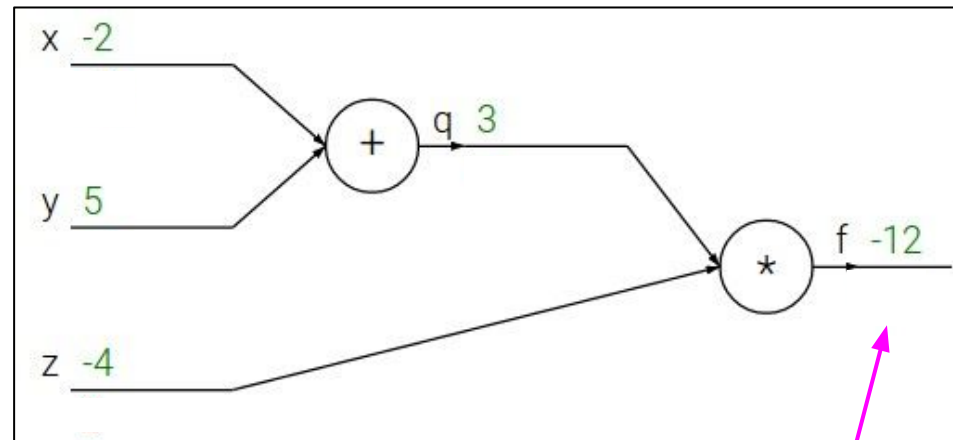
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$$\frac{\partial f}{\partial f}$$

Backpropagation: a simple example

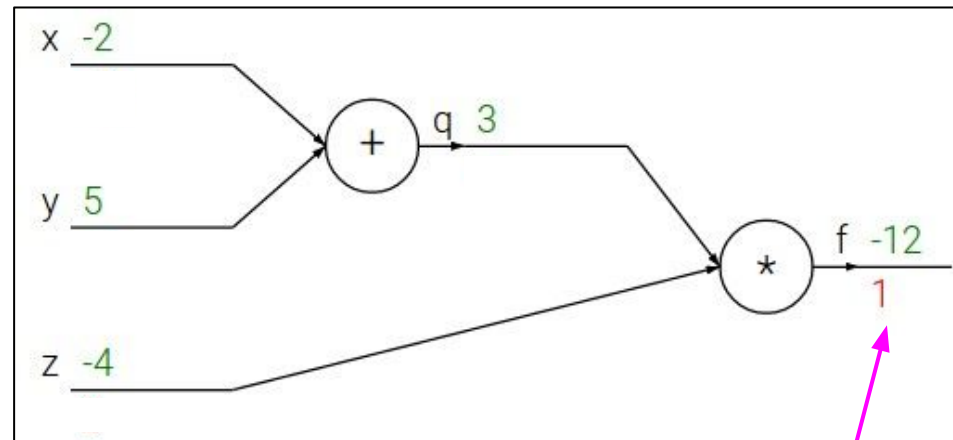
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$$\frac{\partial f}{\partial f}$$

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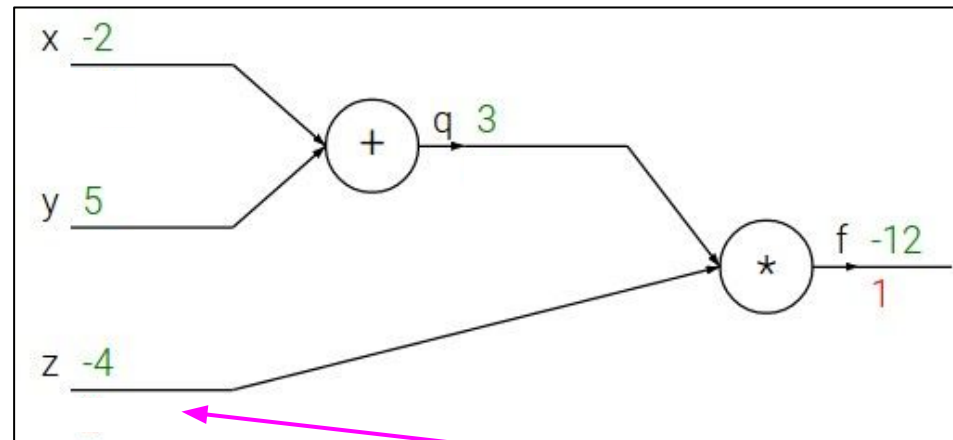
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$$\frac{\partial f}{\partial z}$$

Backpropagation: a simple example

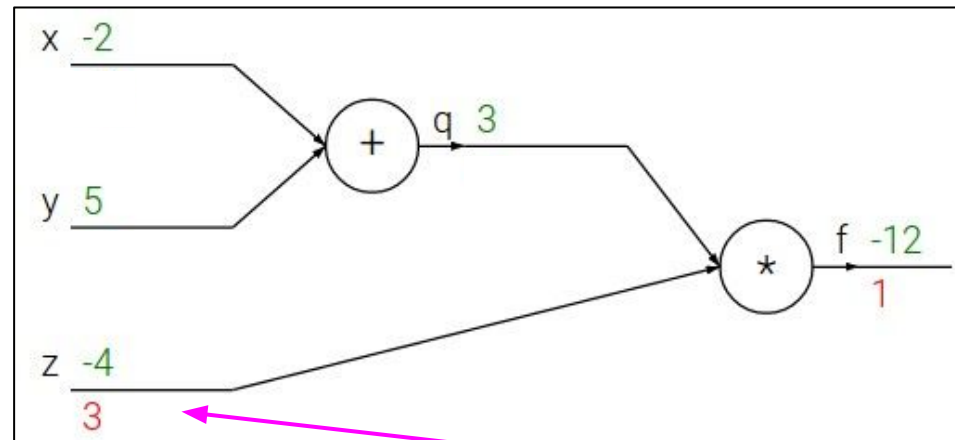
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Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



$$\frac{\partial f}{\partial z}$$

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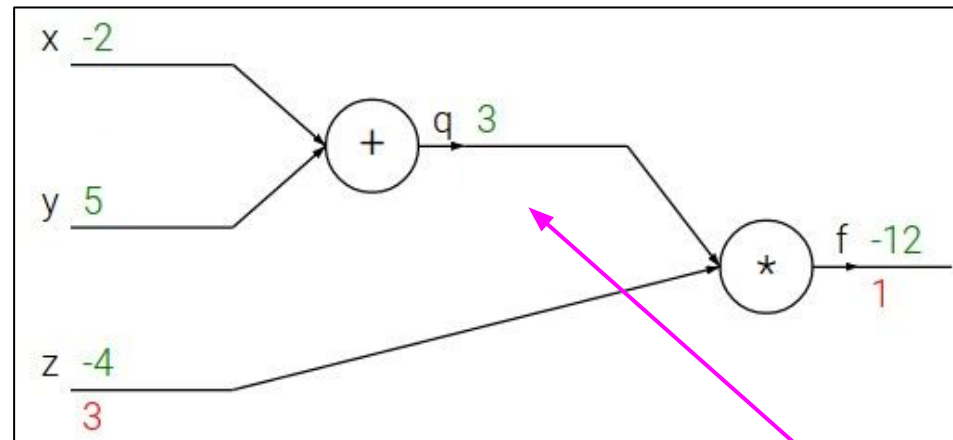
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$$\text{Want: } \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$



$$\frac{\partial f}{\partial q}$$

Backpropagation: a simple example

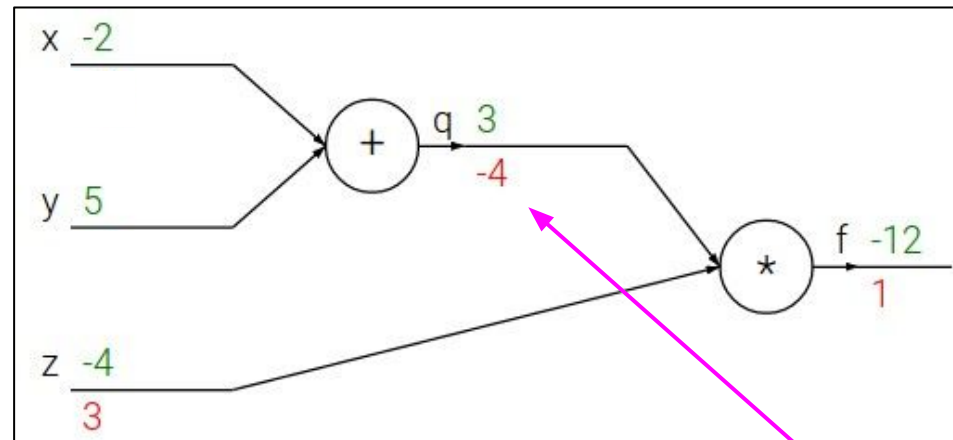
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$$\frac{\partial f}{\partial q}$$

Backpropagation: a simple example

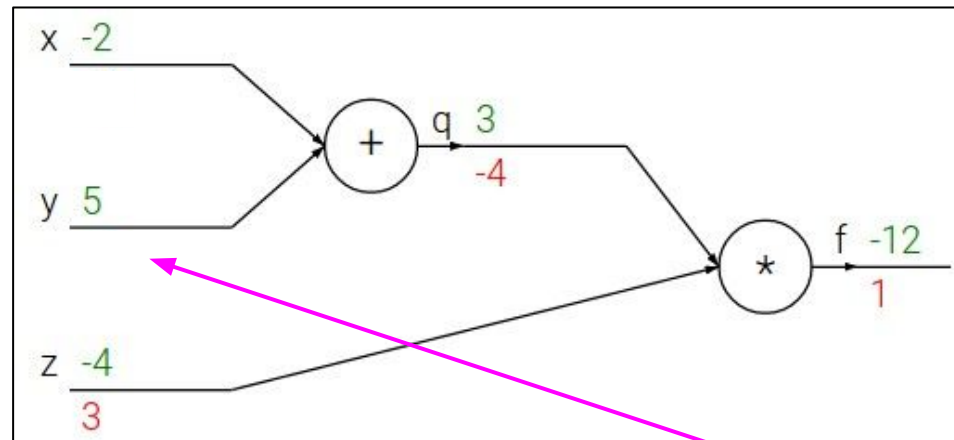
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$$f = qz \quad \frac{\partial f}{\partial q} = z, \frac{\partial f}{\partial z} = q$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial y}$$

Upstream
gradient

Local
gradient

$$\frac{\partial f}{\partial y}$$

Backpropagation: a simple example

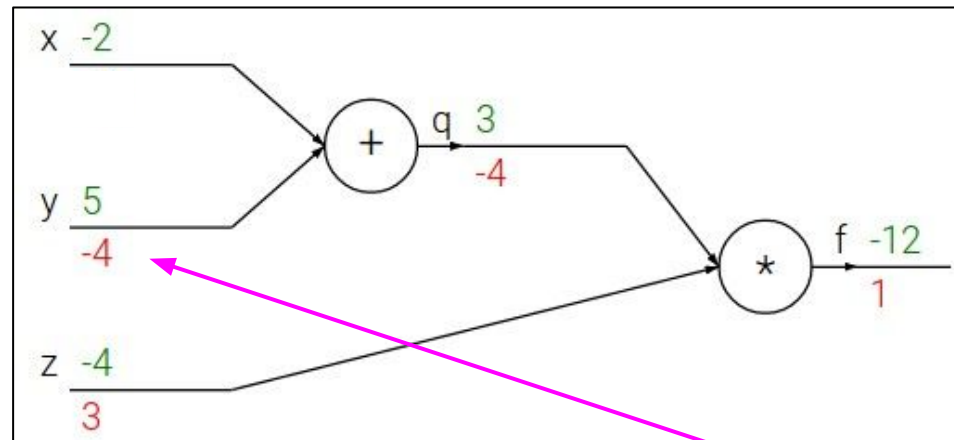
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$$f = qz \quad \frac{\partial f}{\partial q} = z, \frac{\partial f}{\partial z} = q$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



$$\frac{\partial f}{\partial y}$$

Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial y}$$

Upstream gradient

Local gradient

Backpropagation: a simple example

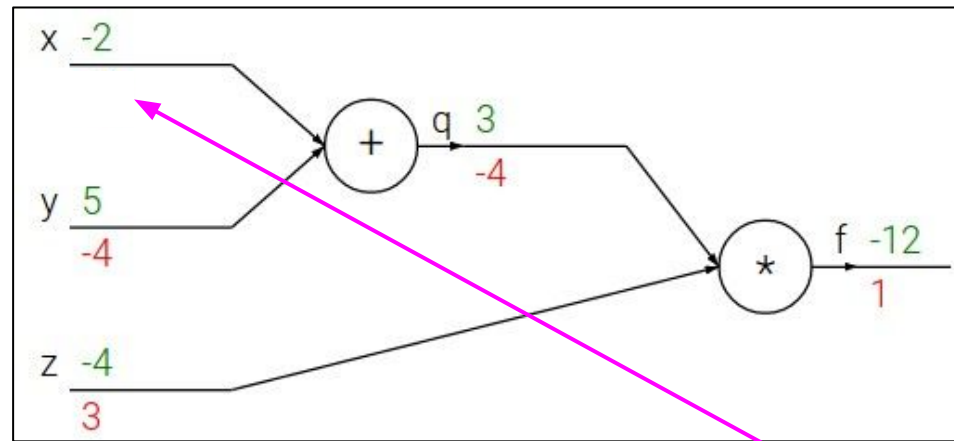
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Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



$$\frac{\partial f}{\partial x}$$

Chain rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial x}$$

Upstream gradient

Local gradient

Backpropagation: a simple example

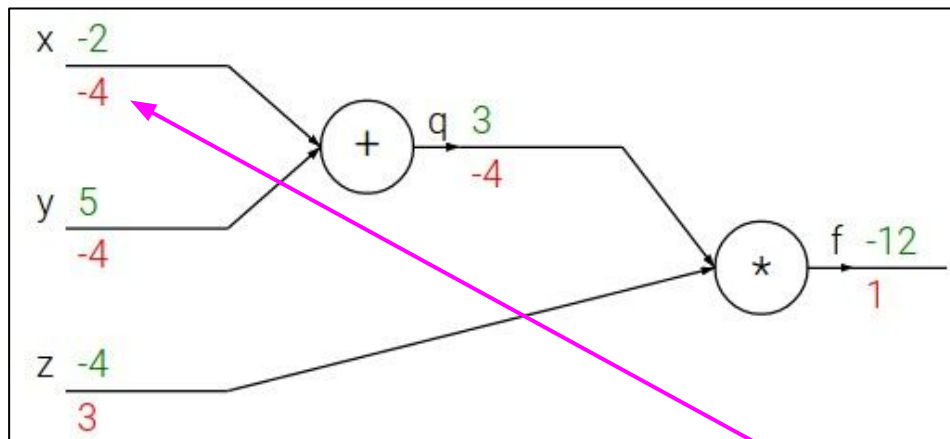
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Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



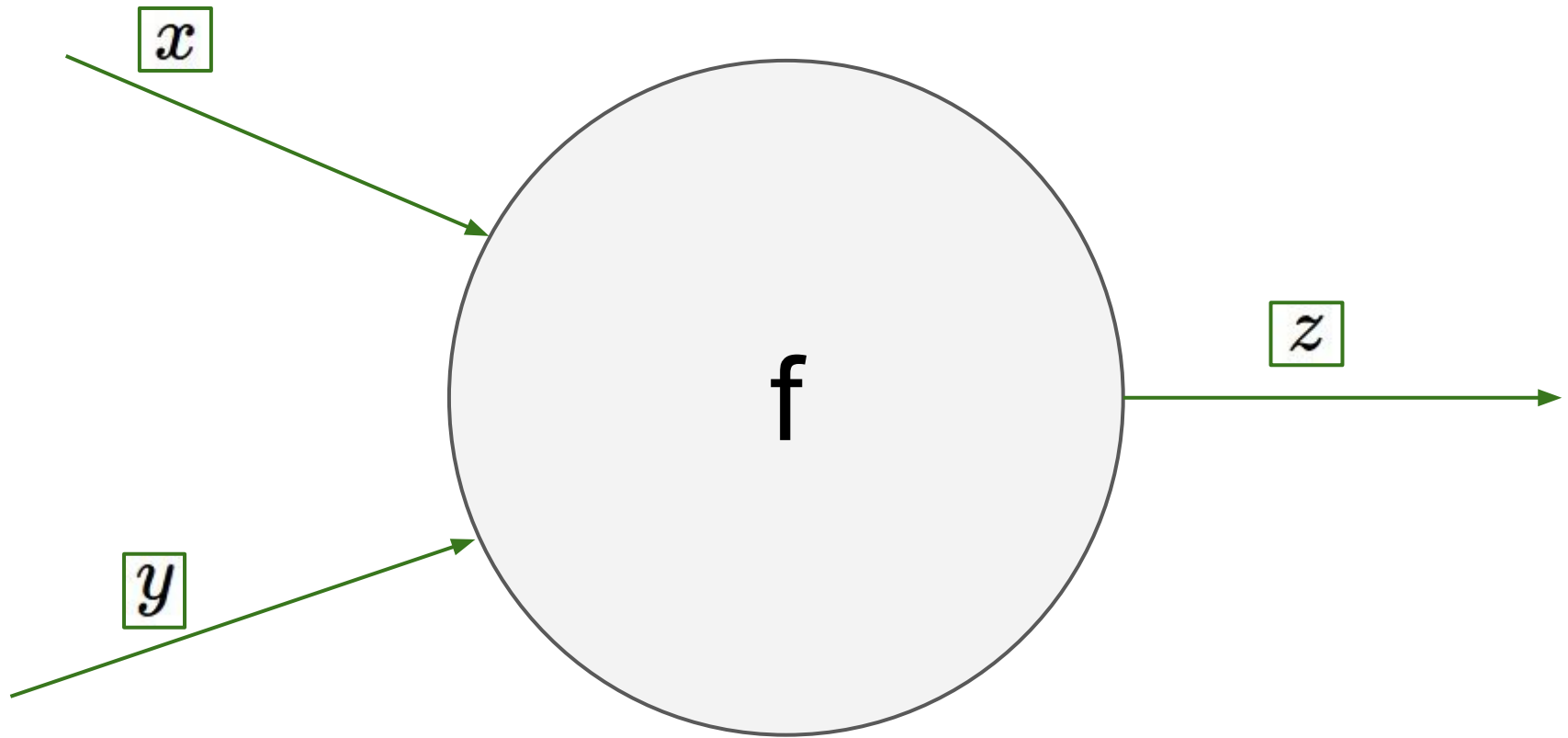
Chain rule:

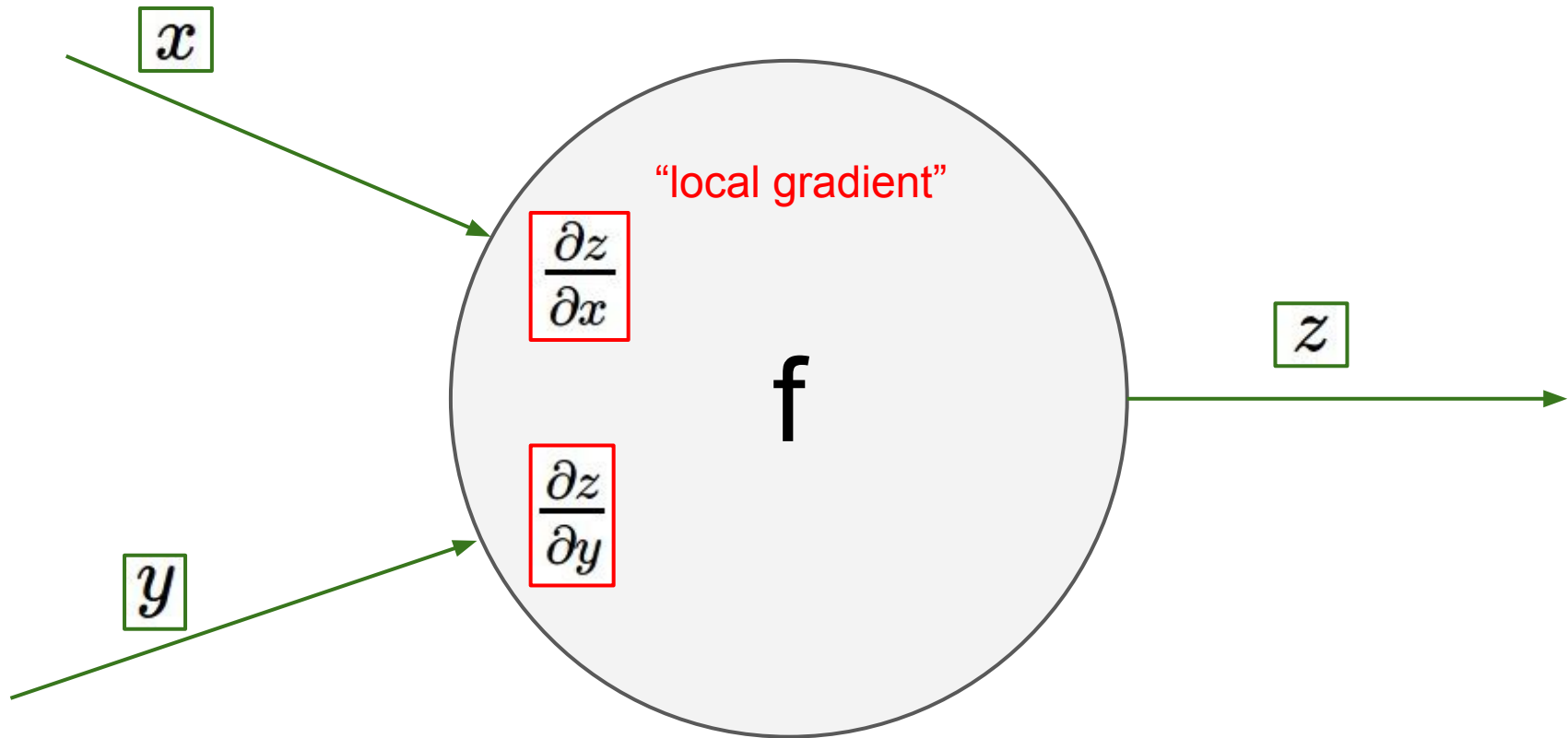
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial x}$$

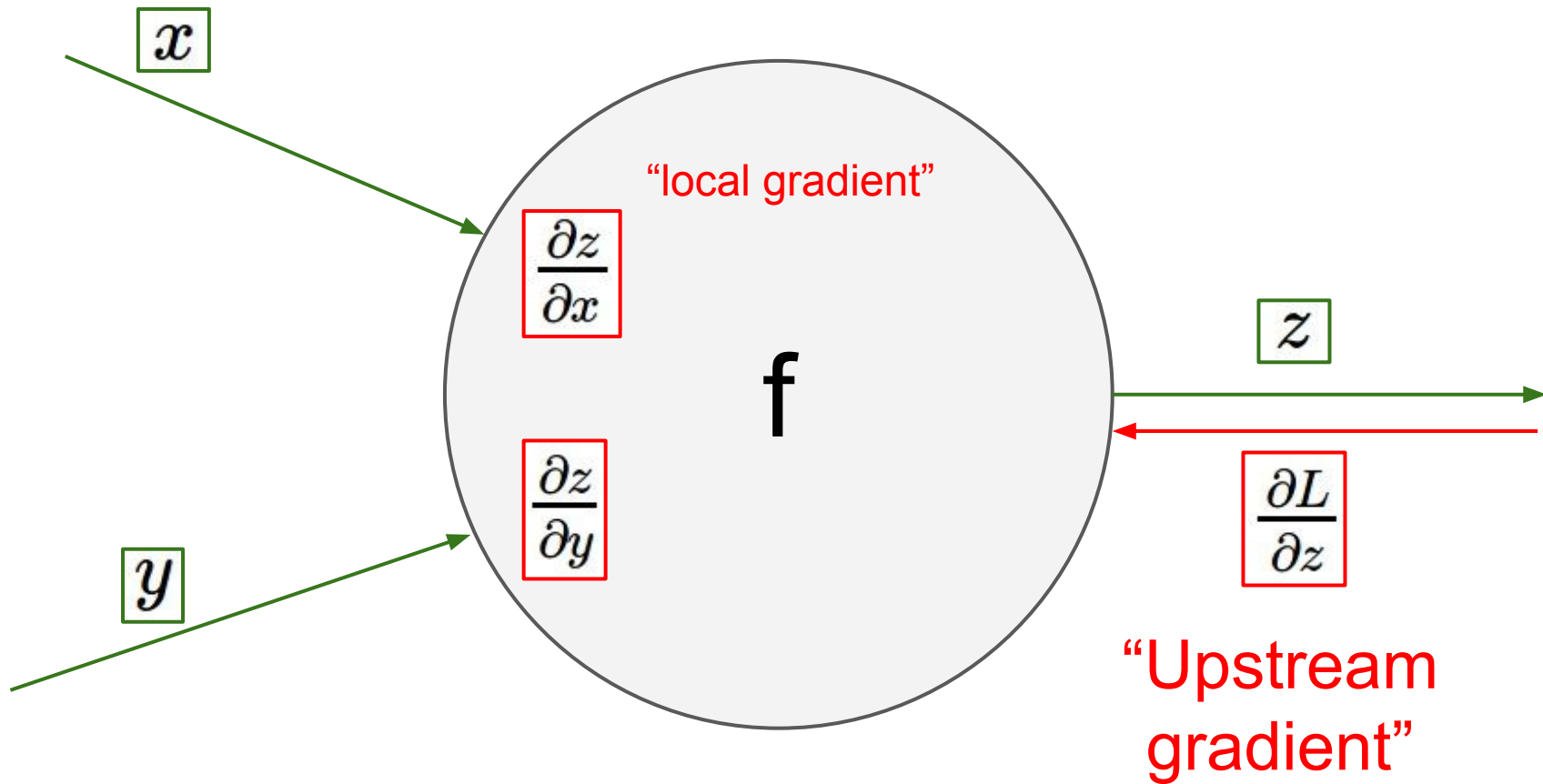
Upstream gradient

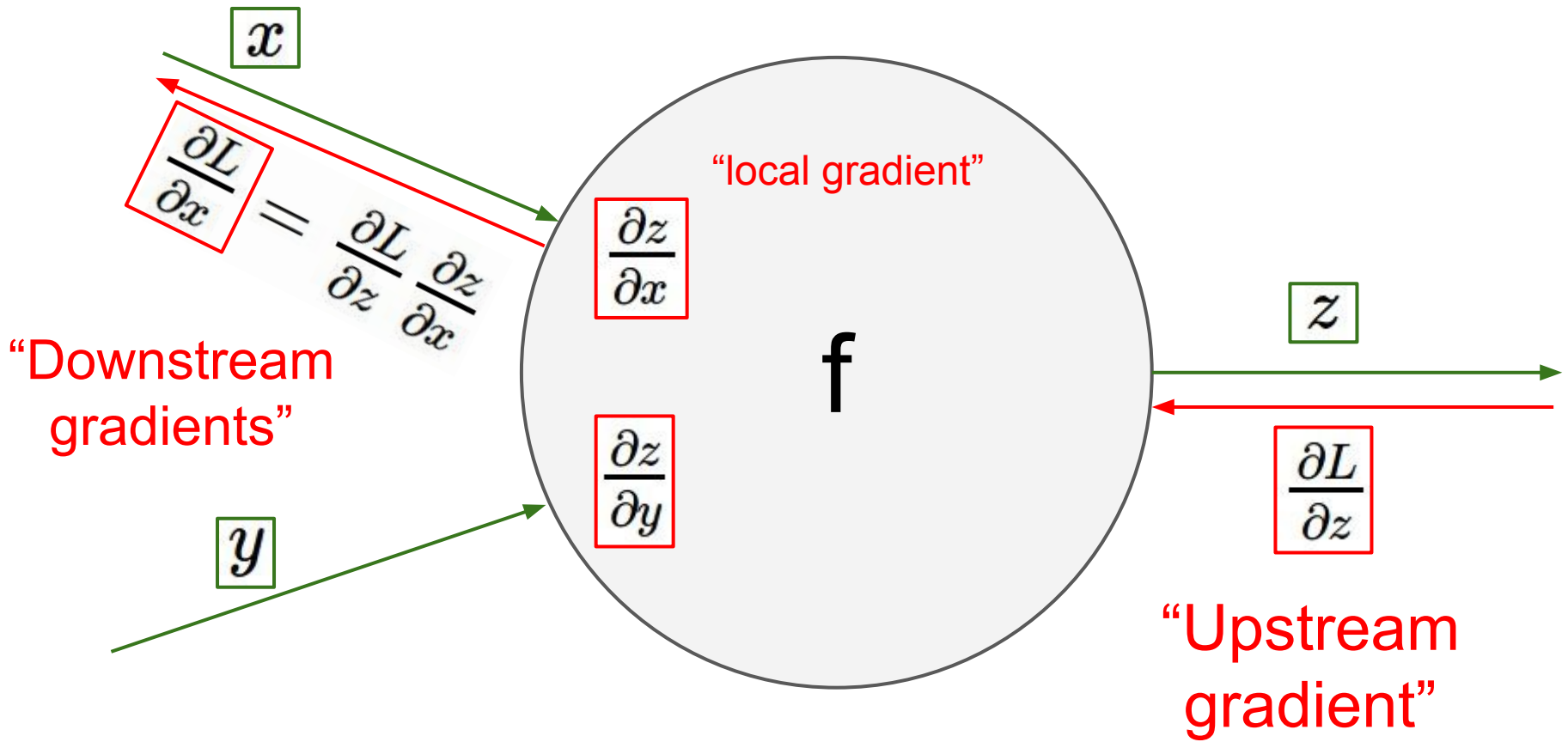
Local gradient

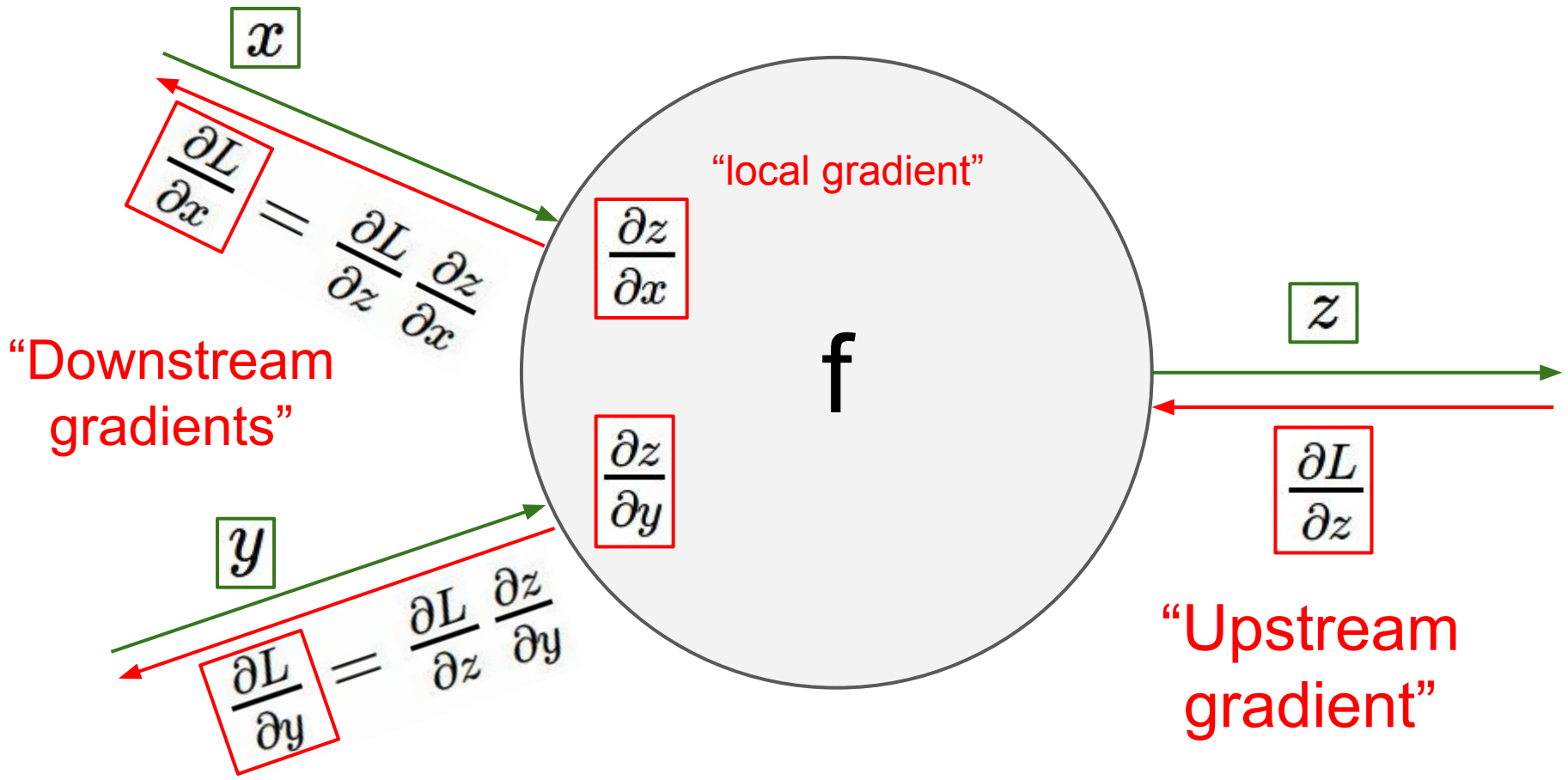
$$\frac{\partial f}{\partial x}$$

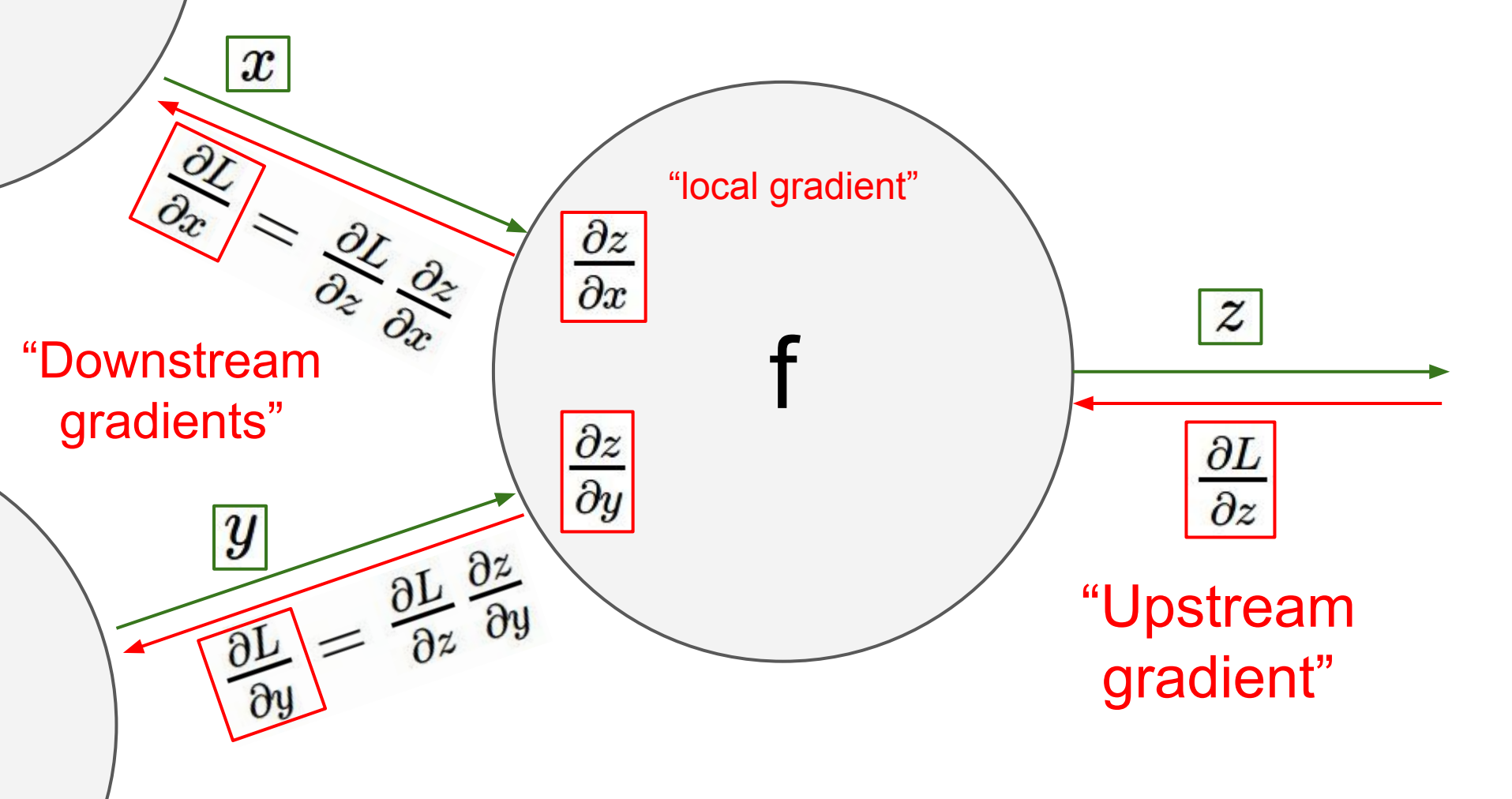






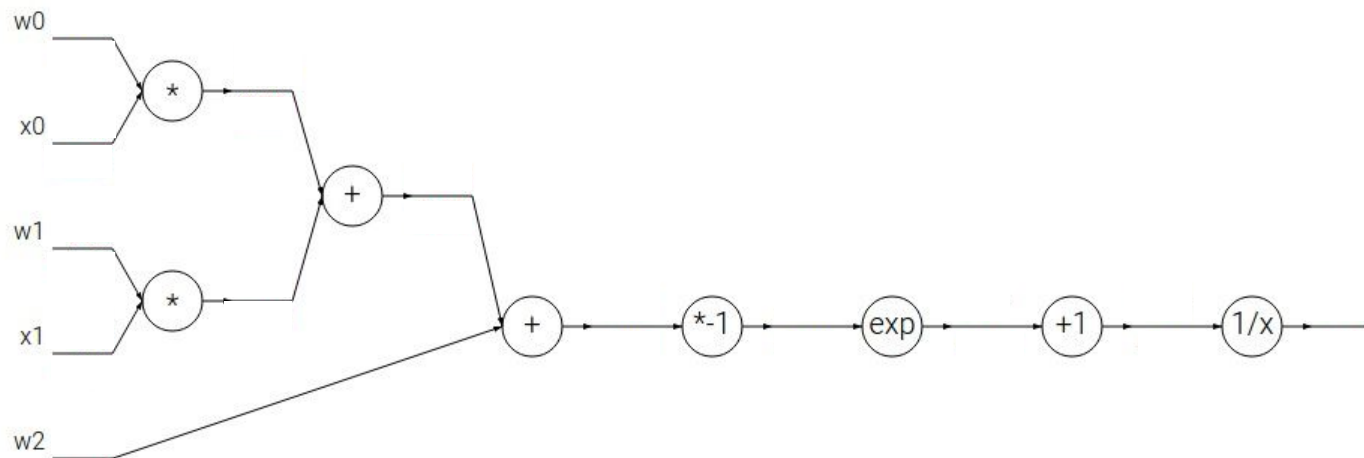






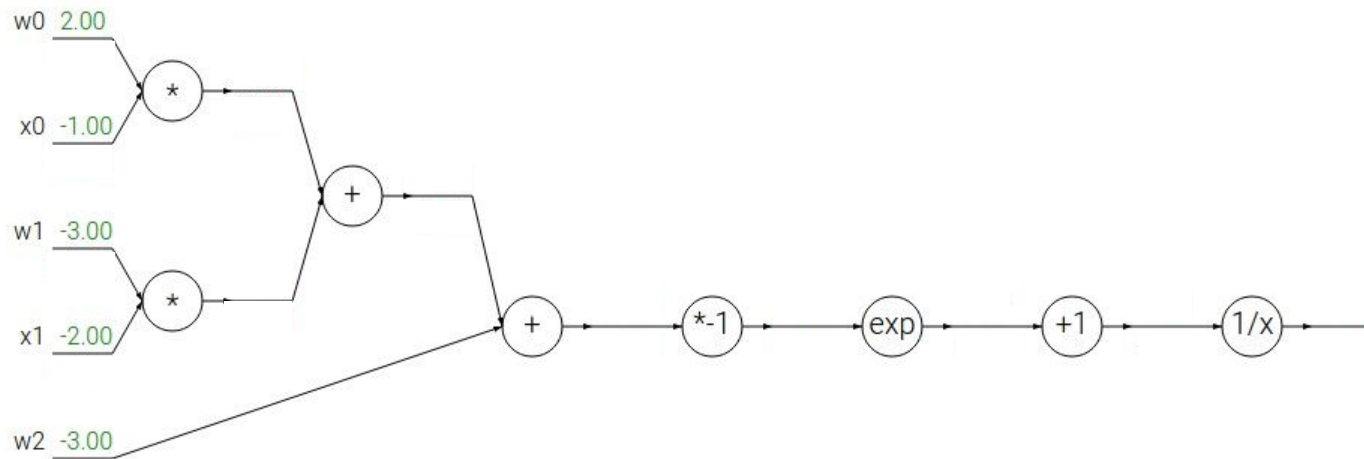
Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



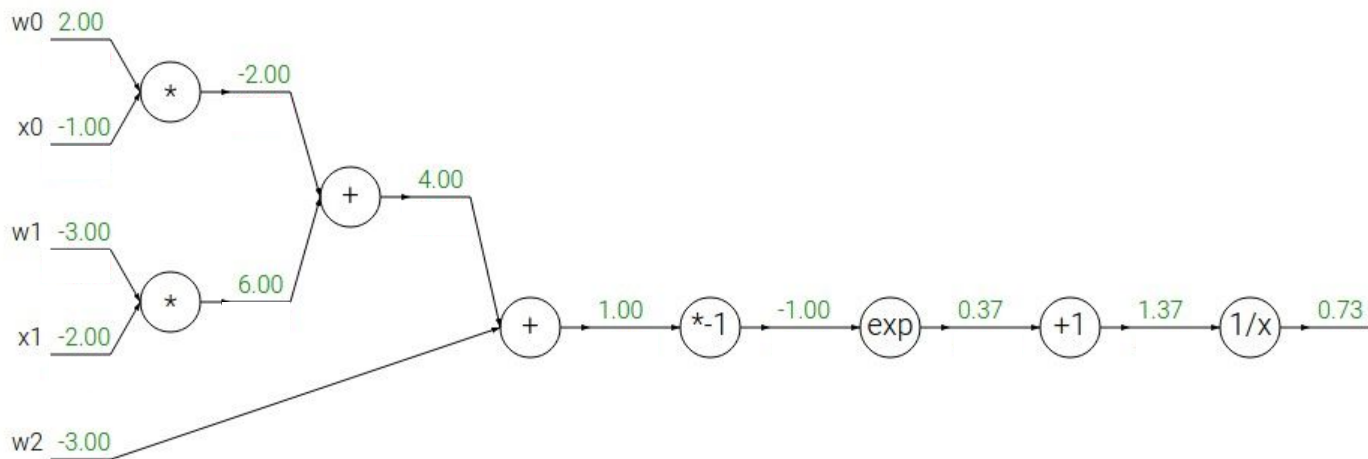
Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



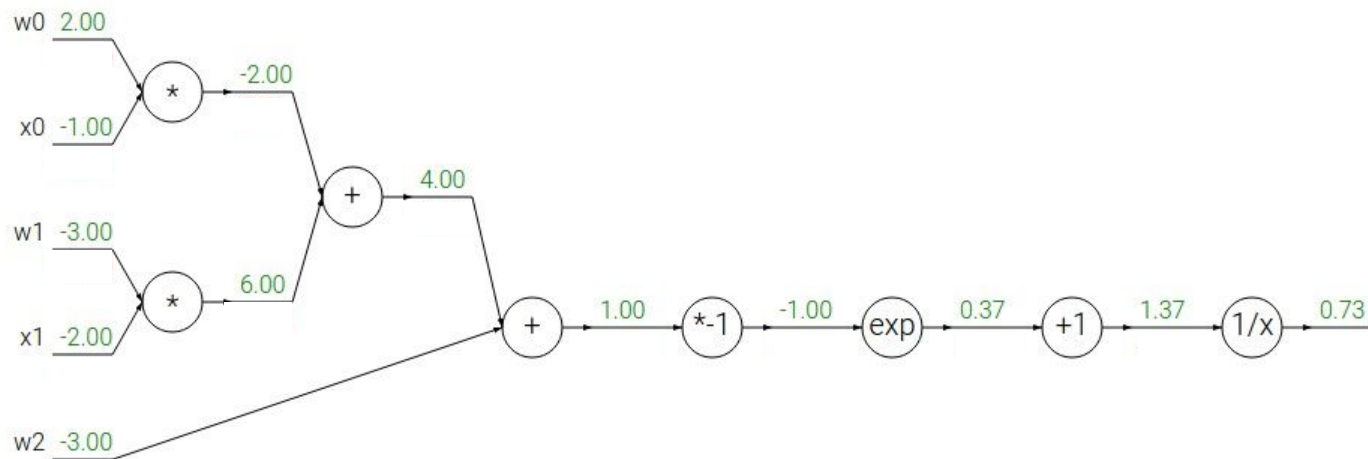
Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



Another example:

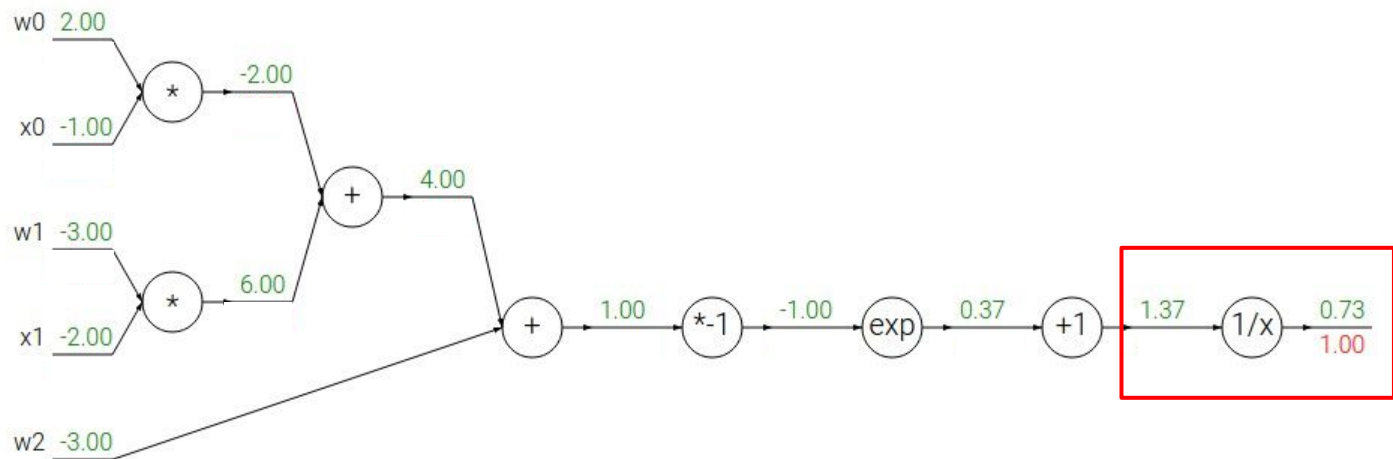
$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$f(x) = e^x$	→	$\frac{df}{dx} = e^x$		$f(x) = \frac{1}{x}$	→	$\frac{df}{dx} = -1/x^2$
$f_a(x) = ax$	→	$\frac{df}{dx} = a$		$f_c(x) = c + x$	→	$\frac{df}{dx} = 1$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x$$

→

$$\frac{df}{dx} = e^x$$

$$f_a(x) = ax$$

→

$$\frac{df}{dx} = a$$

$$f(x) = \frac{1}{x}$$

→

$$\frac{df}{dx} = -1/x^2$$

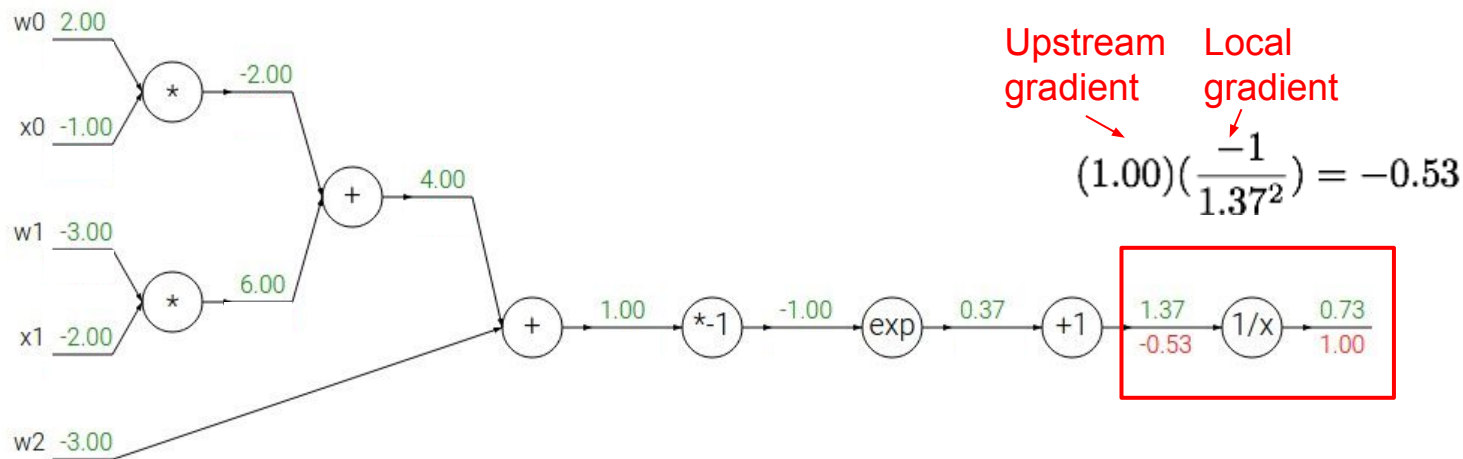
$$f_c(x) = c + x$$

→

$$\frac{df}{dx} = 1$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x$$

→

$$\frac{df}{dx} = e^x$$

$$f_a(x) = ax$$

→

$$\frac{df}{dx} = a$$

$$f(x) = \frac{1}{x}$$

→

$$\frac{df}{dx} = -1/x^2$$

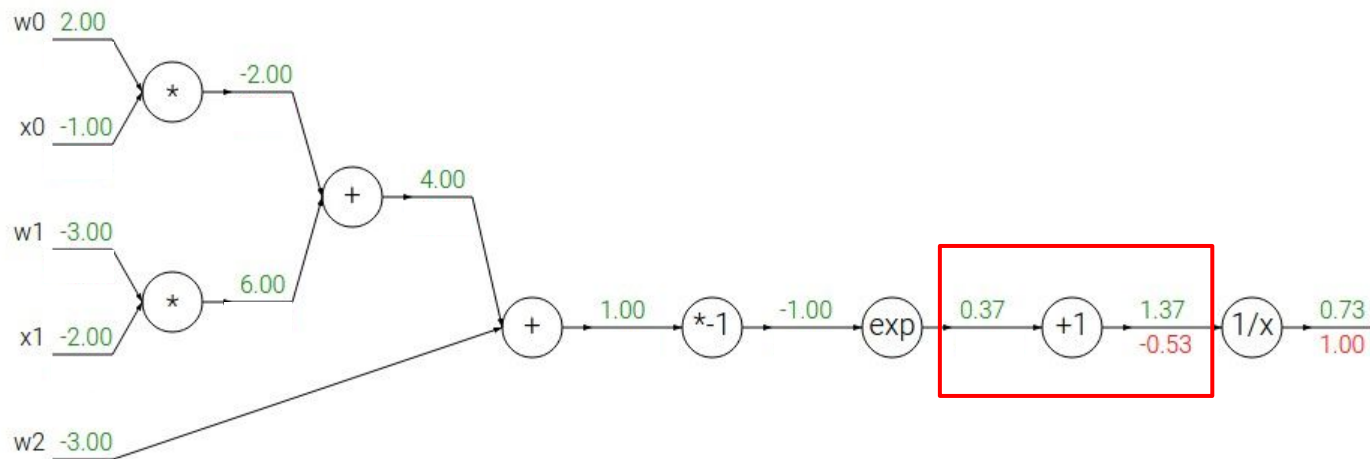
$$f_c(x) = c + x$$

→

$$\frac{df}{dx} = 1$$

Another example:

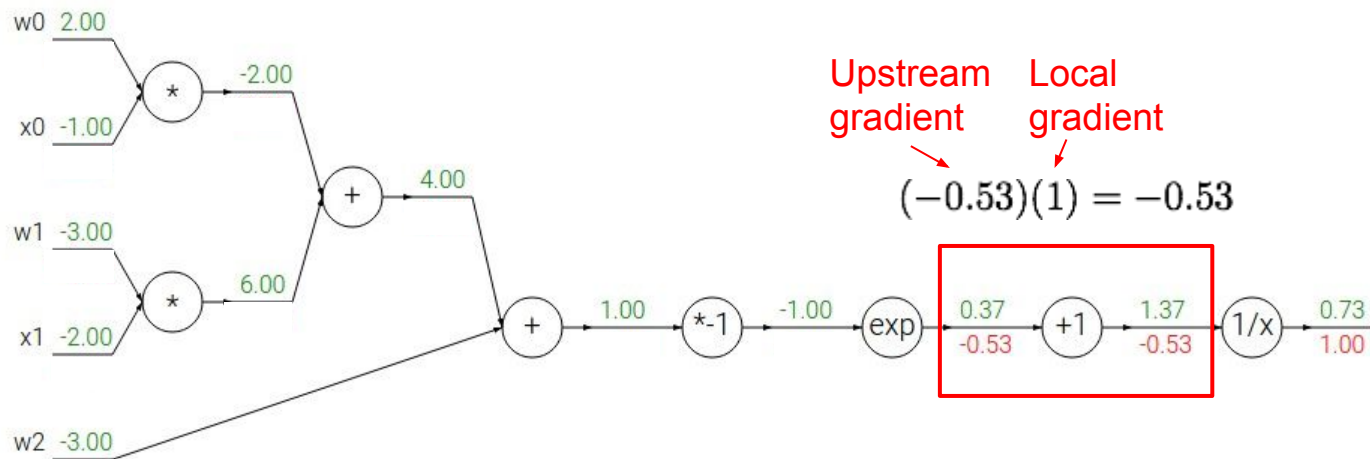
$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$f(x) = e^x$	\rightarrow	$\frac{df}{dx} = e^x$		$f(x) = \frac{1}{x}$	\rightarrow	$\frac{df}{dx} = -1/x^2$
$f_a(x) = ax$	\rightarrow	$\frac{df}{dx} = a$		$f_c(x) = c + x$	\rightarrow	$\frac{df}{dx} = 1$

Another example:

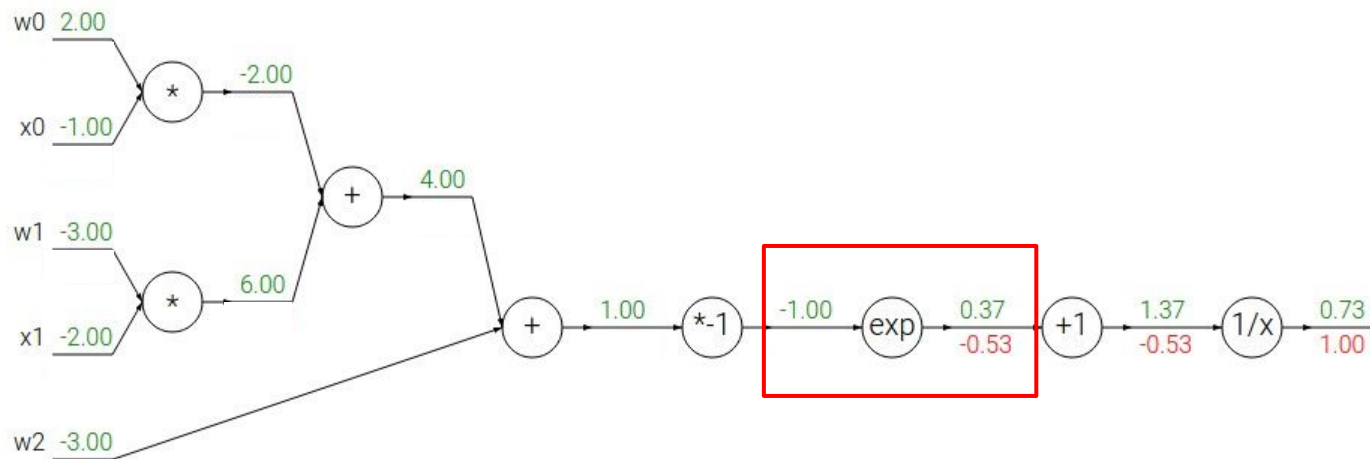
$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$f(x) = e^x$	\rightarrow	$\frac{df}{dx} = e^x$		$f(x) = \frac{1}{x}$	\rightarrow	$\frac{df}{dx} = -1/x^2$
$f_a(x) = ax$	\rightarrow	$\frac{df}{dx} = a$		$f_c(x) = c + x$	\rightarrow	$\frac{df}{dx} = 1$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x \rightarrow \frac{df}{dx} = e^x$$

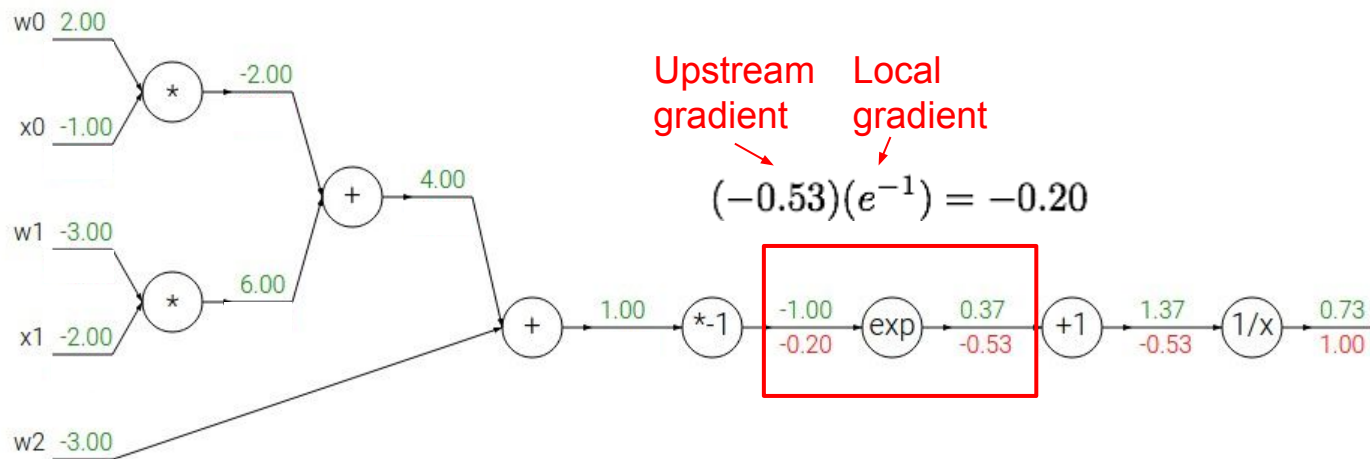
$$f_a(x) = ax \rightarrow \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \rightarrow \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \rightarrow \frac{df}{dx} = 1$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

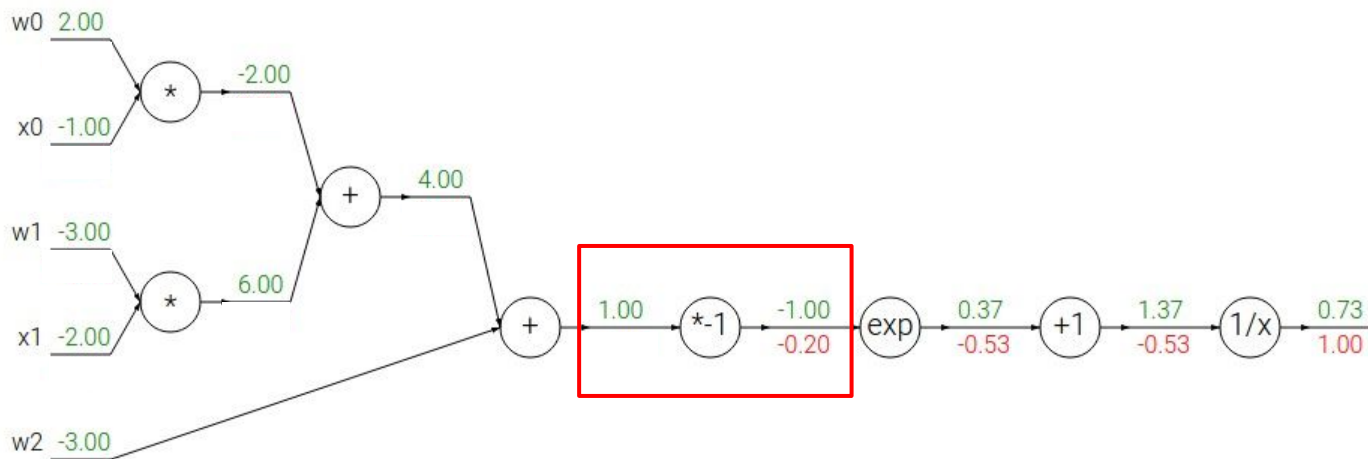
$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

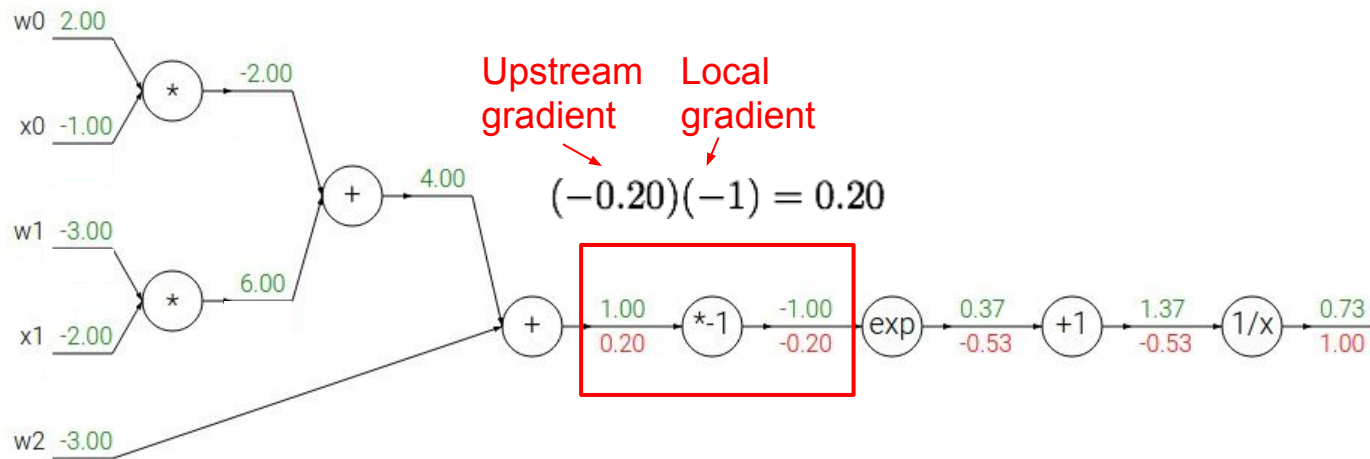
$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x \rightarrow \frac{df}{dx} = e^x$$

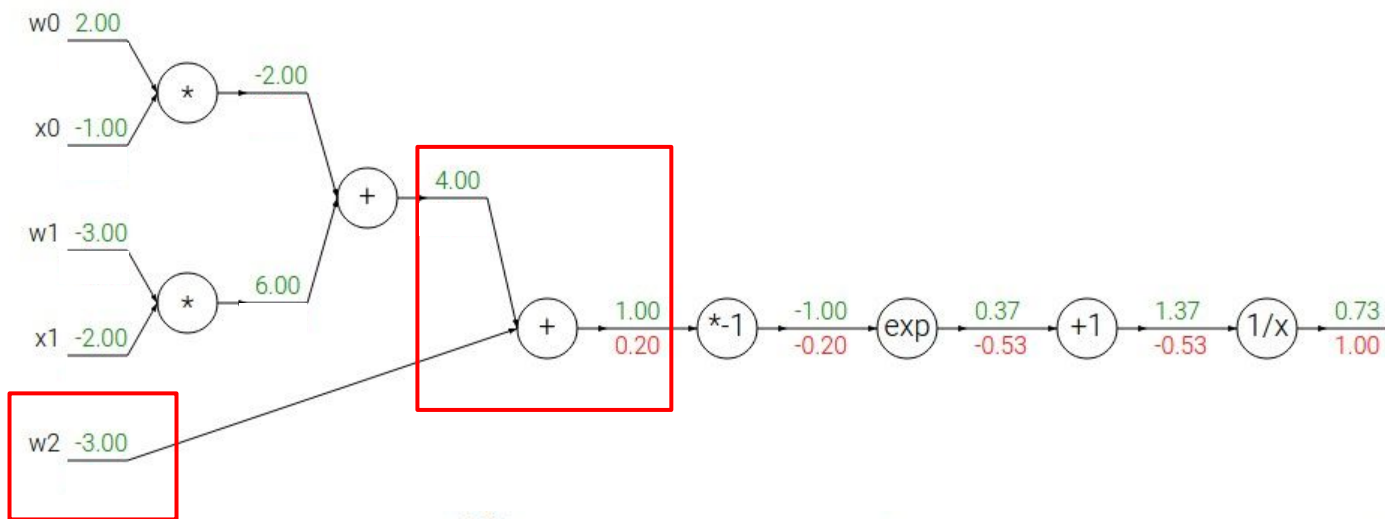
$$f_a(x) = ax \rightarrow \frac{df}{dx} = a$$

$$f(x) = \frac{1}{x} \rightarrow \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \rightarrow \frac{df}{dx} = 1$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$$f(x) = e^x$$

→

$$\frac{df}{dx} = e^x$$

$$f(x) = \frac{1}{x}$$

→

$$\frac{df}{dx} = -1/x^2$$

$$f_a(x) = ax$$

→

$$\frac{df}{dx} = a$$

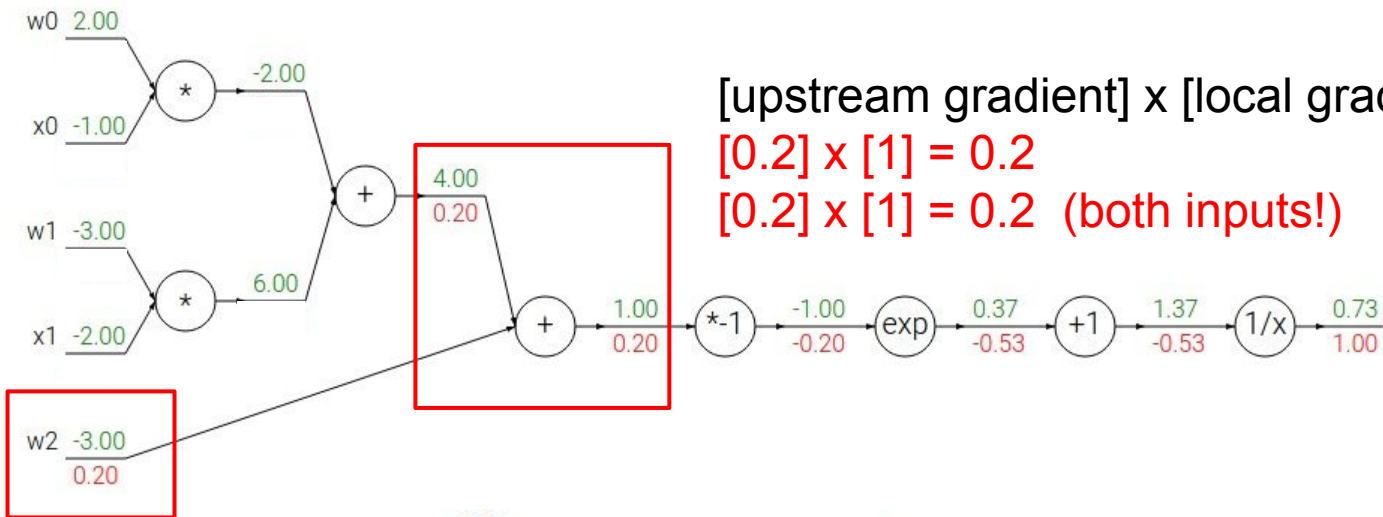
$$f_c(x) = c + x$$

→

$$\frac{df}{dx} = 1$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$f(x) = e^x$

→

$\frac{df}{dx} = e^x$

$f(x) = \frac{1}{x}$

→

$\frac{df}{dx} = -1/x^2$

$f_a(x) = ax$

→

$\frac{df}{dx} = a$

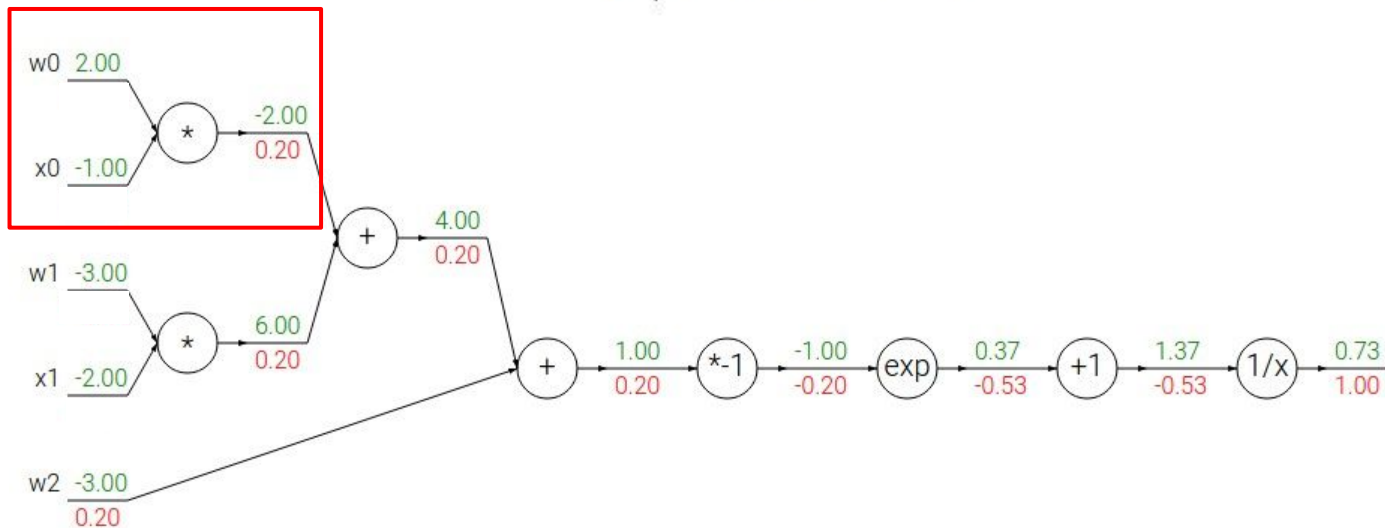
$f_c(x) = c + x$

→

$\frac{df}{dx} = 1$

Another example:

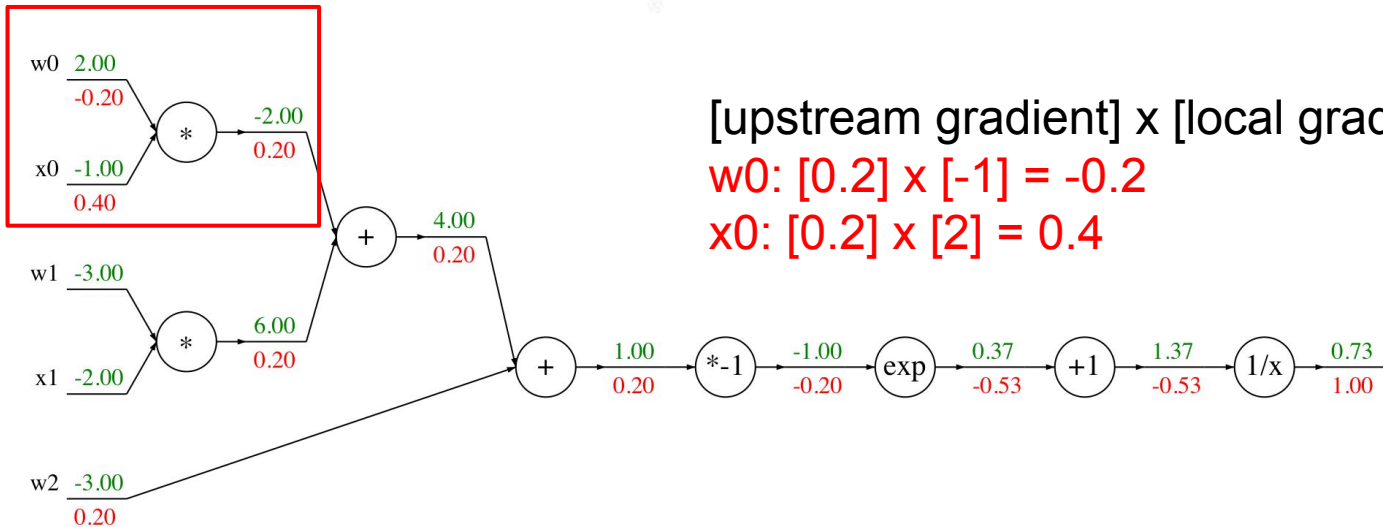
$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



$f(x) = e^x$	→	$\frac{df}{dx} = e^x$		$f(x) = \frac{1}{x}$	→	$\frac{df}{dx} = -1/x^2$
$f_a(x) = ax$	→	$\frac{df}{dx} = a$		$f_c(x) = c + x$	→	$\frac{df}{dx} = 1$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$



[upstream gradient] x [local gradient]

$w_0: [0.2] \times [-1] = -0.2$

$x_0: [0.2] \times [2] = 0.4$

$f(x) = e^x$

→

$\frac{df}{dx} = e^x$

$f(x) = \frac{1}{x}$

→

$\frac{df}{dx} = -1/x^2$

$f_a(x) = ax$

→

$\frac{df}{dx} = a$

$f_c(x) = c + x$

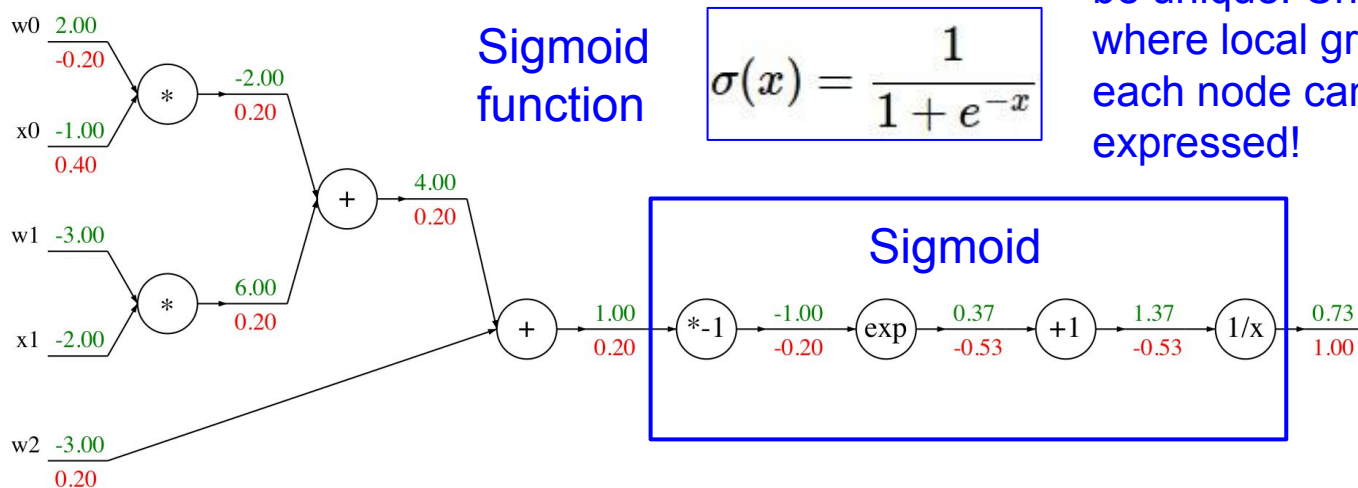
→

$\frac{df}{dx} = 1$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$

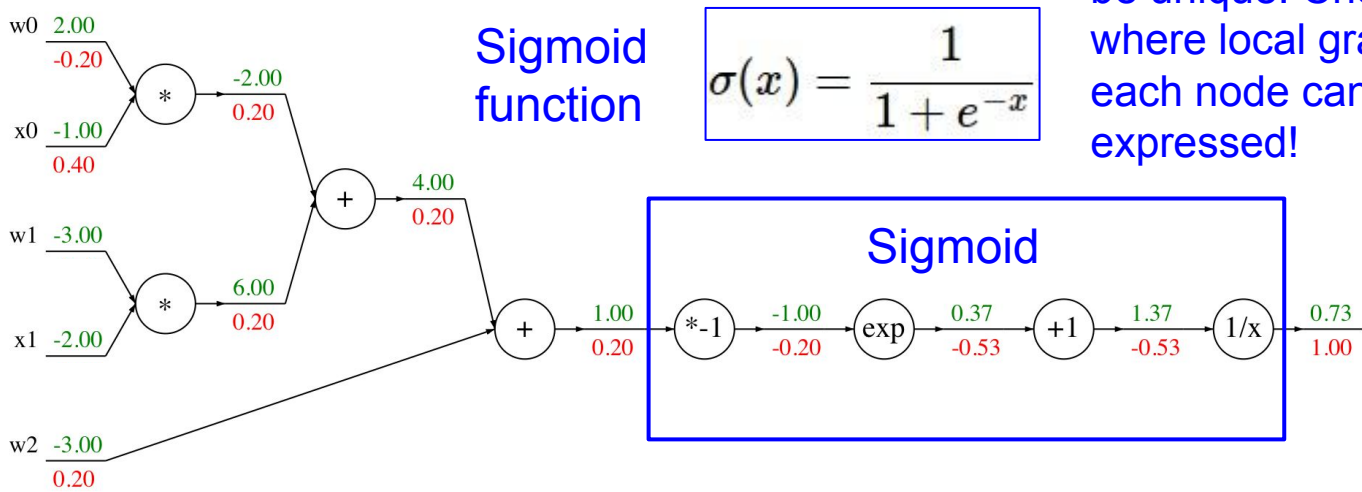
Computational graph representation may not be unique. Choose one where local gradients at each node can be easily expressed!



Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$

Computational graph representation may not be unique. Choose one where local gradients at each node can be easily expressed!



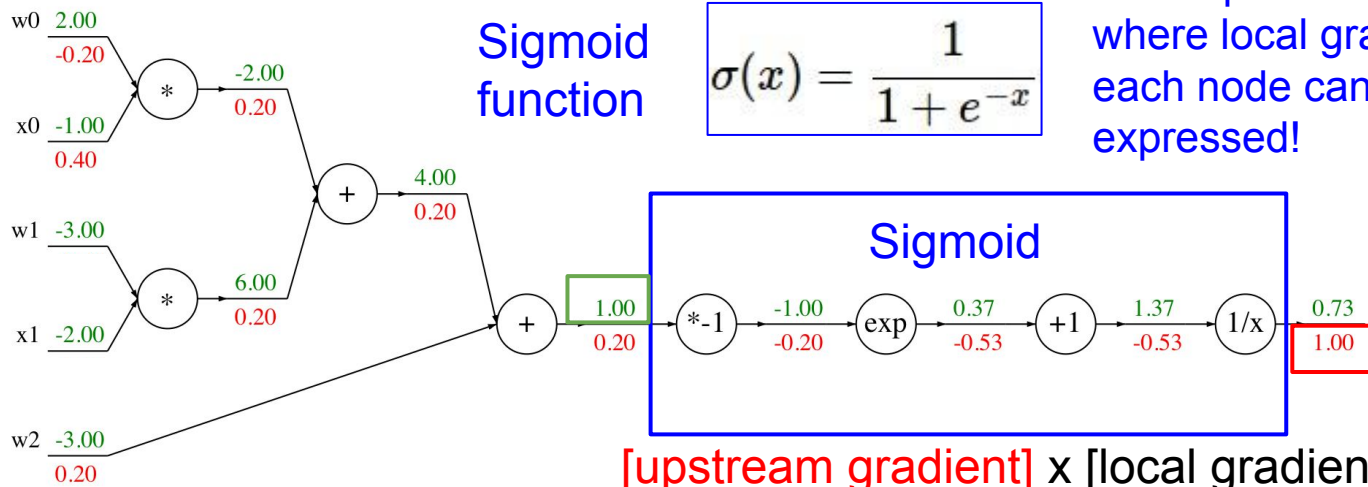
Sigmoid local gradient:

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1 + e^{-x})^2} = \left(\frac{1 + e^{-x} - 1}{1 + e^{-x}} \right) \left(\frac{1}{1 + e^{-x}} \right) = (1 - \sigma(x))\sigma(x)$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$

Computational graph representation may not be unique. Choose one where local gradients at each node can be easily expressed!



Sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

[upstream gradient] x [local gradient]
 [1.00] x [(1 - 1/(1+e¹)) (1/(1+e¹))] = 0.2

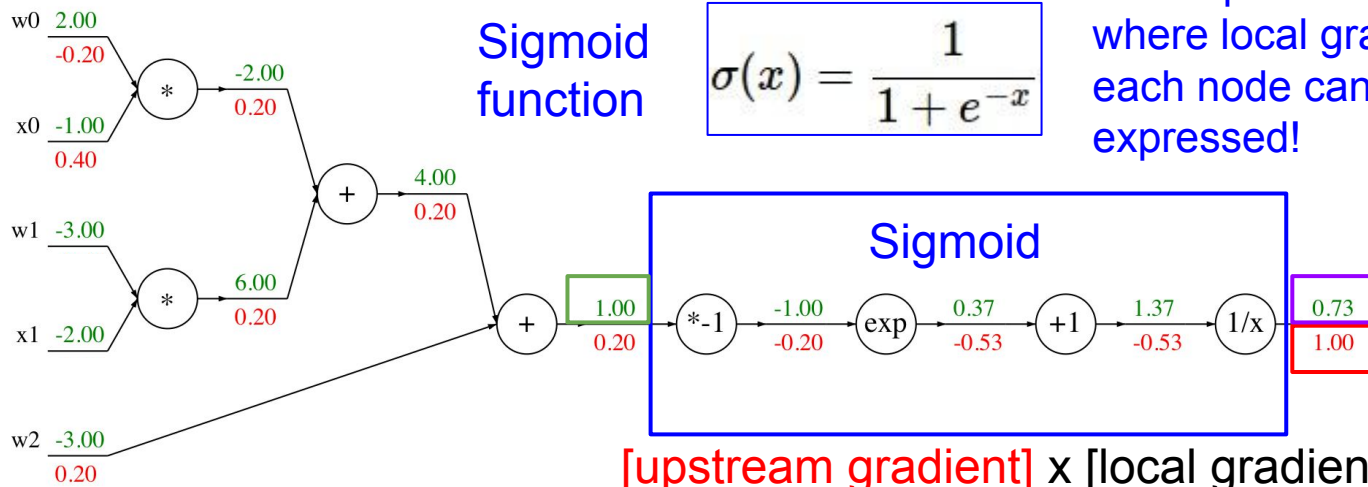
Sigmoid local gradient:

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1 + e^{-x})^2} = \left(\frac{1 + e^{-x} - 1}{1 + e^{-x}} \right) \left(\frac{1}{1 + e^{-x}} \right) = (1 - \sigma(x))\sigma(x)$$

Another example:

$$f(w, x) = \frac{1}{1 + e^{-(w_0x_0 + w_1x_1 + w_2)}}$$

Computational graph representation may not be unique. Choose one where local gradients at each node can be easily expressed!



Sigmoid function

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

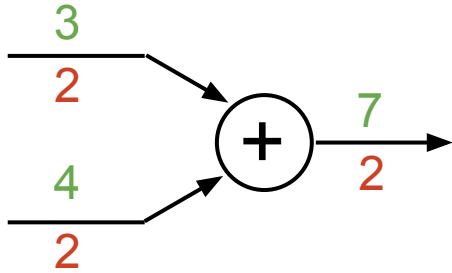
[upstream gradient] x [local gradient]
 [1.00] x [(1 - 0.73) (0.73)] = 0.2

Sigmoid local gradient:

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1 + e^{-x})^2} = \left(\frac{1 + e^{-x} - 1}{1 + e^{-x}} \right) \left(\frac{1}{1 + e^{-x}} \right) = (1 - \sigma(x)) \sigma(x)$$

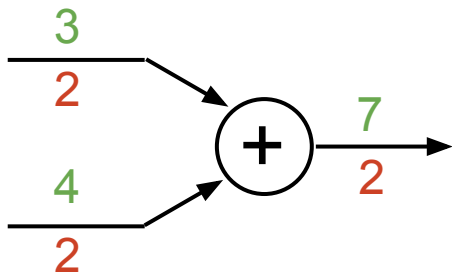
Patterns in gradient flow

add gate: gradient distributor

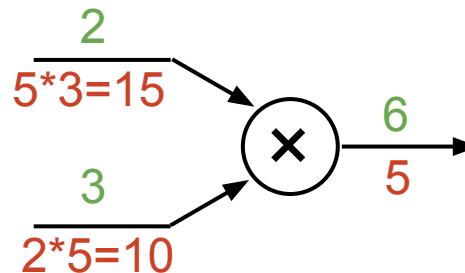


Patterns in gradient flow

add gate: gradient distributor

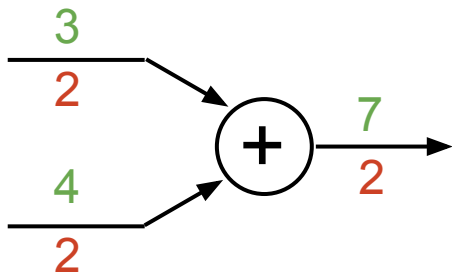


mul gate: “swap multiplier”

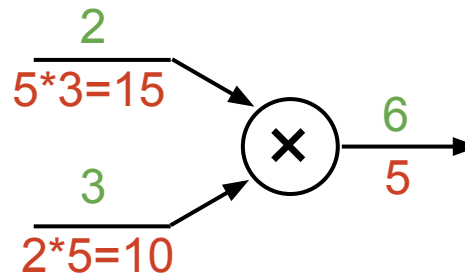


Patterns in gradient flow

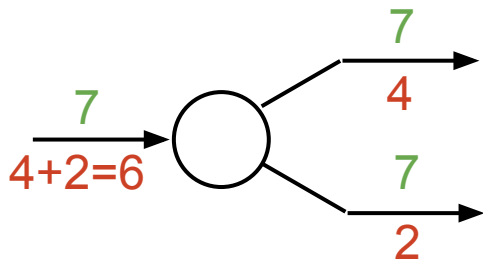
add gate: gradient distributor



mul gate: “swap multiplier”

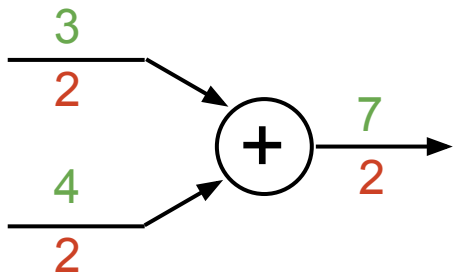


copy gate: gradient adder

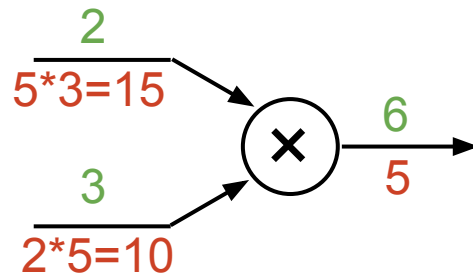


Patterns in gradient flow

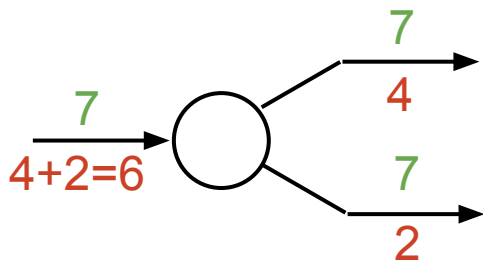
add gate: gradient distributor



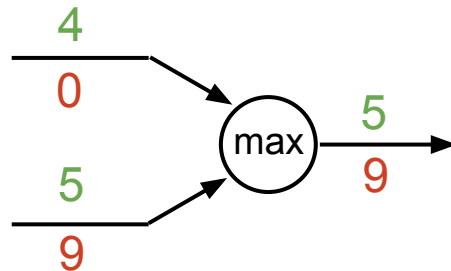
mul gate: “swap multiplier”



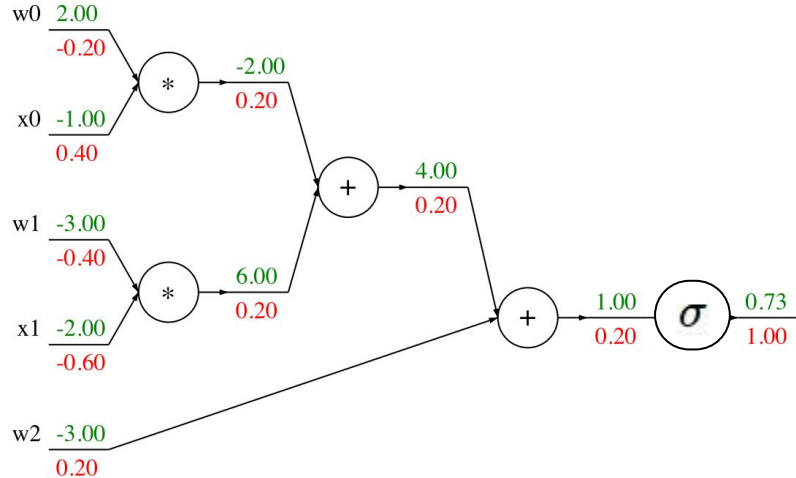
copy gate: gradient adder



max gate: gradient router



Backprop Implementation: “Flat” code



Forward pass:
Compute output

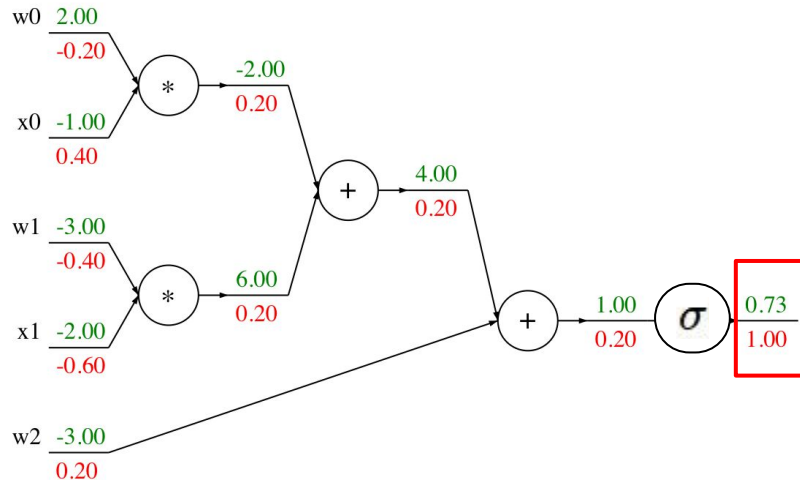
```
def f(w0, x0, w1, x1, w2):
```

```
s0 = w0 * x0  
s1 = w1 * x1  
s2 = s0 + s1  
s3 = s2 + w2  
L = sigmoid(s3)
```

Backward pass:
Compute grads

```
grad_L = 1.0  
grad_s3 = grad_L * (1 - L) * L  
grad_w2 = grad_s3  
grad_s2 = grad_s3  
grad_s0 = grad_s2  
grad_s1 = grad_s2  
grad_w1 = grad_s1 * x1  
grad_x1 = grad_s1 * w1  
grad_w0 = grad_s0 * x0  
grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

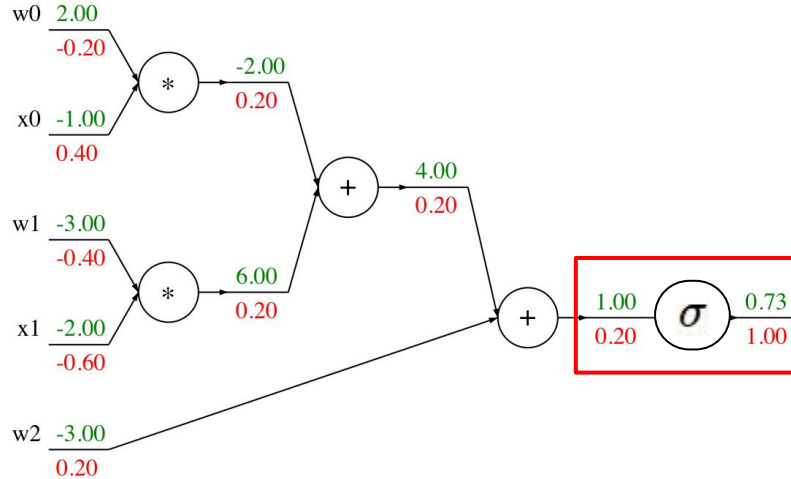
```
def f(w0, x0, w1, x1, w2):
```

```
s0 = w0 * x0  
s1 = w1 * x1  
s2 = s0 + s1  
s3 = s2 + w2  
L = sigmoid(s3)
```

Base case

```
grad_L = 1.0  
grad_s3 = grad_L * (1 - L) * L  
grad_w2 = grad_s3  
grad_s2 = grad_s3  
grad_s0 = grad_s2  
grad_s1 = grad_s2  
grad_w1 = grad_s1 * x1  
grad_x1 = grad_s1 * w1  
grad_w0 = grad_s0 * x0  
grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

```
s0 = w0 * x0
```

```
s1 = w1 * x1
```

```
s2 = s0 + s1
```

```
s3 = s2 + w2
```

```
L = sigmoid(s3)
```

Sigmoid

```
grad_L = 1.0
```

```
grad_s3 = grad_L * (1 - L) * L
```

```
grad_w2 = grad_s3
```

```
grad_s2 = grad_s3
```

```
grad_s0 = grad_s2
```

```
grad_s1 = grad_s2
```

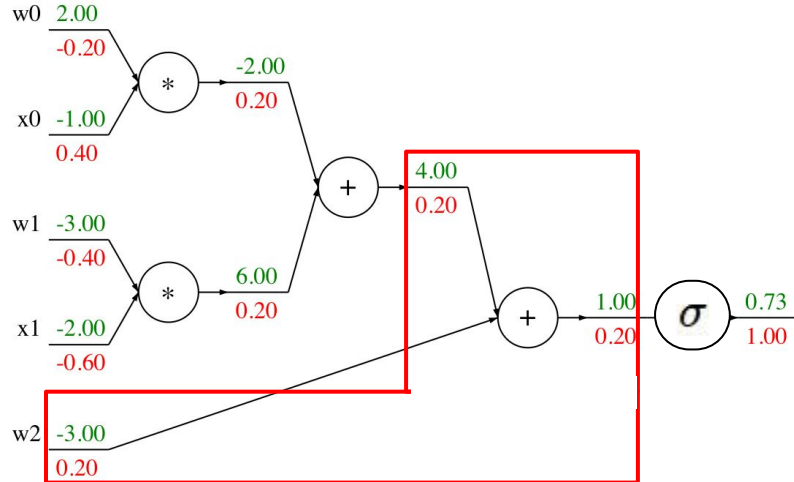
```
grad_w1 = grad_s1 * x1
```

```
grad_x1 = grad_s1 * w1
```

```
grad_w0 = grad_s0 * x0
```

```
grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

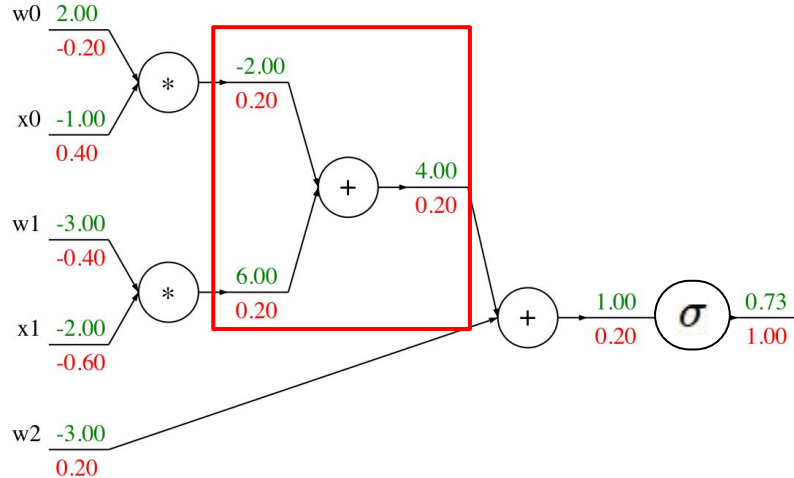
```
def f(w0, x0, w1, x1, w2):
```

```
    s0 = w0 * x0  
    s1 = w1 * x1  
    s2 = s0 + s1  
    s3 = s2 + w2  
    L = sigmoid(s3)
```

Add gate

```
    grad_L = 1.0  
    grad_s3 = grad_L * (1 - L) * L  
    grad_w2 = grad_s3  
    grad_s2 = grad_s3  
    grad_s0 = grad_s2  
    grad_s1 = grad_s2  
    grad_w1 = grad_s1 * x1  
    grad_x1 = grad_s1 * w1  
    grad_w0 = grad_s0 * x0  
    grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

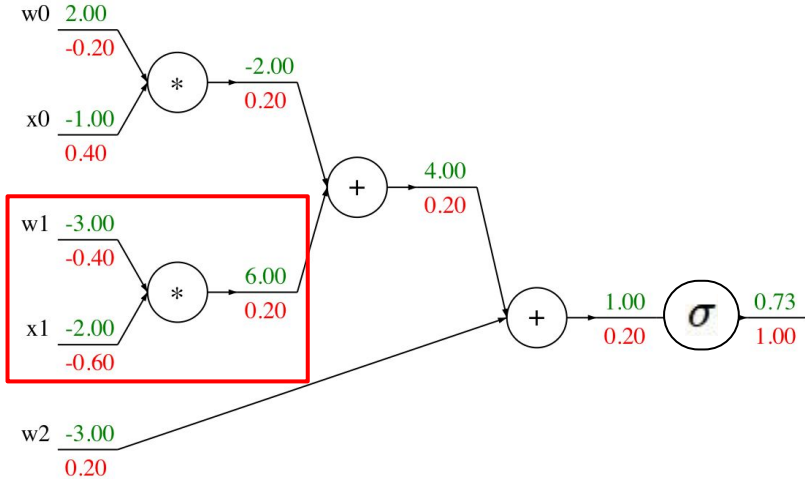
```
def f(w0, x0, w1, x1, w2):
```

```
    s0 = w0 * x0  
    s1 = w1 * x1  
    s2 = s0 + s1  
    s3 = s2 + w2  
    L = sigmoid(s3)
```

```
grad_L = 1.0  
grad_s3 = grad_L * (1 - L) * L  
grad_w2 = grad_s3  
grad_s2 = grad_s3  
grad_s0 = grad_s2  
grad_s1 = grad_s2  
grad_w1 = grad_s1 * x1  
grad_x1 = grad_s1 * w1  
grad_w0 = grad_s0 * x0  
grad_x0 = grad_s0 * w0
```

Add gate

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

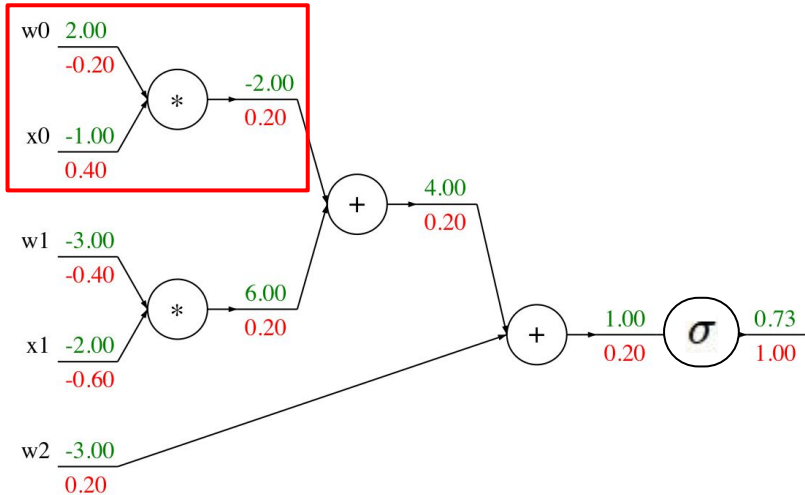
```
s0 = w0 * x0  
s1 = w1 * x1  
s2 = s0 + s1  
s3 = s2 + w2  
L = sigmoid(s3)
```

```
grad_L = 1.0  
grad_s3 = grad_L * (1 - L) * L  
grad_w2 = grad_s3  
grad_s2 = grad_s3  
grad_s0 = grad_s2  
grad_s1 = grad_s2
```

Multiply gate

```
grad_w1 = grad_s1 * x1  
grad_x1 = grad_s1 * w1  
grad_w0 = grad_s0 * x0  
grad_x0 = grad_s0 * w0
```

Backprop Implementation: “Flat” code



Forward pass:
Compute output

```
def f(w0, x0, w1, x1, w2):
```

```
s0 = w0 * x0
```

```
s1 = w1 * x1
```

```
s2 = s0 + s1
```

```
s3 = s2 + w2
```

```
L = sigmoid(s3)
```

```
grad_L = 1.0
```

```
grad_s3 = grad_L * (1 - L) * L
```

```
grad_w2 = grad_s3
```

```
grad_s2 = grad_s3
```

```
grad_s0 = grad_s2
```

```
grad_s1 = grad_s2
```

```
grad_w1 = grad_s1 * x1
```

```
grad_x1 = grad_s1 * w1
```

```
grad_w0 = grad_s0 * x0
```

```
grad_x0 = grad_s0 * w0
```

Multiply gate

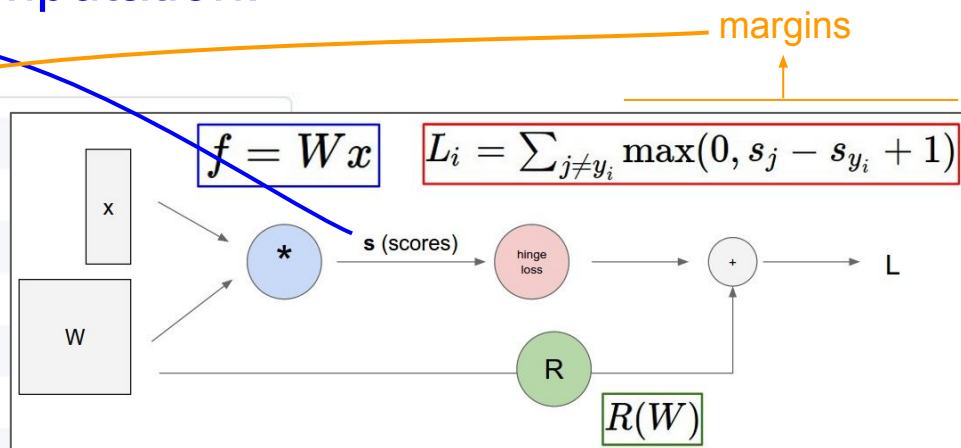
“Flat” Backprop: Do this for assignment 2!

Stage your forward/backward computation!

E.g. for the SVM:

```
# receive W (weights), X (data)
# forward pass (we have 8 lines)
scores = #...
margins = #...
data_loss = #...
reg_loss = #...
loss = data_loss + reg_loss

# backward pass (we have 5 lines)
dmargins = # ... (optionally, we go direct to dscores)
dscores = #...
dW = #...
```



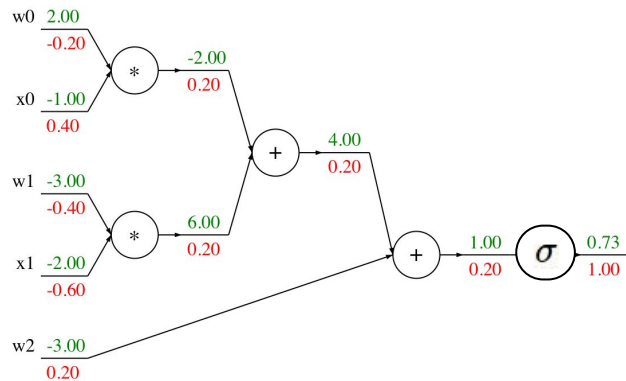
“Flat” Backprop: Do this for assignment 1!

E.g. for two-layer neural net:

```
# receive W1,W2,b1,b2 (weights/biases), X (data)
# forward pass:
h1 = #... function of X,W1,b1
scores = #... function of h1,W2,b2
loss = #... (several lines of code to evaluate Softmax loss)
# backward pass:
dscores = #...
dh1,dW2,db2 = #...
dW1,db1 = #...
```

Backprop Implementation: Modularized API

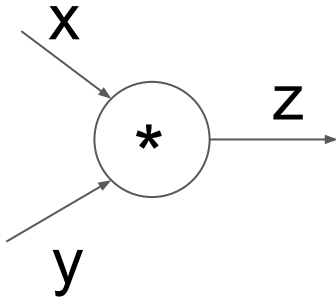
Graph (or Net) object (*rough pseudo code*)



```
class ComputationalGraph(object):  
    #...  
    def forward(inputs):  
        # 1. [pass inputs to input gates...]  
        # 2. forward the computational graph:  
        for gate in self.graph.nodes_topologically_sorted():  
            gate.forward()  
        return loss # the final gate in the graph outputs the loss  
    def backward():  
        for gate in reversed(self.graph.nodes_topologically_sorted()):  
            gate.backward() # little piece of backprop (chain rule applied)  
        return inputs_gradients
```

Modularized implementation: forward / backward API

Gate / Node / Function object: Actual PyTorch code



(x,y,z are scalars)

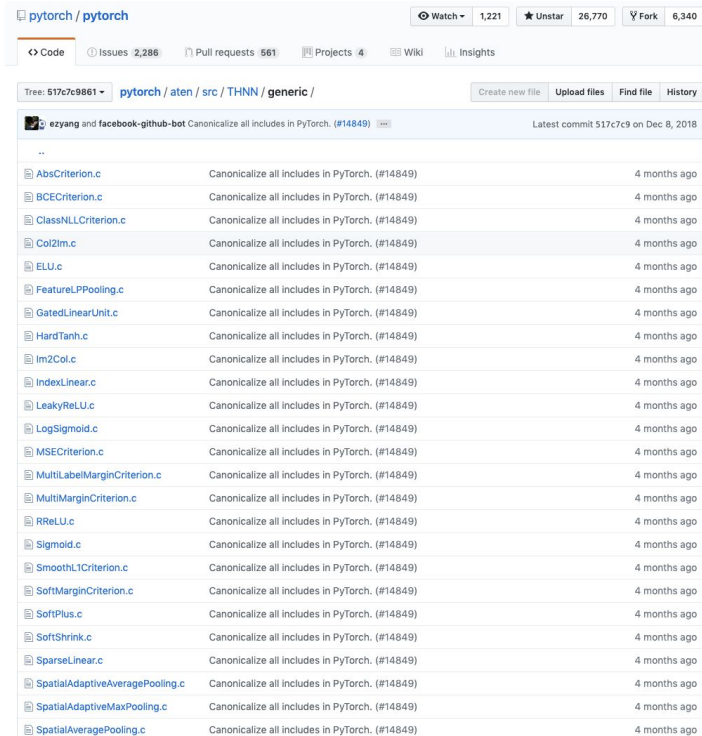
```
class Multiply(torch.autograd.Function):
    @staticmethod
    def forward(ctx, x, y):
        ctx.save_for_backward(x, y)
        z = x * y
        return z
    @staticmethod
    def backward(ctx, grad_z):
        x, y = ctx.saved_tensors
        grad_x = y * grad_z # dz/dx * dL/dz
        grad_y = x * grad_z # dz/dy * dL/dz
        return grad_x, grad_y
```

Need to stash
some values for
use in backward

Upstream
gradient

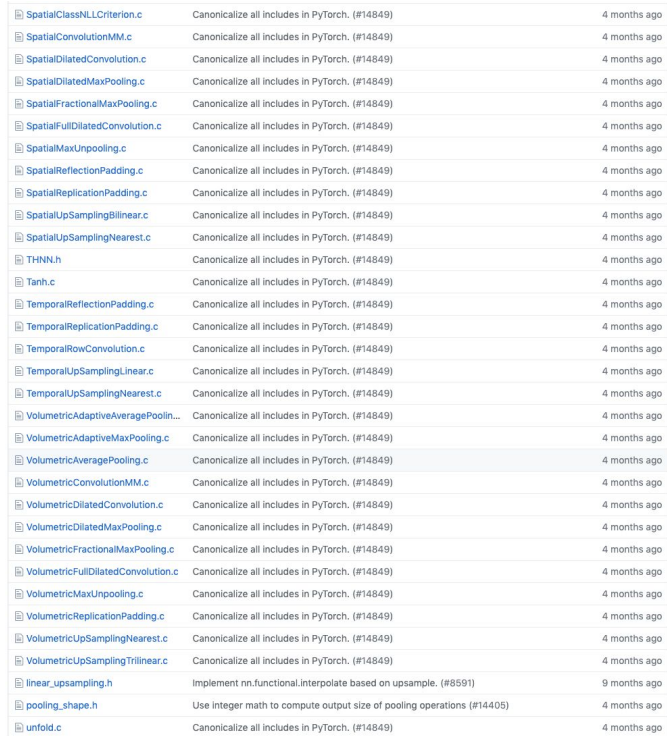
Multiply upstream
and local gradients

Example: PyTorch operators



The screenshot shows the GitHub repository for PyTorch, specifically the 'aten/src/THNN/generic/' directory. The repository has 1,221 watches, 26,770 stars, and 6,340 forks. The current commit is by 'ezyang and facebook-github-bot' on Dec 8, 2018. The table below lists various operators, each with a description and a commit date.

Operator	Description	Commit Date
AbsCriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
BCECriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
ClassNLLCriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
Col2Im.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
ELU.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
FeatureLPPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
GatedLinearUnit.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
HardTanh.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
Im2Col.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
IndexLinear.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
LeakyReLU.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
LogSigmoid.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
MSECriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
MultiLabelMarginCriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
MultiMarginCriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
RReLU.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
Sigmoid.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SmoothL1Criterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SoftMarginCriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SoftPlus.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SoftShrink.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SparseLinear.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialAdaptiveAveragePooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialAdaptiveMaxPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialAveragePooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago



The screenshot shows the GitHub repository for PyTorch, specifically the 'aten/src/THNN/generic/' directory. The repository has 1,221 watches, 26,770 stars, and 6,340 forks. The current commit is by 'ezyang and facebook-github-bot' on Dec 8, 2018. The table below lists various operators, each with a description and a commit date.

SpatialClassNLLCriterion.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialConvolutionMM.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialDilatedConvolution.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialDilatedMaxPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialFractionalMaxPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialFullDilatedConvolution.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialMaxUnpooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialReflectionPadding.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialReplicationPadding.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialUpsamplingBilinear.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
SpatialUpsamplingNearest.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
THNN.h	Canonicalize all includes in PyTorch. (#14849)	4 months ago
Tanh.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
TemporalReflectionPadding.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
TemporalReplicationPadding.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
TemporalRowConvolution.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
TemporalUpsamplingLinear.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
TemporalUpsamplingNearest.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricAdaptiveAveragePooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricAdaptiveMaxPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricAveragePooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricConvolutionMM.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricDilatedConvolution.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricDilatedMaxPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricFractionalMaxPooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricFullDilatedConvolution.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricMaxUnpooling.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricReplicationPadding.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricUpsamplingNearest.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
VolumetricUpsamplingTrilinear.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago
linear_upsampling.h	Implement nn.functional.interpolate based on upsample. (#8591)	9 months ago
pooling_shape.h	Use integer math to compute output size of pooling operations (#14405)	4 months ago
unfold.c	Canonicalize all includes in PyTorch. (#14849)	4 months ago

PyTorch sigmoid layer

```
1 #ifndef TH_GENERIC_FILE
2 #define TH_GENERIC_FILE "THNN/generic/Sigmoid.c"
3 #else
```

```
4
5 void THNN_(Sigmoid_updateOutput)(
6     THNNState *state,
7     THTensor *input,
8     THTensor *output)
9 {
10     THTensor_(sigmoid)(output, input);
11 }
```

Forward

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

```
12
13 void THNN_(Sigmoid_updateGradInput)(
14     THNNState *state,
15     THTensor *gradOutput,
16     THTensor *gradInput,
17     THTensor *output)
18 {
19     THNN_CHECK_NELEMENT(output, gradOutput);
20     THTensor_(resizeAs)(gradInput, output);
21     TH_TENSOR_APPLY3(scalar_t, gradInput, scalar_t, gradOutput, scalar_t, output,
22         scalar_t z = *output_data;
23         *gradInput_data = *gradOutput_data * (1. - z) * z;
24     );
25 }
26
27 #endif
```

[Source](#)

PyTorch sigmoid layer

```
1 #ifndef TH_GENERIC_FILE
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19     THNN_CHECK_NELEMENT(output, gradOutput);
20     THTensor_(resizeAs)(gradInput, output);
21     TH_TENSOR_APPLY3(scalar_t, gradInput, scalar_t, gradOutput, scalar_t, output,
22         scalar_t z = *output_data;
23         *gradInput_data = *gradOutput_data * (1. - z) * z;
24     );
25 }
26
27 #endif
```

```
static void sigmoid_kernel(TensorIterator& iter) {
    AT_DISPATCH_FLOATING_TYPES(iter.dtype(), "sigmoid_cpu", [&]() {
        unary_kernel_vec(
            iter,
            [=](scalar_t a) -> scalar_t { return (1 / (1 + std::exp((-a)))); },
            [=](Vec256<scalar_t> a) {
                a = Vec256<scalar_t>((scalar_t)(0)) - a;
                a = a.exp();
                a = Vec256<scalar_t>((scalar_t)(1)) + a;
                a = a.reciprocal();
                return a;
            });
    });
}
```

Forward actually defined elsewhere...

```
return (1 / (1 + std::exp((-a))));
```

[Source](#)

PyTorch sigmoid layer

```
1 #ifndef TH_GENERIC_FILE
2 #define TH_GENERIC_FILE "THNN/generic/Sigmoid.c"
3 #else
```

```
4
5 void THNN_(Sigmoid_updateOutput)(
6     THNNState *state,
7     THTensor *input,
8     THTensor *output)
9 {
10     THTensor_(sigmoid)(output, input);
11 }
```

Forward

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

```
12
13 void THNN_(Sigmoid_updateGradInput)(
14     THNNState *state,
15     THTensor *gradOutput,
16     THTensor *gradInput,
17     THTensor *output)
18 {
19     THNN_CHECK_NELEMENT(output, gradOutput);
20     THTensor_(resizeAs)(gradInput, output);
21     TH_TENSOR_APPLY3(scalar_t, gradInput, scalar_t, gradOutput, scalar_t, output,
22         scalar_t z = *output_data;
23         *gradInput_data = *gradOutput_data * (1. - z) * z;
24     );
25 }
```

Backward

$$(1 - \sigma(x)) \sigma(x)$$

```
static void sigmoid_kernel(TensorIterator& iter) {
    AT_DISPATCH_FLOATING_TYPES(iter.dtype(), "sigmoid_cpu", [&]() {
        unary_kernel_vec(
            iter,
            [=](scalar_t a) -> scalar_t { return (1 / (1 + std::exp((-a)))); },
            [=](Vec256<scalar_t> a) {
                a = Vec256<scalar_t>((scalar_t)(0)) - a;
                a = a.exp();
                a = Vec256<scalar_t>((scalar_t)(1)) + a;
                a = a.reciprocal();
                return a;
            });
    });
}
```

Forward actually defined [elsewhere...](#)

[Source](#)

Summary for today:

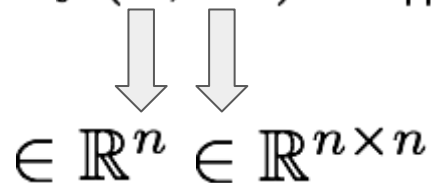
- **(Fully-connected) Neural Networks** are stacks of linear functions and nonlinear activation functions; they have much more representational power than linear classifiers
- **backpropagation** = recursive application of the chain rule along a computational graph to compute the gradients of all inputs/parameters/intermediates
- implementations maintain a graph structure, where the nodes implement the **forward()** / **backward()** API
- **forward**: compute result of an operation and save any intermediates needed for gradient computation in memory
- **backward**: apply the chain rule to compute the gradient of the loss function with respect to the inputs

So far: backprop with scalars

Next time: vector-valued functions!

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

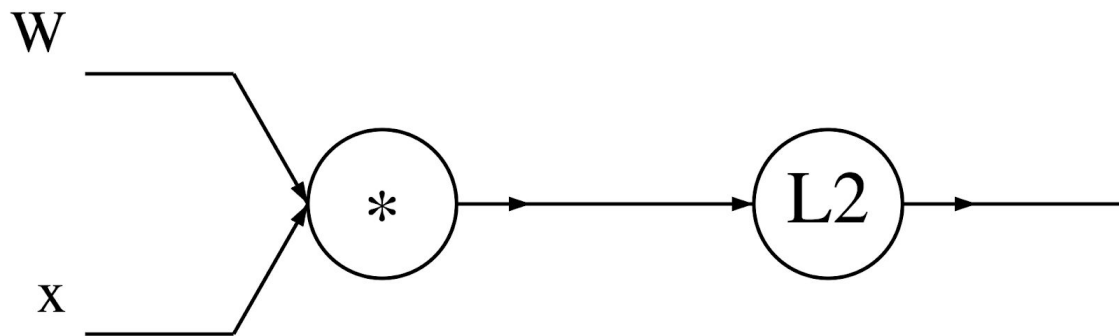
A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$



The diagram shows two grey arrows pointing downwards from the variables x and W in the equation above. The arrow from x points to the expression $\in \mathbb{R}^n$, and the arrow from W points to the expression $\in \mathbb{R}^{n \times n}$.

$$\in \mathbb{R}^n \quad \in \mathbb{R}^{n \times n}$$

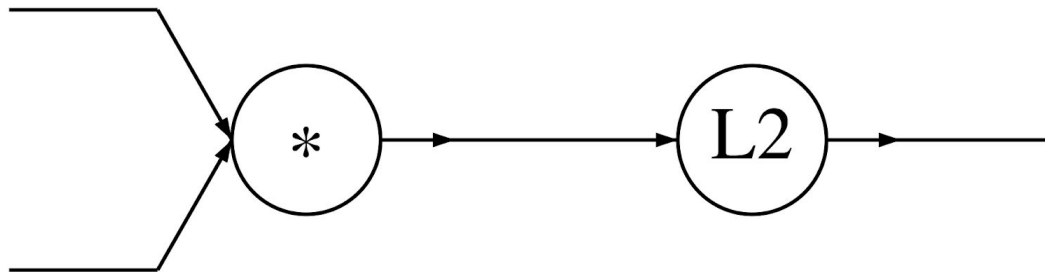
A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$



A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$

$$\begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} x$$

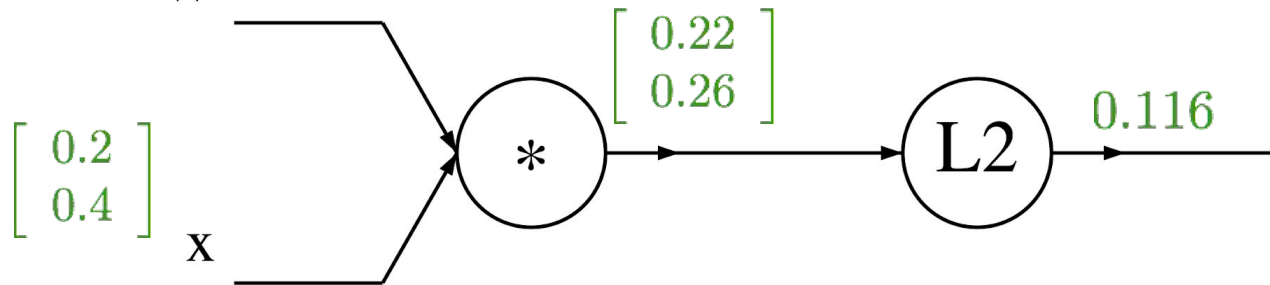


$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$



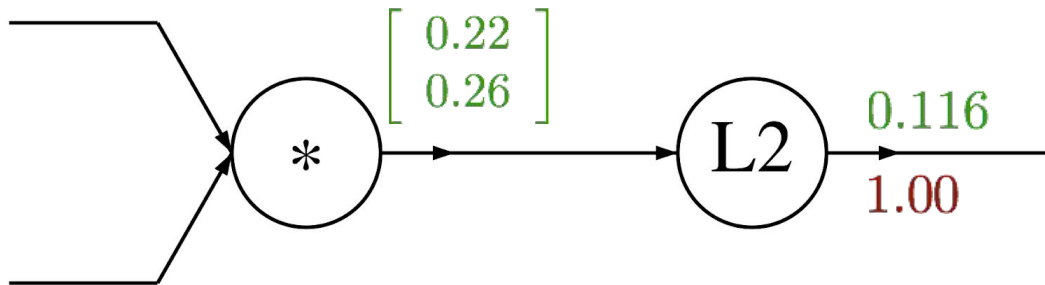
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$$f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$

$$\begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix} x$$

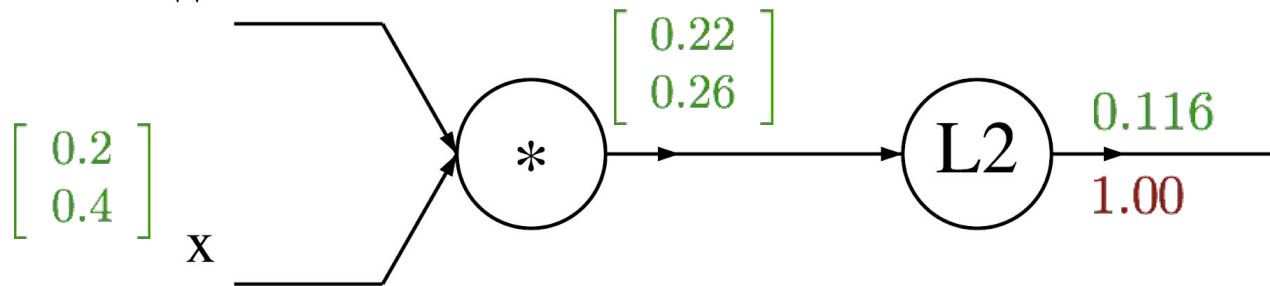


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A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$



$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

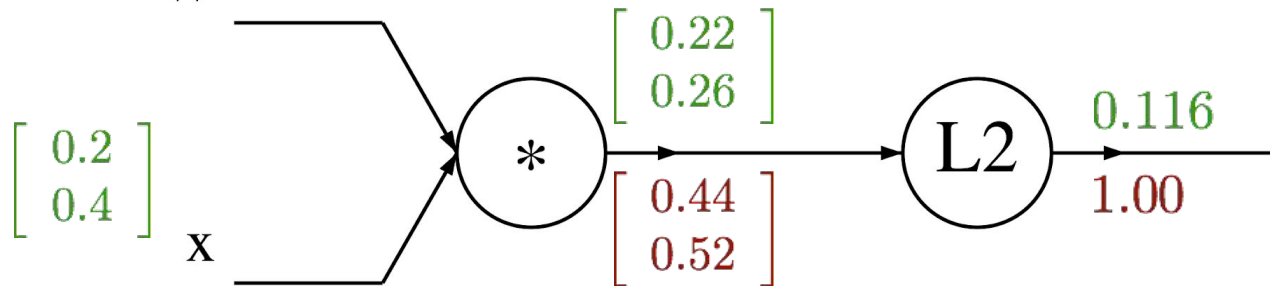
$$f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$$

$$\frac{\partial f}{\partial q_i} = 2q_i$$

$$\nabla_q f = 2q$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$



$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

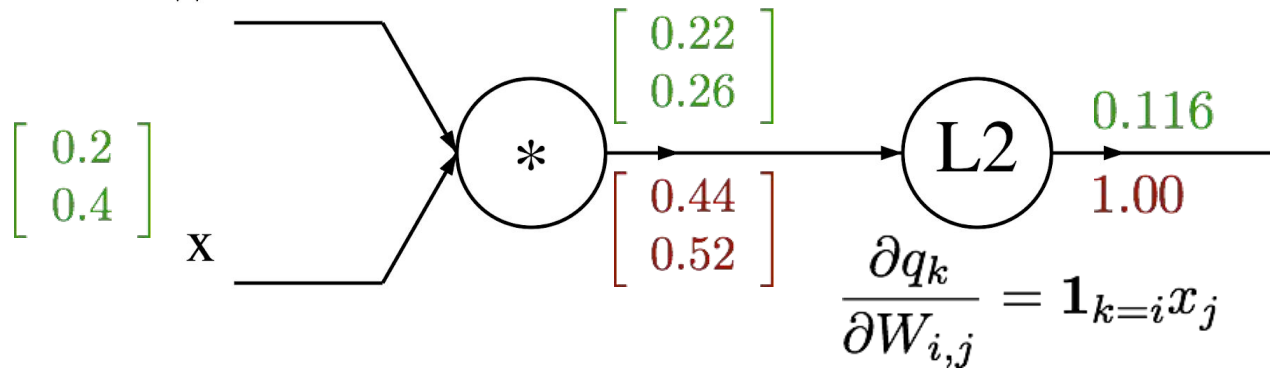
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A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$

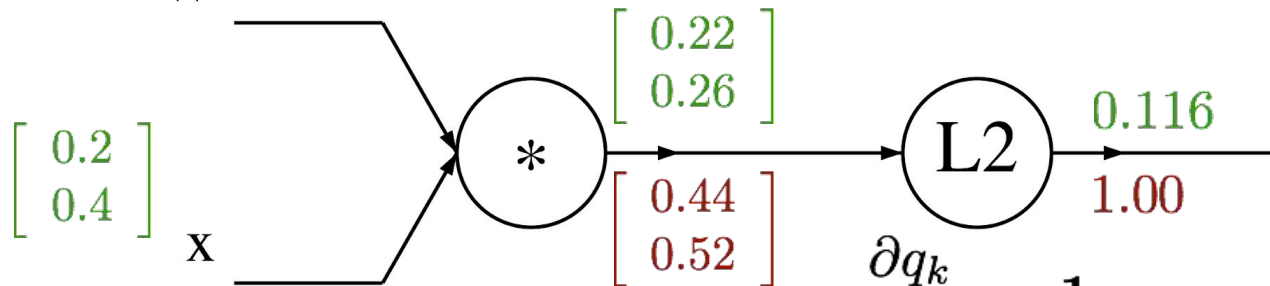


$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$\begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix} W$$



$$\frac{\partial q_k}{\partial W_{i,j}} = \mathbf{1}_{k=i} x_j$$

$$\frac{\partial f}{\partial W_{i,j}} = \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial W_{i,j}}$$

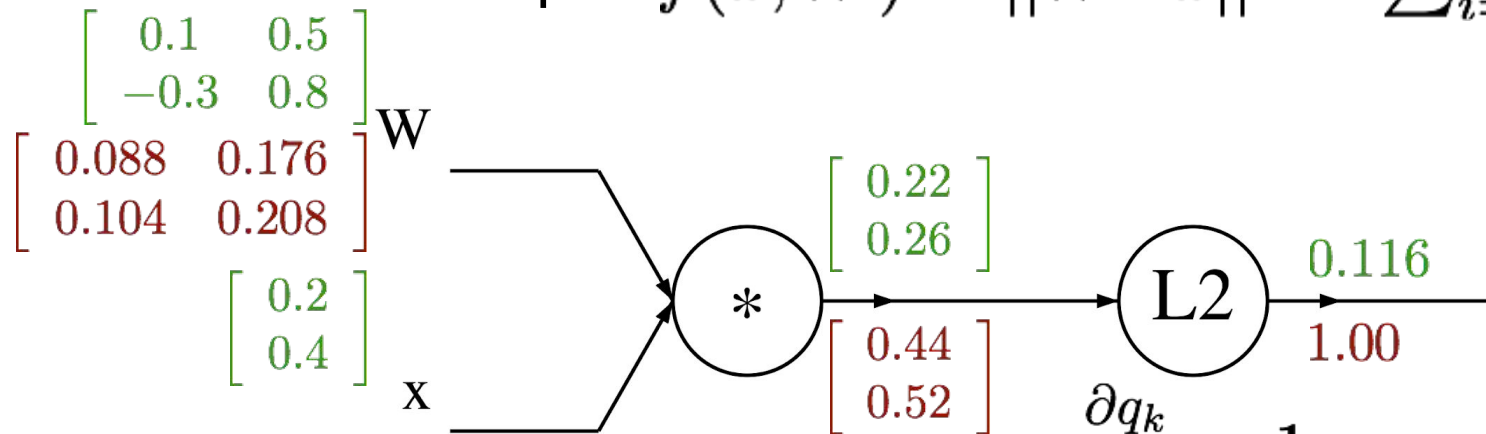
$$= \sum_k (2q_k) (\mathbf{1}_{k=i} x_j)$$

$$= 2q_i x_j$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \dots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \dots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = \|q\|^2 = q_1^2 + \dots + q_n^2$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

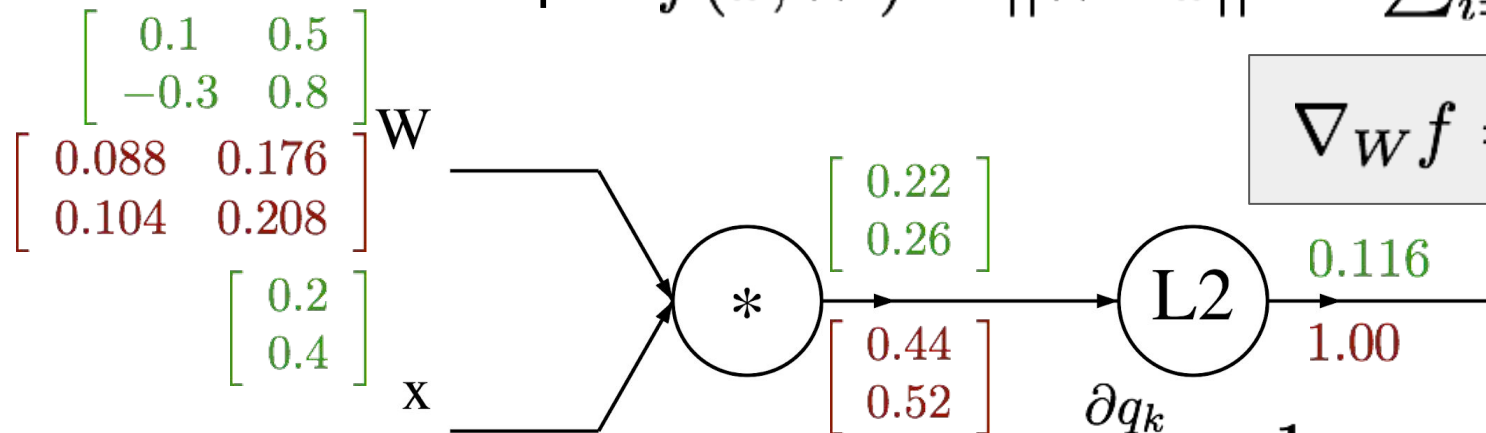


$$\begin{aligned} \frac{\partial q_k}{\partial W_{i,j}} &= \mathbf{1}_{k=i} x_j \\ \frac{\partial f}{\partial W_{i,j}} &= \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial W_{i,j}} \\ &= \sum_k (2q_k) (\mathbf{1}_{k=i} x_j) \\ &= 2q_i x_j \end{aligned}$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$



$$\nabla_W f = 2q \cdot x^T$$

$$\frac{\partial q_k}{\partial W_{i,j}} = \mathbf{1}_{k=i} x_j$$

$$\frac{\partial f}{\partial W_{i,j}} = \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial W_{i,j}}$$

$$= \sum_k (2q_k) (\mathbf{1}_{k=i} x_j)$$

$$= 2q_i x_j$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \dots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \dots + W_{n,n}x_n \end{pmatrix}$$

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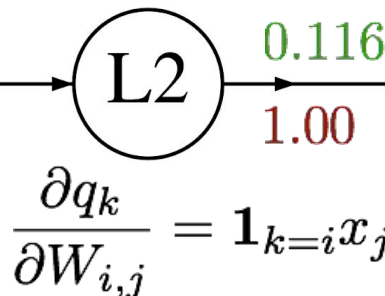
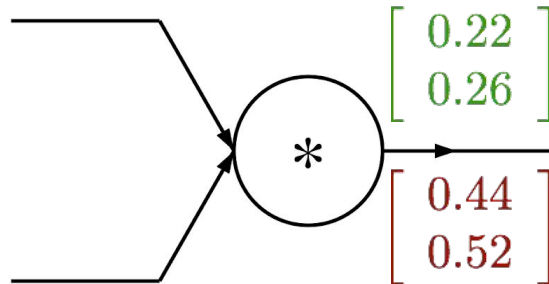
A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

$$W = \begin{bmatrix} 0.1 & 0.5 \\ -0.3 & 0.8 \end{bmatrix}$$

$$x = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$$

W

x



$$\nabla_W f = 2q \cdot x^T$$

Always check: The gradient with respect to a variable should have the same shape as the variable

$$\frac{\partial q_k}{\partial W_{i,j}} = \mathbf{1}_{k=i} x_j$$

$$\frac{\partial f}{\partial W_{i,j}} = \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial W_{i,j}}$$

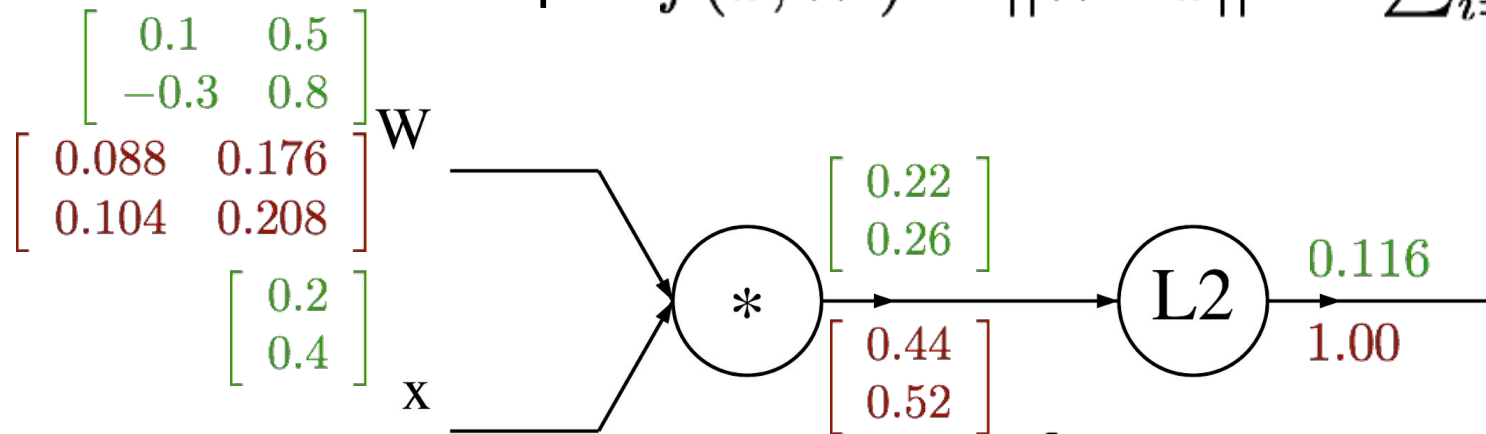
$$= \sum_k (2q_k) (\mathbf{1}_{k=i} x_j)$$

$$= 2q_i x_j$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \dots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \dots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = \|q\|^2 = q_1^2 + \dots + q_n^2$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$

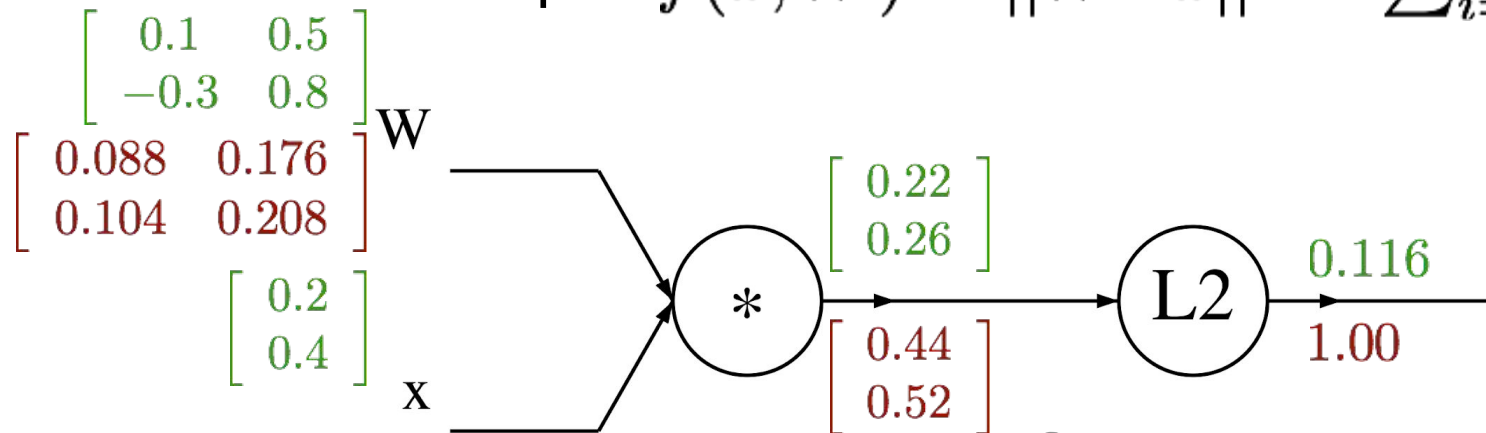


$$\frac{\partial q_k}{\partial x_i} = W_{k,i}$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \cdots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \cdots + W_{n,n}x_n \end{pmatrix}$$

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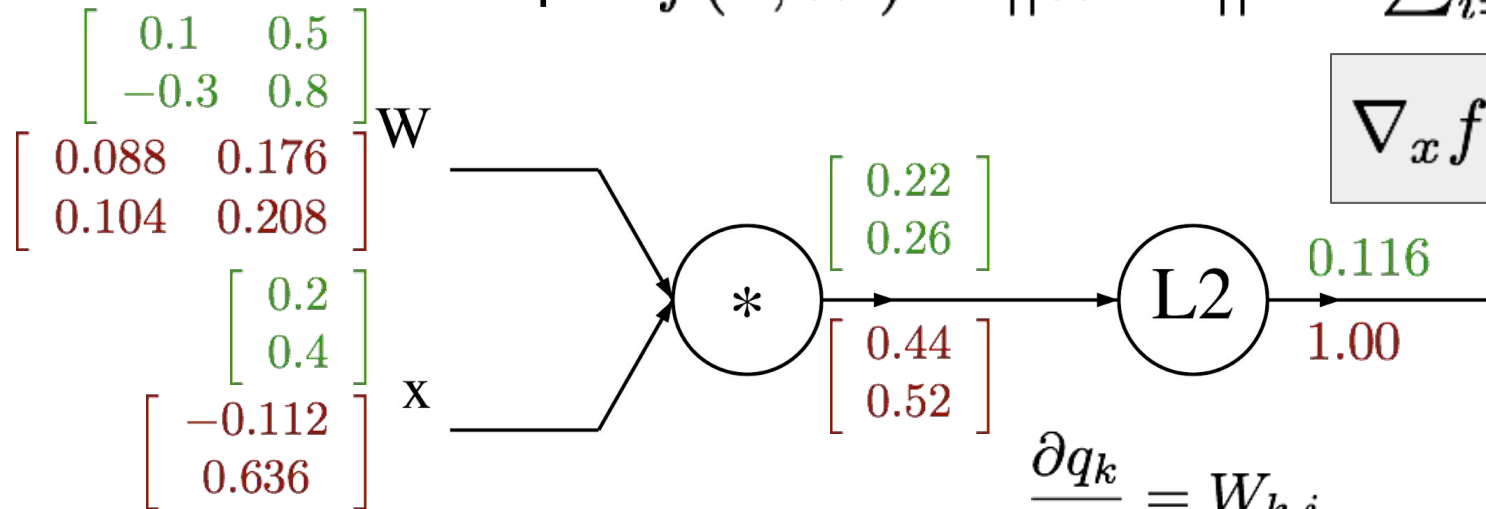


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$$f(q) = \|q\|^2 = q_1^2 + \cdots + q_n^2$$

$$\begin{aligned} \frac{\partial q_k}{\partial x_i} &= W_{k,i} \\ \frac{\partial f}{\partial x_i} &= \sum_k \frac{\partial f}{\partial q_k} \frac{\partial q_k}{\partial x_i} \\ &= \sum_k 2q_k W_{k,i} \end{aligned}$$

A vectorized example: $f(x, W) = \|W \cdot x\|^2 = \sum_{i=1}^n (W \cdot x)_i^2$



$$\nabla_x f = 2W^T \cdot q$$

$$q = W \cdot x = \begin{pmatrix} W_{1,1}x_1 + \dots + W_{1,n}x_n \\ \vdots \\ W_{n,1}x_1 + \dots + W_{n,n}x_n \end{pmatrix}$$

$$f(q) = \|q\|^2 = q_1^2 + \dots + q_n^2$$

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In discussion section: A matrix example...

$$z_1 = XW_1$$

$$h_1 = \text{ReLU}(z_1)$$

$$\hat{y} = h_1 W_2$$

$$L = \|\hat{y}\|_2^2$$

$$\frac{\partial L}{\partial W_2} = ?$$

$$\frac{\partial L}{\partial W_1} = ?$$

