Lecture 3:
Loss Functions

Administrative: Assignment 0

- Due **tomorrow night (Apr 8)** by 11:59pm
- Easy assignment
- Hardest part is learning how to use colab and how to submit on gradescope
- Worth 0% of your grade
- Used to evaluate how prepared you are to take this course

Administrative: Assignment 1

Due 4/16 11:59pm

- K-Nearest Neighbor
- Linear classifiers: SVM, Softmax

Administrative: EdStem

Please make sure to check and read all pinned EdStem posts.

Administrative: Fridays

This Friday 9:30-10:30am and again 12:30-1:30pm

Project Design & Backprop

Administrative: Course Project

Project proposal due 4/29 11:59pm

“Is X a valid project for 493G1?”

- Anything related to deep learning or computer vision
- Maximum of 3 students per team
- Make a EdStem private post or come to TA Office Hours

More info on the website

Last time: Image Classification: A core task in Computer Vision


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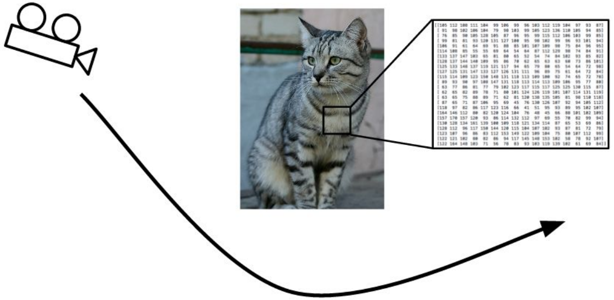
(assume given a set of labels)
{dog, cat, truck, plane, ...}

→

cat
dog
bird
deer
truck

Recall from last time: Challenges of recognition

Viewpoint



Illumination

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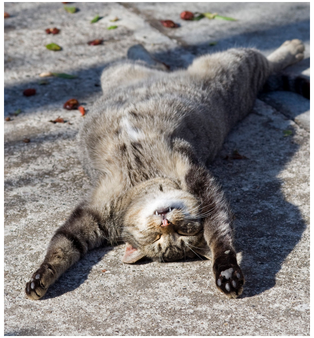
Deformation



This image by Umberto Salvagnin is licensed under CC-BY 2.0

Occlusion

This image by jonsson is licensed under CC-BY 2.0

Clutter

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Intraclass Variation

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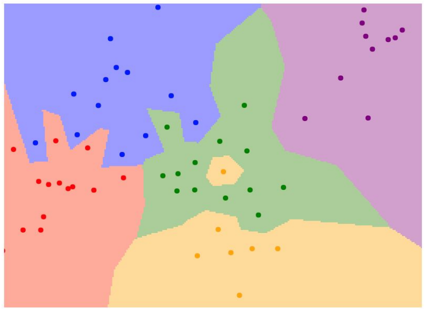
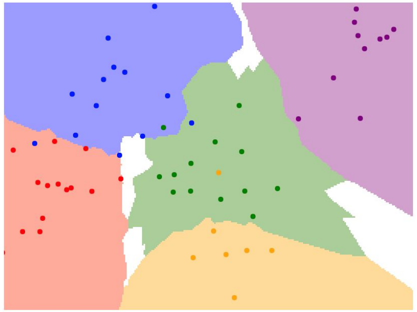
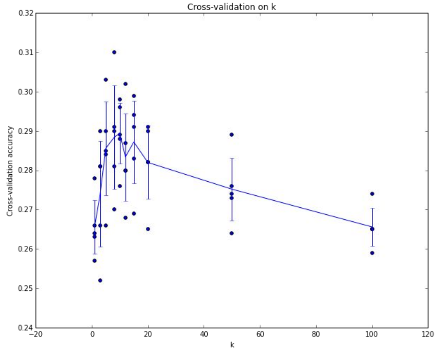
Recall from last time: data-driven approach, kNN

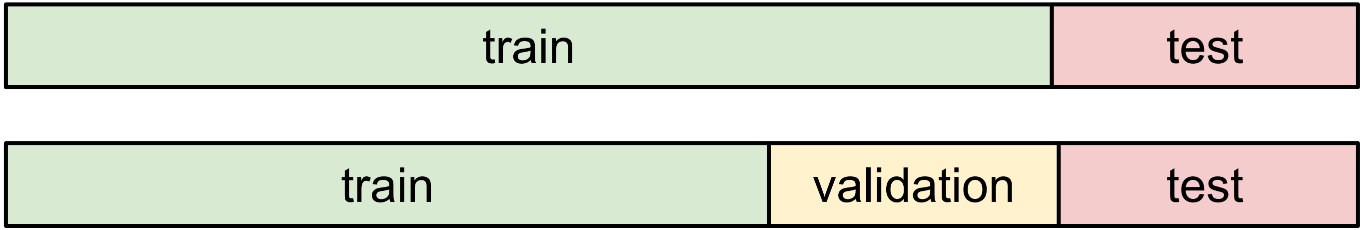
1-NN classifier



5-NN classifier







train

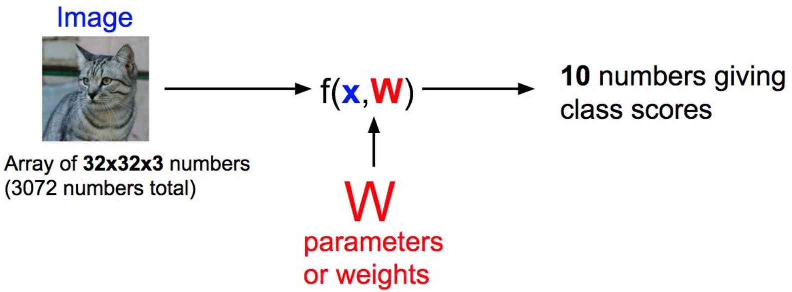
test

train

validation

test

Recall from last time: Linear Classifier



Image

Array of **32x32x3** numbers (3072 numbers total)

f(**x**,**W**)

10 numbers giving class scores

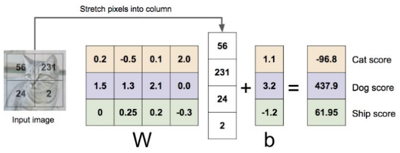
W

parameters or weights

$$f(x,W) = Wx + b$$

Algebraic Viewpoint

$f(x,W) = Wx$

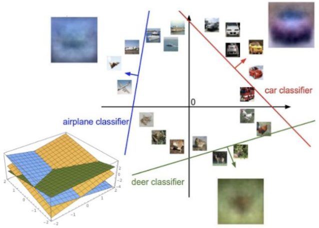


Visual Viewpoint

One template per class

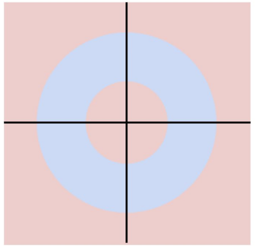
Geometric Viewpoint

Hyperplanes cutting up space



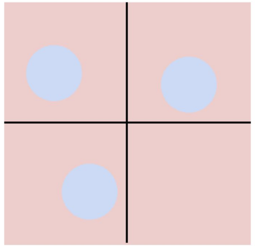
Class 1:
1 <= L2 norm <= 2

Class 2:
Everything else

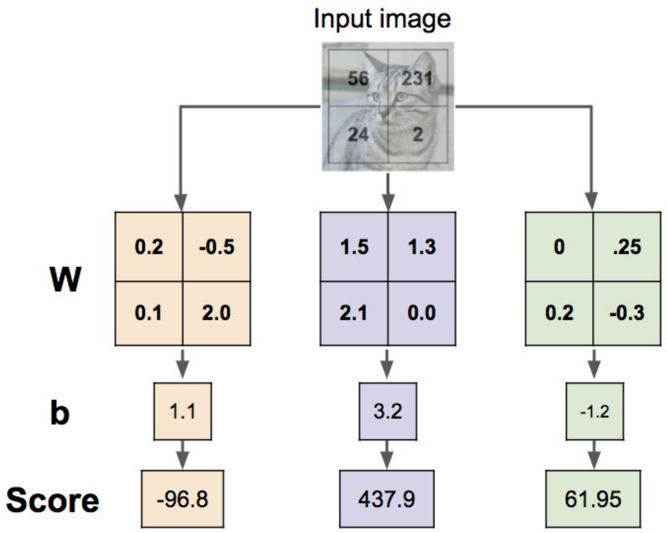


Class 1:
Three modes

Class 2:
Everything else



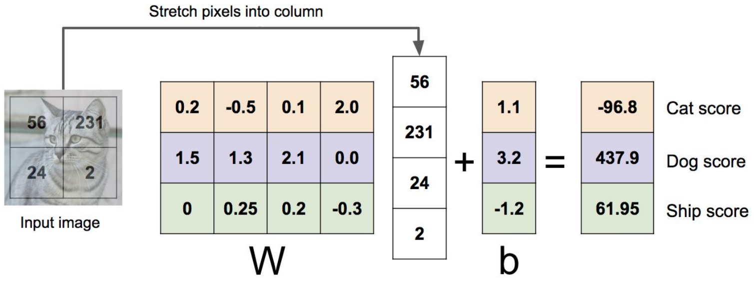
Interpreting a Linear Classifier: Visual Viewpoint



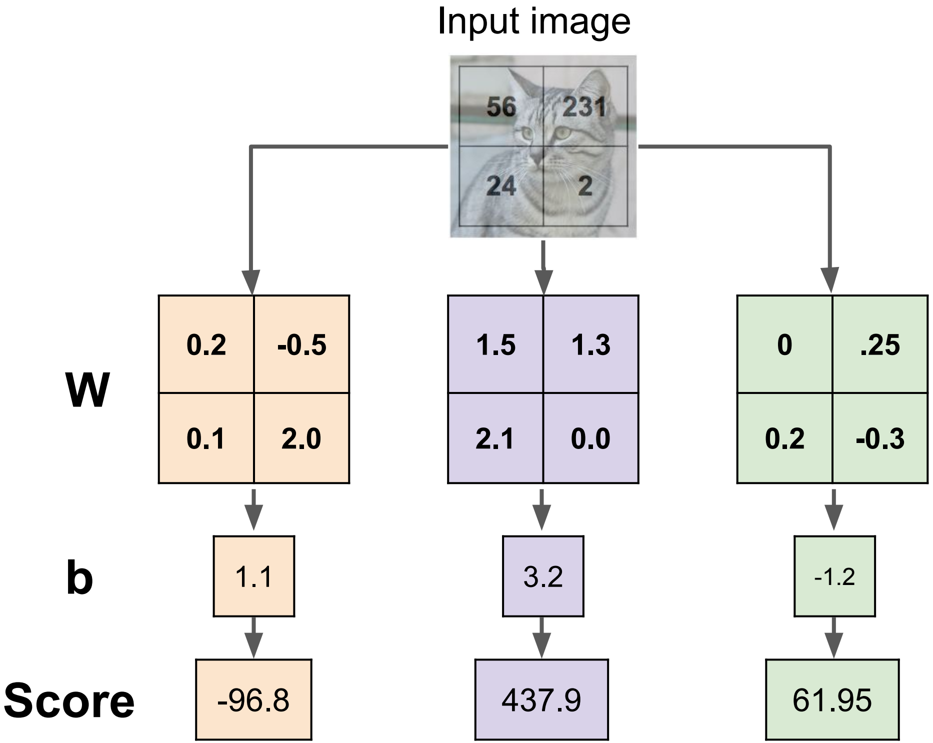
Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Algebraic Viewpoint

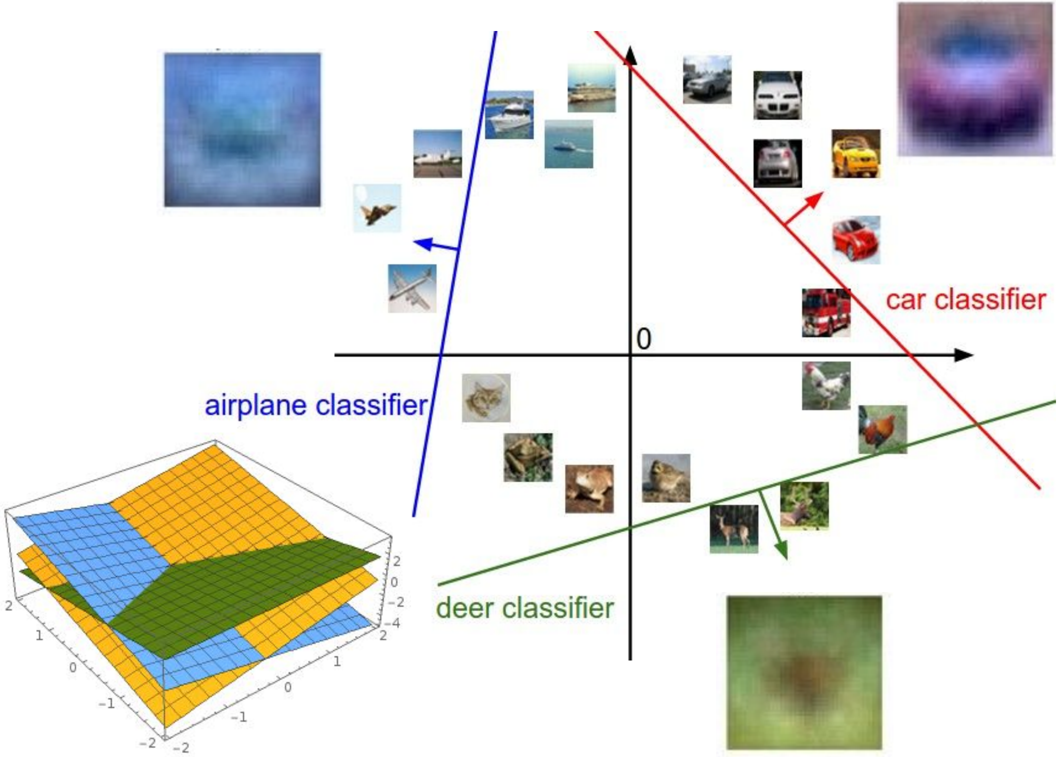
f(x,W) = Wx



Visual Viewpoint



Interpreting a Linear Classifier: Geometric Viewpoint



$$f(x,W) = Wx + b$$

Array of **32x32x3** numbers
(3072 numbers total)

Plot created using Wolfram Cloud

Linear Classifier

Parametric Approach

Image

Array of **32x32x3** numbers
(3072 numbers total)

$f(\mathbf{x},\mathbf{W})$

10 numbers giving
class scores

W

parameters
or weights

Parametric Approach: Linear Classifier

$$f(x,W) = Wx$$

Image

Array of **32x32x3** numbers
(3072 numbers total)

$f(\mathbf{x},\mathbf{W})$

W

parameters
or weights

10 numbers giving
class scores

Parametric Approach: Linear Classifier

$$\underset{10x1}{\boxed{f(x,W)}} = \underset{10x3072}{\boxed{W}}\overset{3072x1}{\boxed{x}}$$

Image

Array of **32x32x3** numbers
(3072 numbers total)

→ f(**x**,**W**) →

10 numbers giving
class scores

↑

W

parameters
or weights

Parametric Approach: Linear Classifier

$$\underset{10x1}{\boxed{f(x,W)}} = \underset{10x3072}{\boxed{W}}\overset{3072x1}{\boxed{x}} + \boxed{b}\ \ 10x1$$

Image

Array of **32x32x3** numbers
(3072 numbers total)

$f(x, W) \rightarrow$ **10** numbers giving class scores

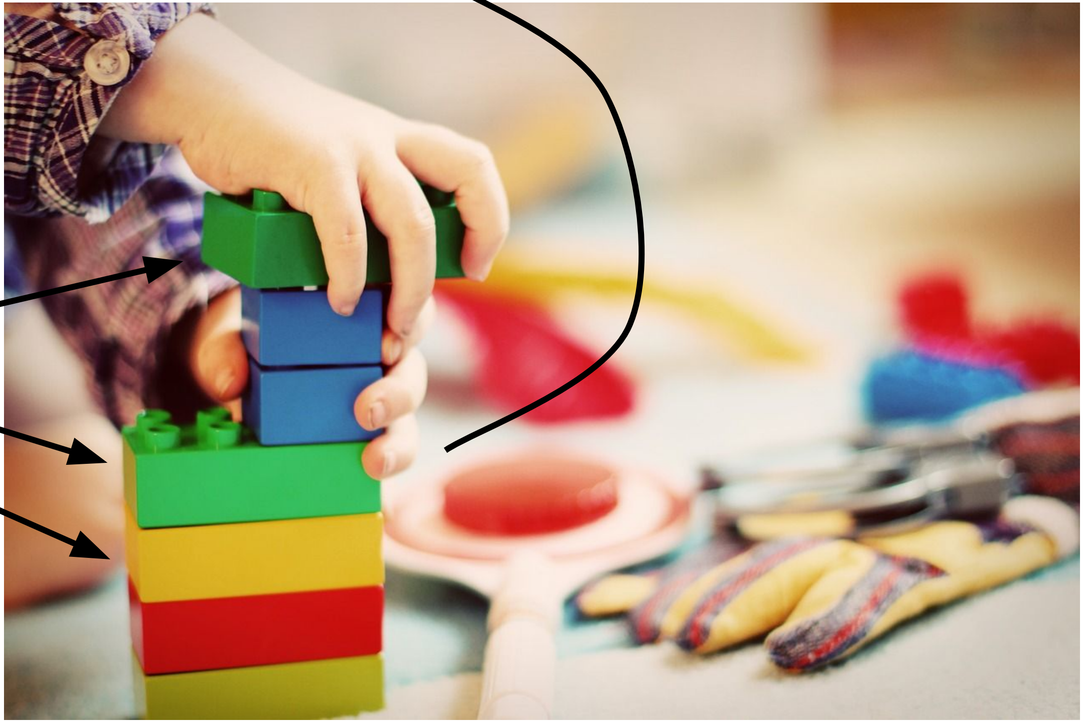
$\uparrow$

W

parameters
or weights

Neural Network

Linear classifiers



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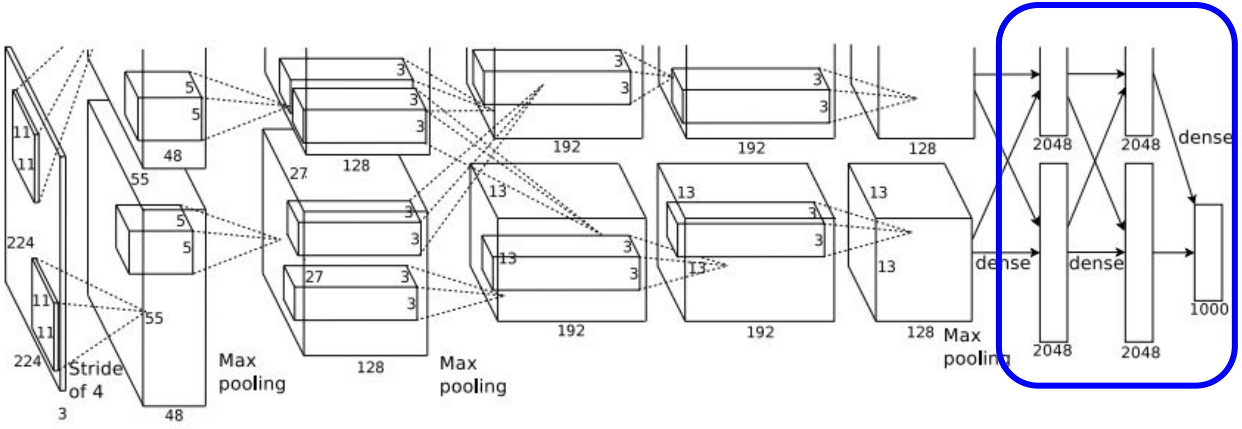
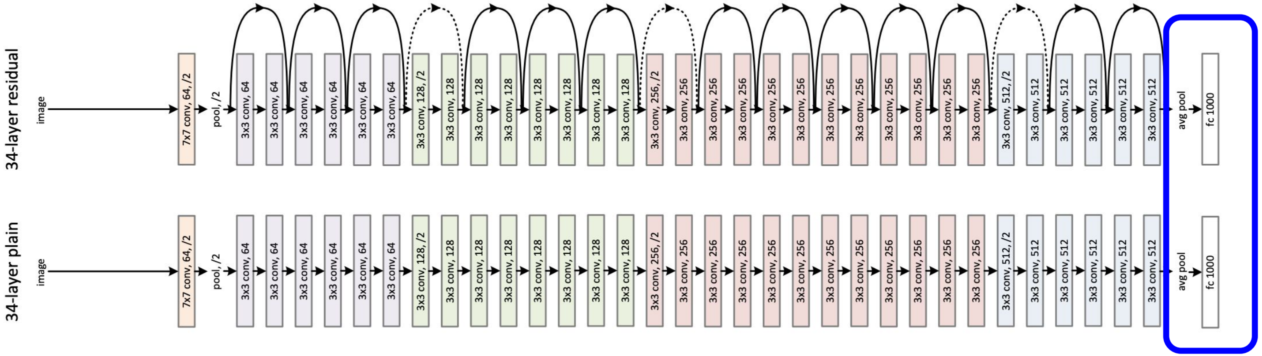


[Krizhevsky et al. 2012]

Linear layers



[He et al. 2015]

Recall CIFAR10

50,000 training images
each image is **32x32x3**

10,000 test images.

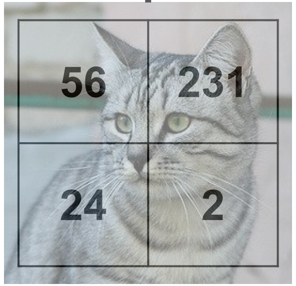
Algebraic viewpoint: Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

Flatten tensors into a vector

Input image

Algebraic viewpoint: Example with an image with 4 pixels, and 3 classes (cat/dog/ship)

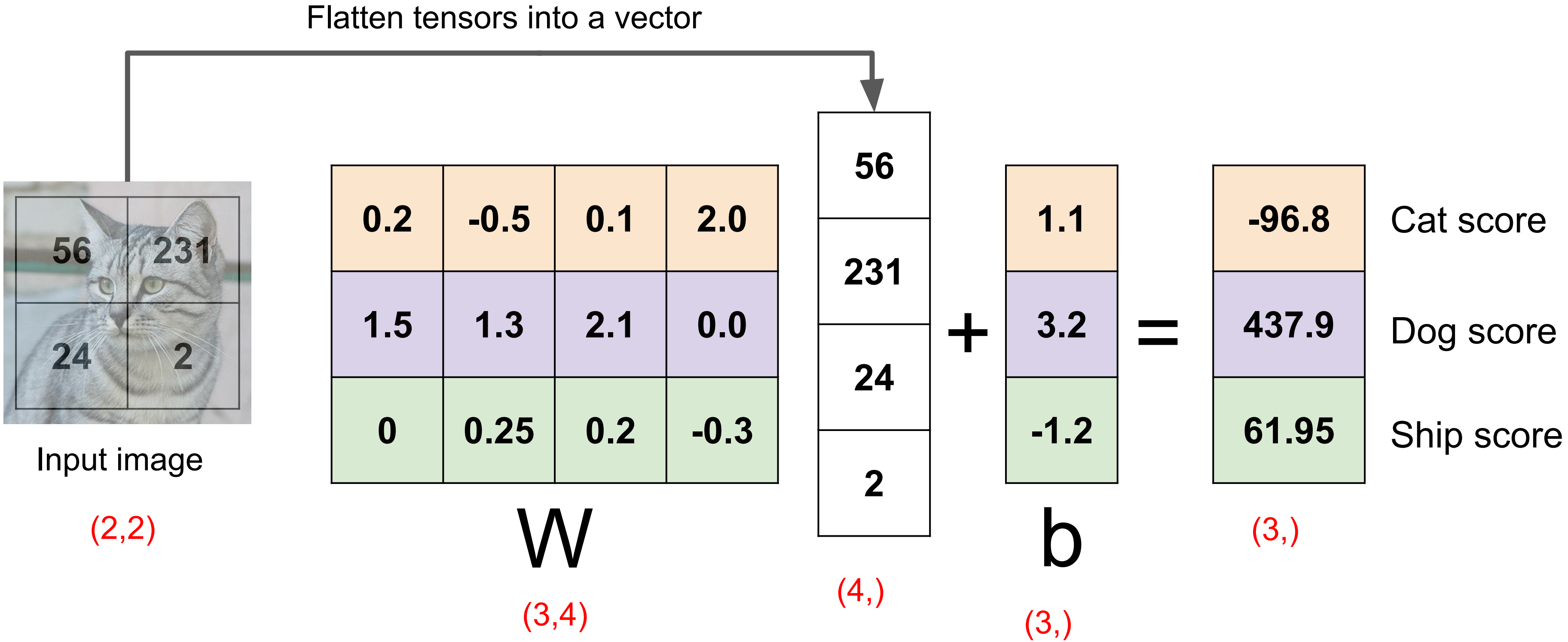
Flatten tensors into a vector



Input image

$$\underset{W}{\begin{bmatrix}0.2 & -0.5 & 0.1 & 2.0\\ 1.5 & 1.3 & 2.1 & 0.0\\ 0 & 0.25 & 0.2 & -0.3\end{bmatrix}}\begin{bmatrix}56\\ 231\\ 24\\ 2\end{bmatrix}+\underset{b}{\begin{bmatrix}1.1\\ 3.2\\ -1.2\end{bmatrix}}=\begin{bmatrix}-96.8\\ 437.9\\ 61.95\end{bmatrix}\begin{array}{l}\text{Cat score}\\ \text{Dog score}\\ \text{Ship score}\end{array}$$

Algebraic viewpoint: Example with an image with 4 pixels, and 3 classes (cat/dog/ship)



Flatten tensors into a vector

Input image

(2,2)

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix}}_{(4,)} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b} = \underbrace{\begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}}_{(3,)}$$

(3,4)

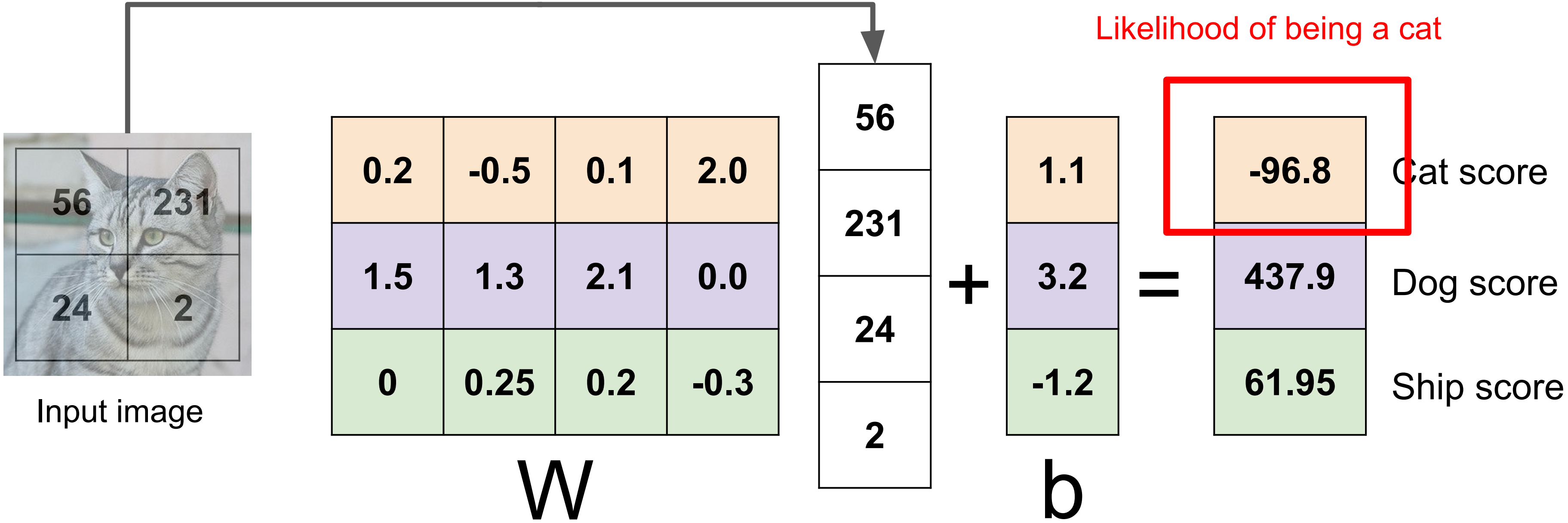
(3,)

Cat score

Dog score

Ship score

Algebraic viewpoint: Example with an image with 4 pixels, and 3 classes (cat/dog/ship)



Flatten tensors into a vector

Input image

$$\underbrace{\begin{bmatrix}0.2 & -0.5 & 0.1 & 2.0\\ 1.5 & 1.3 & 2.1 & 0.0\\ 0 & 0.25 & 0.2 & -0.3\end{bmatrix}}_{W}\begin{bmatrix}56\\ 231\\ 24\\ 2\end{bmatrix}+\underbrace{\begin{bmatrix}1.1\\ 3.2\\ -1.2\end{bmatrix}}_{b}=\begin{bmatrix}-96.8\\ 437.9\\ 61.95\end{bmatrix}$$

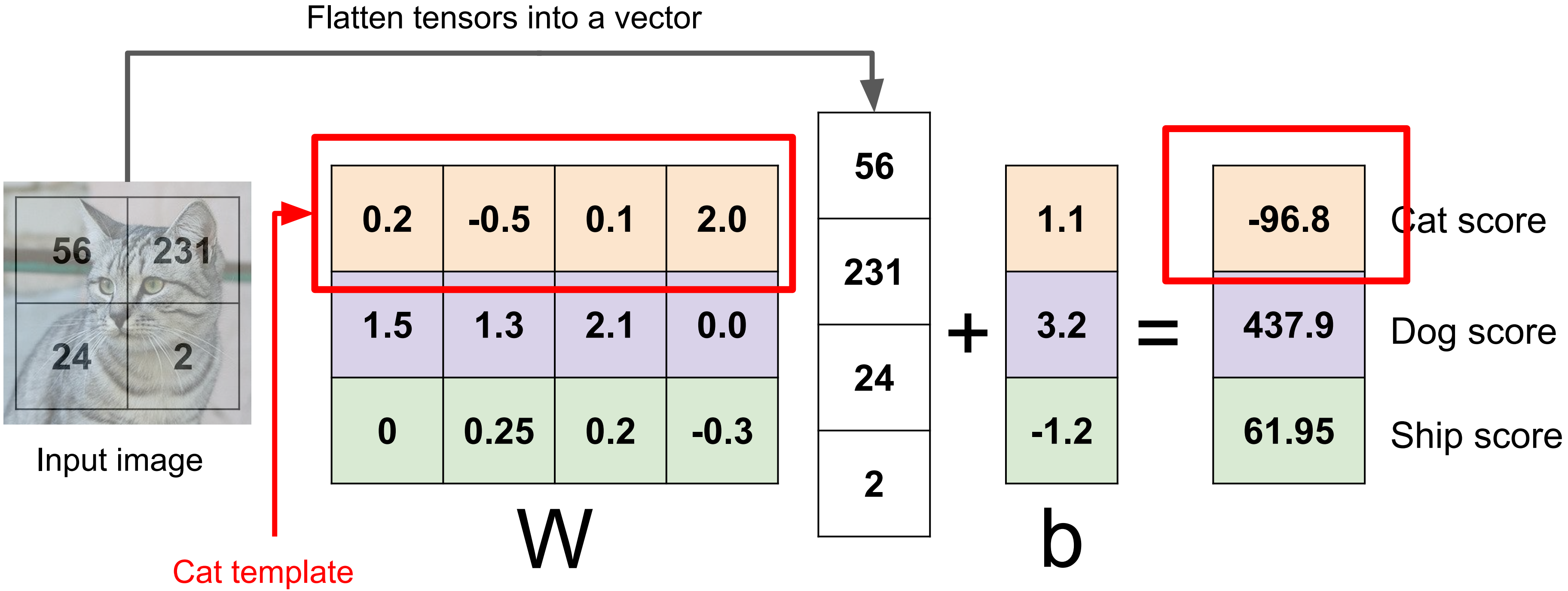
Likelihood of being a cat

Cat score

Dog score

Ship score

Algebraic viewpoint: Example with an image with 4 pixels, and 3 classes (cat/dog/ship)



Flatten tensors into a vector

56	231
24	2

Input image

$$\underbrace{\begin{bmatrix} 0.2 & -0.5 & 0.1 & 2.0 \\ 1.5 & 1.3 & 2.1 & 0.0 \\ 0 & 0.25 & 0.2 & -0.3 \end{bmatrix}}_{W} \begin{bmatrix} 56 \\ 231 \\ 24 \\ 2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1.1 \\ 3.2 \\ -1.2 \end{bmatrix}}_{b} = \begin{bmatrix} -96.8 \\ 437.9 \\ 61.95 \end{bmatrix}$$

Cat template

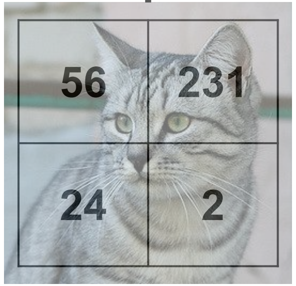
Cat score

Dog score

Ship score

Algebraic viewpoint: Bias trick to simply computation

Flatten tensors into a vector

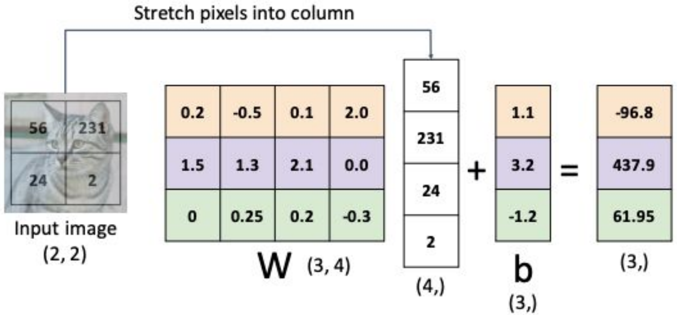


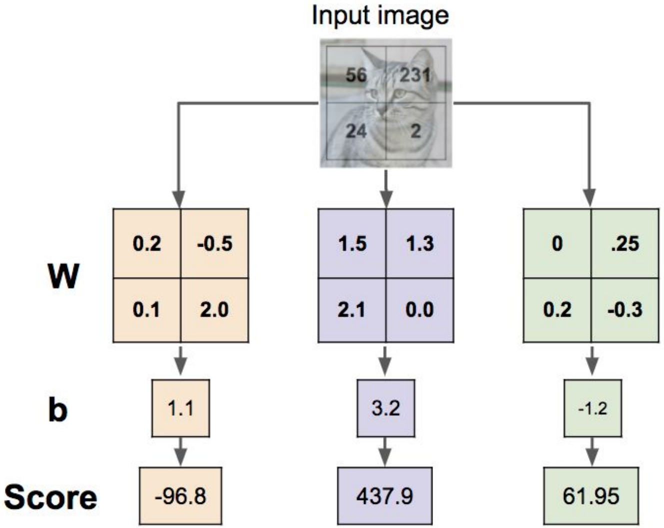
Input image

$$\underset{(3,5)}{\underset{W;b}{\left[\begin{array}{cccc|c} 0.2 & -0.5 & 0.1 & 2.0 & 1.1 \\ 1.5 & 1.3 & 2.1 & 0.0 & 3.2 \\ 0 & 0.25 & 0.2 & -0.3 & -1.2 \end{array}\right]}} \underset{(5,)}{\left[\begin{array}{c} 56 \\ 231 \\ 24 \\ 2 \\ \hline 1 \end{array}\right]} = \begin{array}{cl} \left[\begin{array}{c} -96.8 \\ 437.9 \\ 61.95 \end{array}\right] & \begin{array}{l} \text{Cat score} \\ \text{Dog score} \\ \text{Ship score} \end{array} \end{array}$$

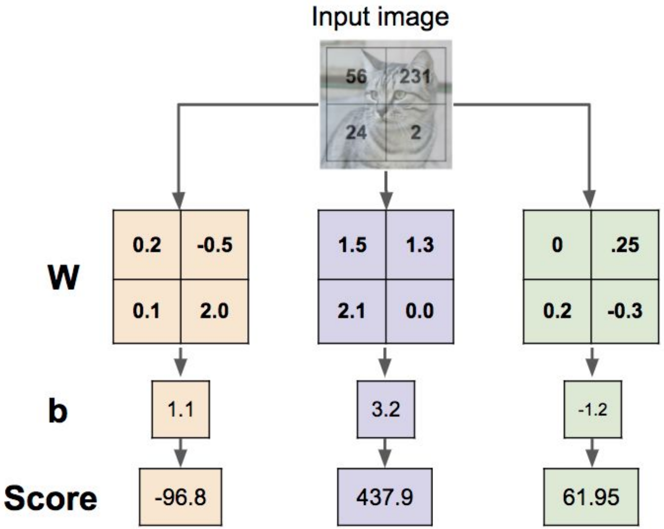
Visual Viewpoint: learning templates

Algebraic viewpoint:

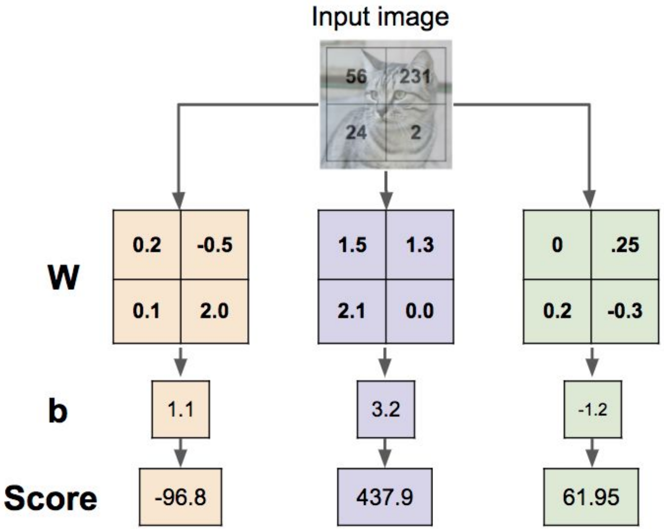




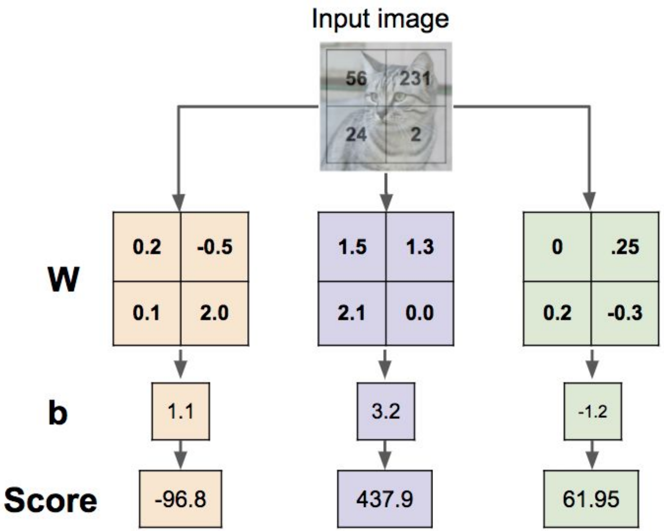
Visual Viewpoint: learning templates

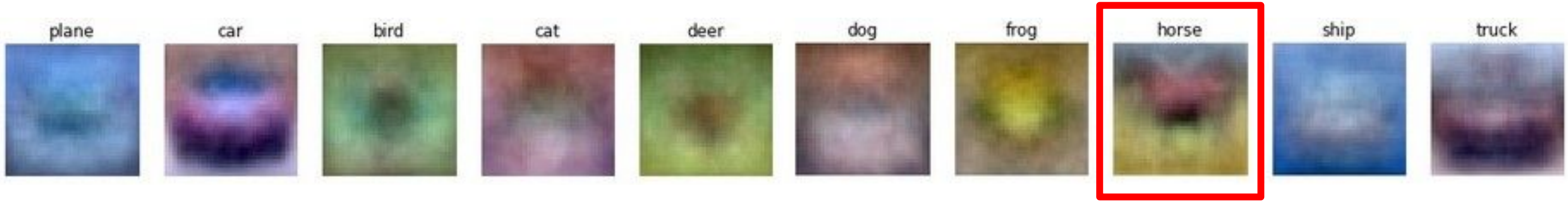


Visual Viewpoint: learning templates

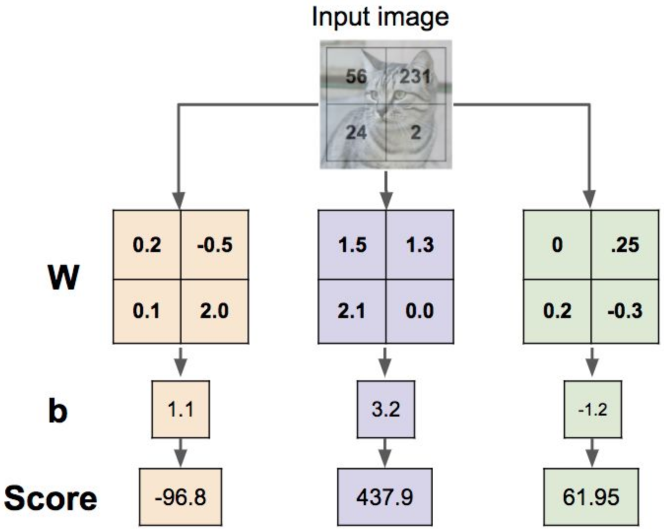


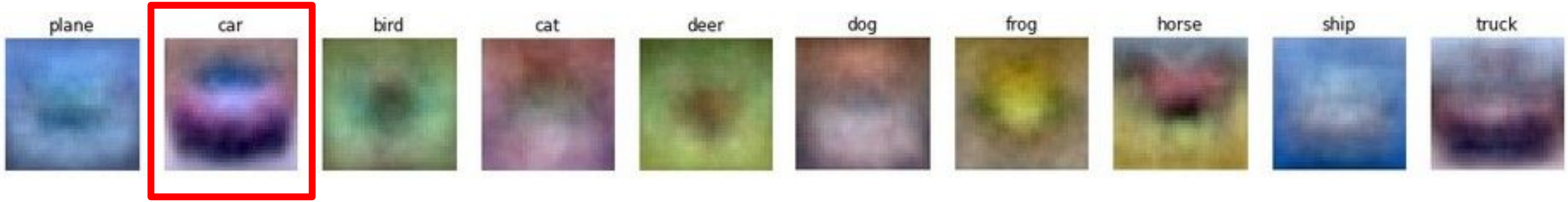
Visual Viewpoint: learning templates



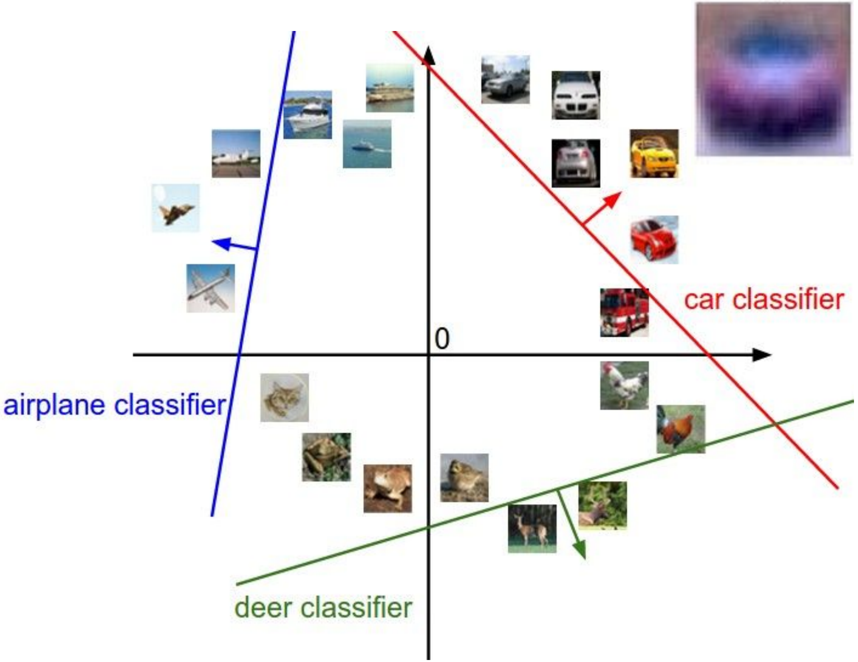


Visual Viewpoint: learning templates





Geometric Viewpoint: linear decision boundaries



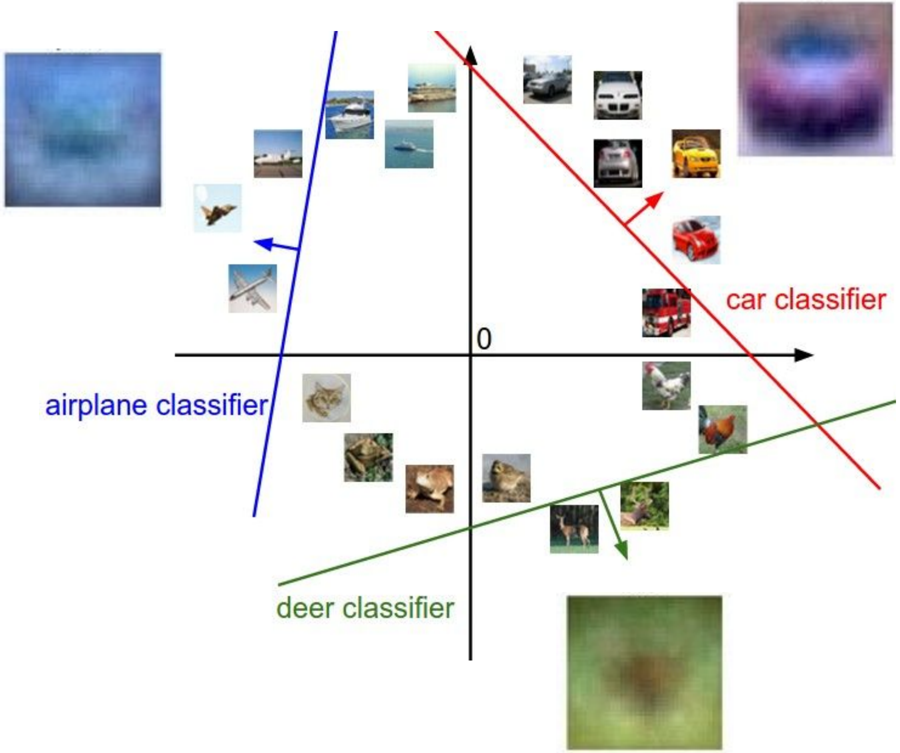
$$f(x,W) = Wx + b$$

Array of **32x32x3** numbers
(3072 numbers total)

Plot created using Wolfram Cloud

Cat image by Nikita is licensed under CC-BY 2.0

Geometric Viewpoint: linear decision boundaries



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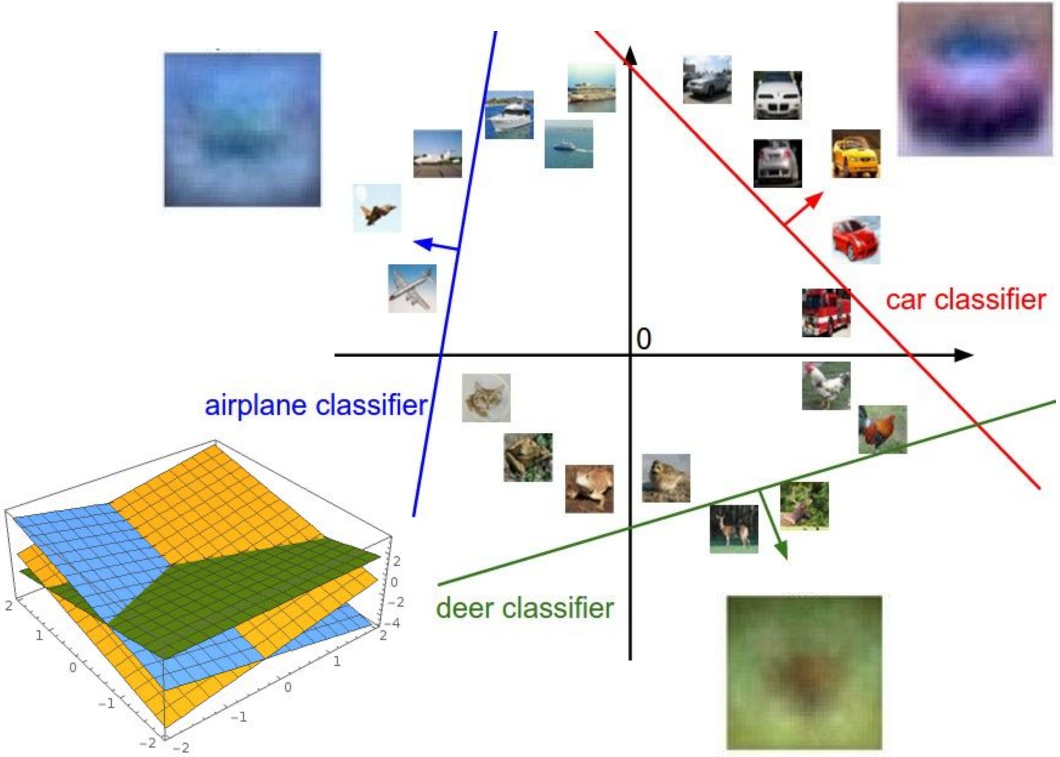
Geometric Viewpoint: linear decision boundaries



$$f(x,W) = Wx + b$$

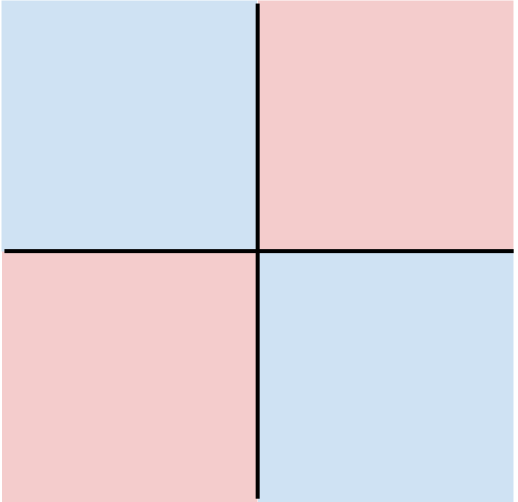
Array of **32x32x3** numbers
(3072 numbers total)

Plot created using Wolfram Cloud

Hard cases for a linear classifier

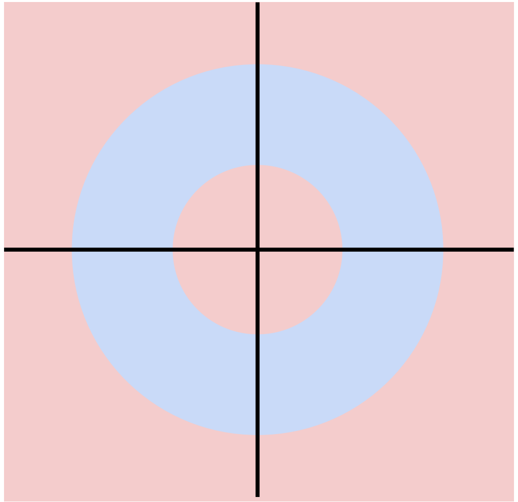
Class 1:
First and third quadrants

Class 2:
Second and fourth quadrants



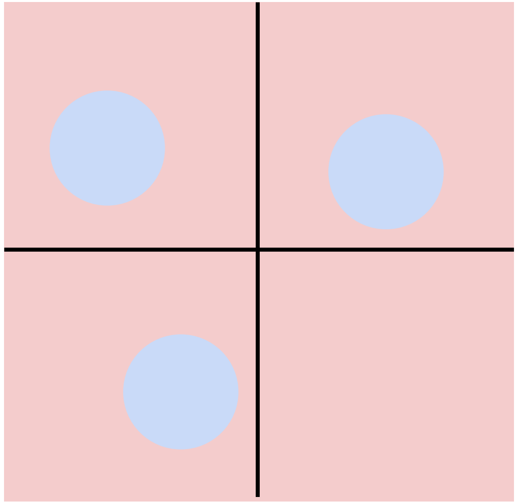
Class 1:
1 <= L2 norm <= 2

Class 2:
Everything else



Class 1:
Three modes

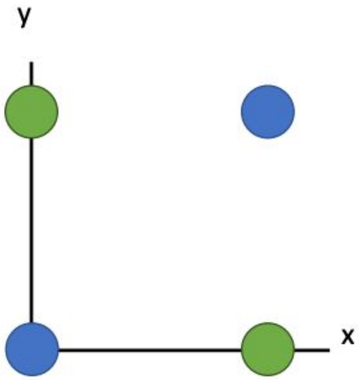
Class 2:
Everything else

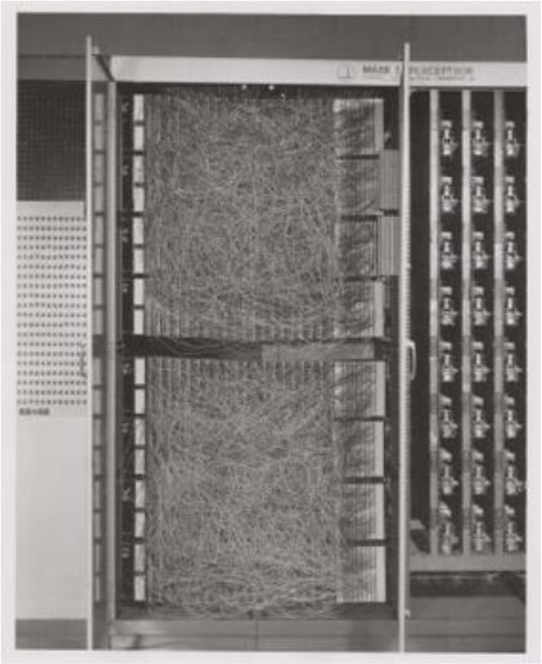


Recall the Minsky report 1969 from last lecture

Unable to learn the XNOR function

X	Y	F(x,y)
0	0	0
0	1	1
1	0	1
1	1	0

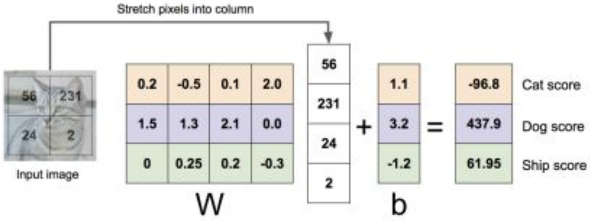




Three viewpoints for interpreting linear classifiers

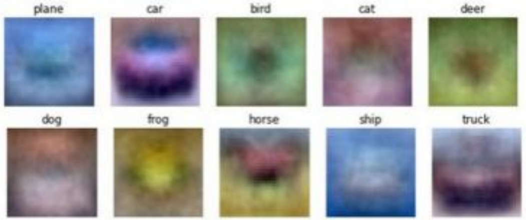
Algebraic Viewpoint

f(x,W) = Wx



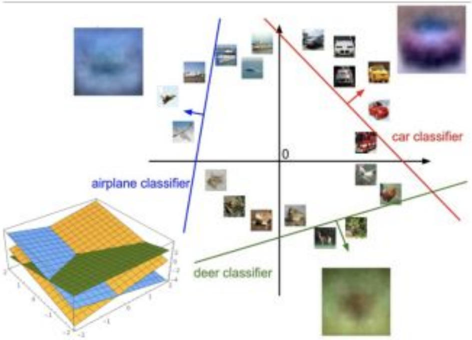
Visual Viewpoint

One template
per class



Geometric Viewpoint

Hyperplanes
cutting up space



Next: How to train the weights in a Linear Classifier

TODO:

1. Define a **loss function** that quantifies our unhappiness with the scores across the training data.

2. Come up with a way of efficiently finding the parameters that minimize the loss function. **(optimization)**

Example output for CIFAR-10:

airplane	-3.45	-0.51	3.42
automobile	-8.87	**6.04**	4.64
bird	0.09	5.31	2.65
cat	**2.9**	-4.22	5.1
deer	4.48	-4.19	2.64
dog	8.02	3.58	5.55
frog	3.78	4.49	**-4.34**
horse	1.06	-4.37	-1.5
ship	-0.36	-2.09	-4.79
truck	-0.72	-2.93	6.14

Cat image by Nikita is licensed under CC-BY 2.0; Car image is CC0 1.0 public domain; Frog image is in the public domain

- A random W produces the following 10 scores for the 3 images to the left.
- 10 scores because there are 10 classes.
- First column bad because dog is highest.
- Second column good.
- Third column bad because frog is highest

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

			
cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

A **loss function** tells how good our current classifier is

cat	**3.2**	1.3	2.2
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A **loss function** tells how good our current classifier is

Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where x_i is image and
y_i is (integer) label

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

A **loss function** tells how good our current classifier is

Given a dataset of examples

$$\{(x_i, y_i)\}_{i=1}^{N}$$

Where x_i is image and
y_i is (integer) label

Loss over the dataset is a average of loss over examples:

$$L = \frac{1}{N} \sum_i L_i(f(x_i, W), y_i)$$

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
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Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
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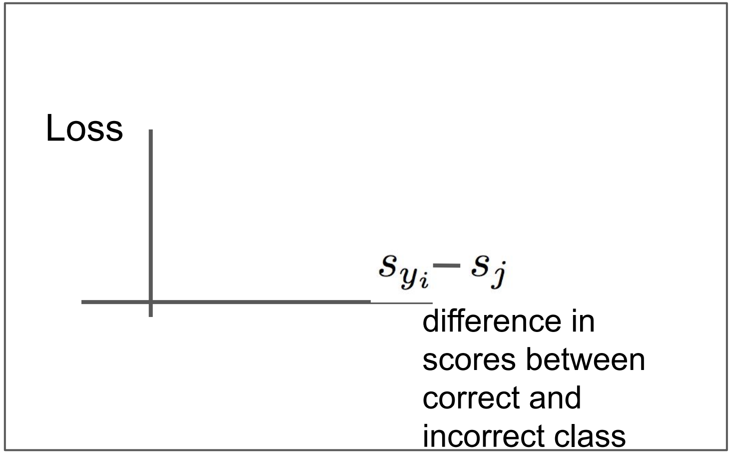
Interpreting Multiclass SVM loss:



difference in scores between correct and incorrect class

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

$$= \boxed{\sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

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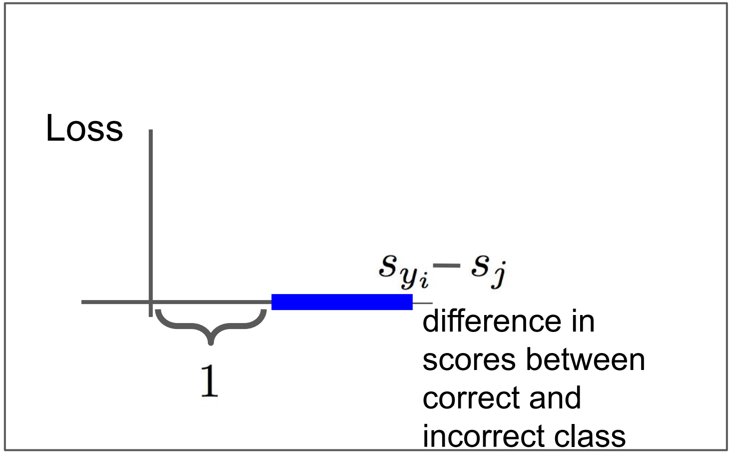

Loss

$s_{y_i} - s_j$

1

difference in scores between correct and incorrect class

$$L_i = \sum_{j \neq y_i} \begin{cases} 0 & \text{if } s_{y_i} \geq s_j + 1 \\ s_j - s_{y_i} + 1 & \text{otherwise} \end{cases}$$

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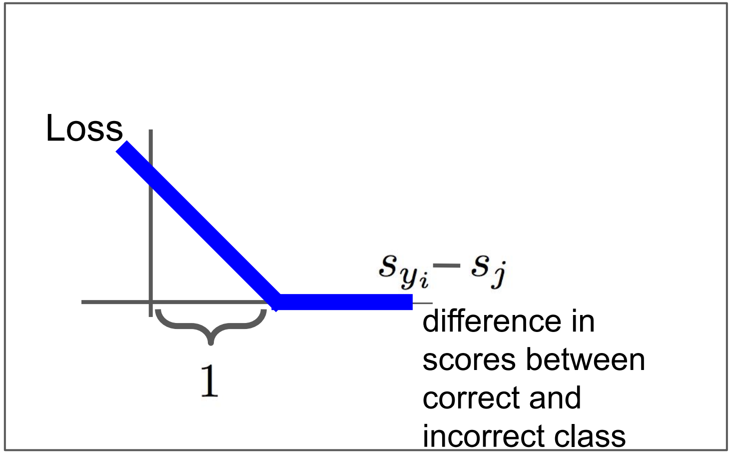

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Losses:	2.9		

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
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and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

$$= \max(0, \boxed{5.1} - \boxed{3.2} + 1)$$
$$+ \max(0, -1.7 - 3.2 + 1)$$

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

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the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

$$= \max(0, 5.1 - 3.2 + 1)$$
$$+\max(0, -\boxed{1.7} - \boxed{3.2} + 1)$$

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

cat	**3.2**	1.3	2.2
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the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

= max(0, 5.1 - 3.2 + 1)
+max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

cat	**3.2**	1.3	2.2
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and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

= max(0, 5.1 - 3.2 + 1)
+max(0, -1.7 - 3.2 + 1)
= max(0, 2.9) + max(0, -3.9)
= 2.9 + 0
= 2.9

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	

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Given an example (x_i, y_i)
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the SVM loss has the form:

$$\boxed{L_i = \sum_{j\neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

$$= \max(0, \boxed{1.3} - \boxed{4.9} + 1)$$
$$+\max(0, \boxed{2.0} - \boxed{4.9} + 1)$$
$$= \max(0, -2.6) + \max(0, -1.9)$$
$$= 0 + 0$$
$$= 0$$

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
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and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

= max(0, 2.2 - (-3.1) + 1)
+max(0, 2.5 - (-3.1) + 1)
= max(0, 6.3) + max(0, 6.6)
= 6.3 + 6.6
= 12.9

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
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the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Loss over full dataset is average:

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i$$

L = (2.9 + 0 + 12.9)/3
= **5.27**

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

Multiclass SVM loss:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q1: What happens to loss if car scores decrease by 0.5 for this training example?

cat	1.3
car	**4.9**
frog	2.0
Losses:	0

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

Multiclass SVM loss:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

cat	1.3
car	**4.9**
frog	2.0
Losses:	0

Q1: What happens to loss if car scores decrease by 0.5 for this training example?

Q2: what is the min/max possible SVM loss L_i?

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

Multiclass SVM loss:

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

cat	1.3
car	**4.9**
frog	2.0
Losses:	0

Q1: What happens to loss if car scores decrease by 0.5 for this training example?

Q2: what is the min/max possible SVM loss L_i?

Q3: At initialization W is small so all s ≈ 0. What is the loss L_i, assuming N examples and C classes?

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x, W) = Wx$ are:

cat	**3.2**	1.3	2.2
car	5.1	**4.9**	2.5
frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	12.9

Multiclass SVM loss:

Given an example (x_i, y_i)
where x_i is the image and
where y_i is the (integer) label,

and using the shorthand for the scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q4: What if the sum was over all classes? (including j = y_i)

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

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Multiclass SVM loss:

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scores vector: $s = f(x_i, W)$

the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q5: What if we used mean instead of sum?

Suppose: 3 training examples, 3 classes.
With some W the scores $f(x,W) = Wx$ are:

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the SVM loss has the form:

$$\boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

Q6: What if we used

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

Suppose: 3 training examples, 3 classes.
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cat	**3.2**	1.3	2.2
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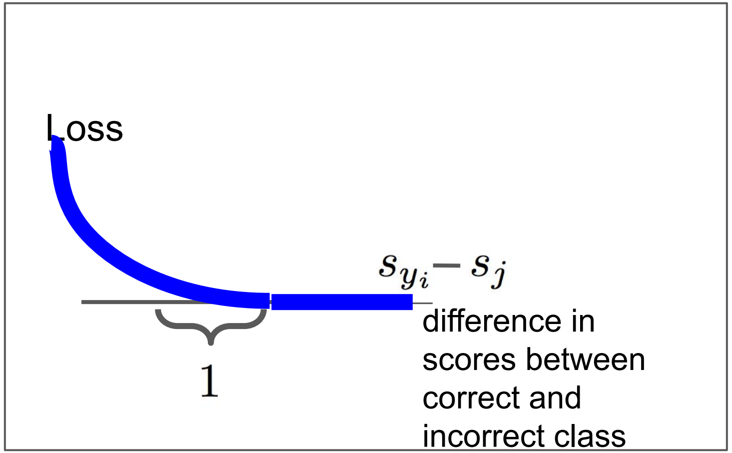
Multiclass SVM loss:



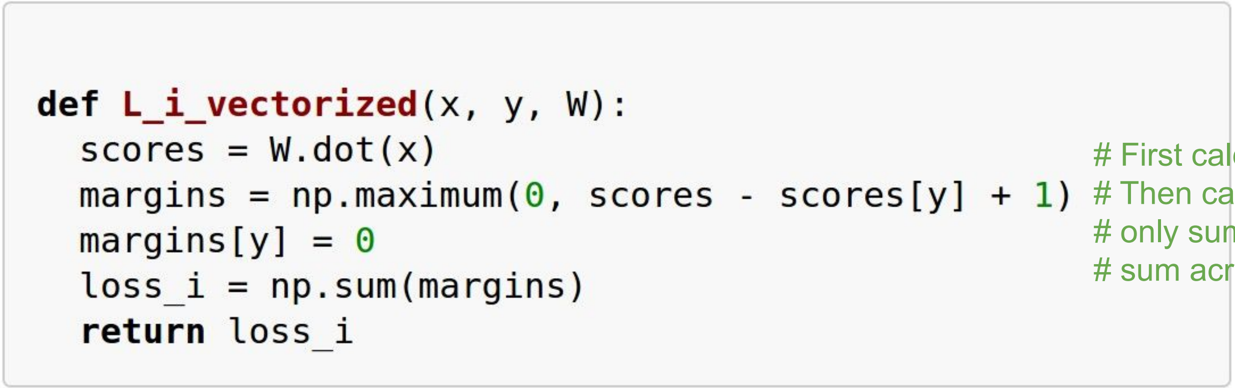
difference in scores between correct and incorrect class

Q6: What if we used

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)^2$$

Multiclass SVM Loss: Example code

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$



```
def L_i_vectorized(x, y, W):
  scores = W.dot(x)
  margins = np.maximum(0, scores - scores[y] + 1)
  margins[y] = 0
  loss_i = np.sum(margins)
  return loss_i
```

First calculate scores
Then calculate the margins s_j - s_{yi} + 1
only sum j is not y_i, so when j = y_i, set to zero.
sum across all j

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

Q7. Suppose that we found a W such that L = 0. Is this W unique?

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a W such that L = 0. Is this W unique?

No! 2W is also has L = 0!

Suppose: 3 training examples, 3 classes.
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frog	-1.7	2.0	**-3.1**
Losses:	2.9	0	

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Before:

= max(0, 1.3 - 4.9 + 1)
+max(0, 2.0 - 4.9 + 1)
= max(0, -2.6) + max(0, -1.9)
= 0 + 0
= 0

With W twice as large:

= max(0, 2.6 - 9.8 + 1)
+max(0, 4.0 - 9.8 + 1)
= max(0, -6.2) + max(0, -4.8)
= 0 + 0
= 0

$$f(x, W) = Wx$$

$$L = \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq y_i} \max(0, f(x_i; W)_j - f(x_i; W)_{y_i} + 1)$$

E.g. Suppose that we found a W such that L = 0.
Is this W unique?

No! 2W is also has L = 0!
How do we choose between W and 2W?

Regularization

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}$$

Data loss: Model predictions should match training data

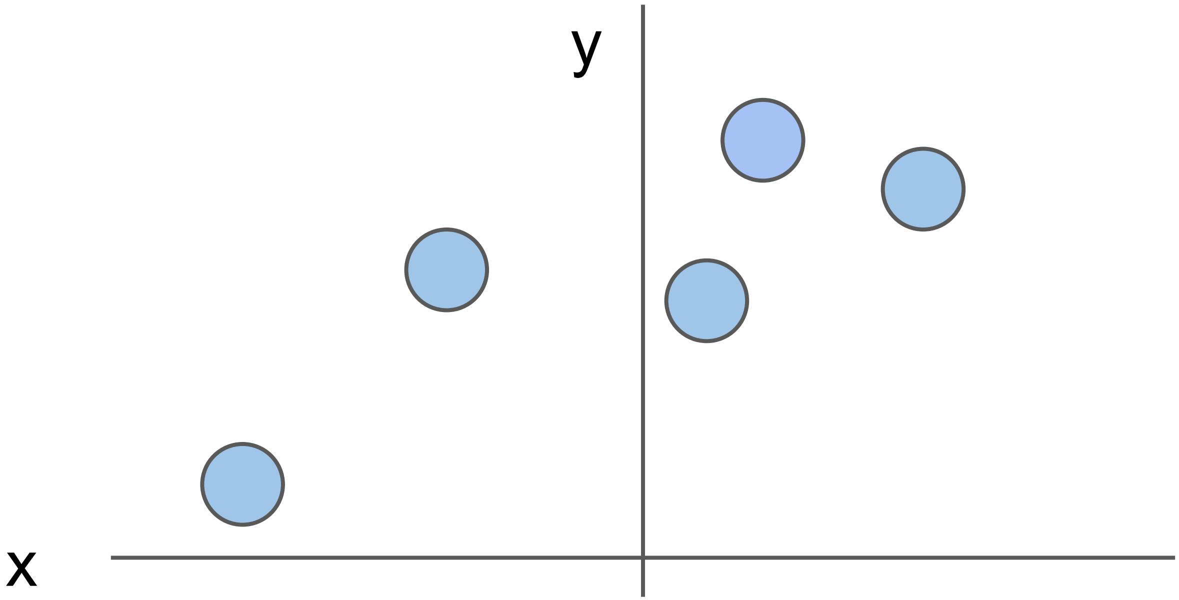
Regularization

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)} + \underbrace{\lambda R(W)}$$

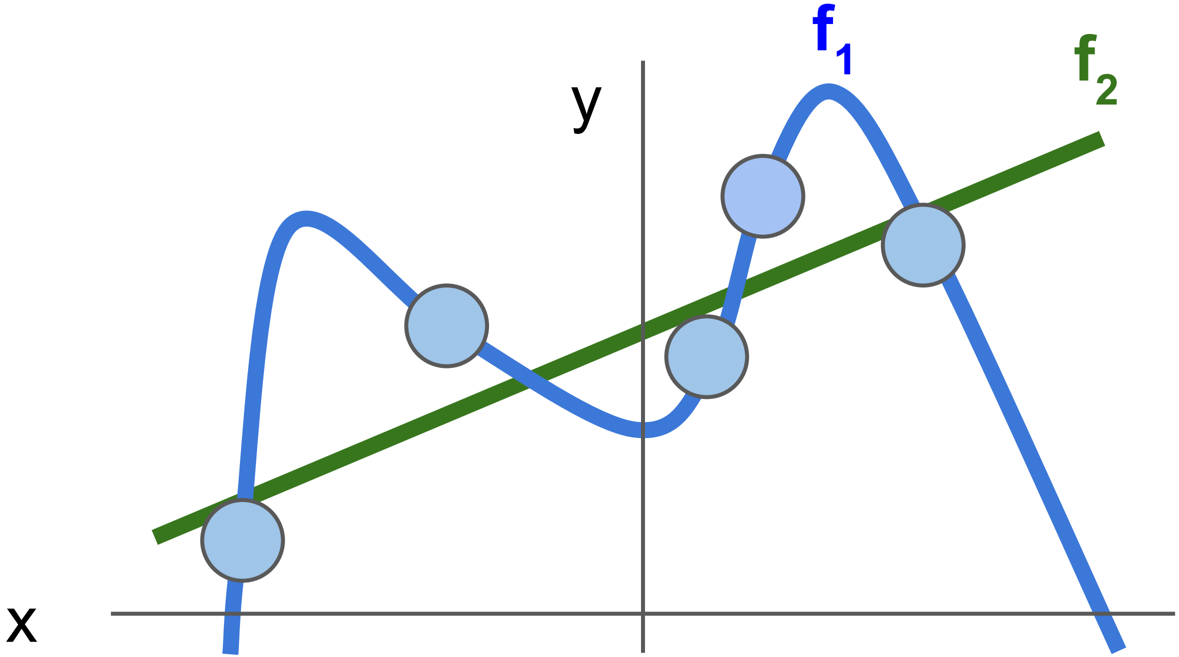
Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

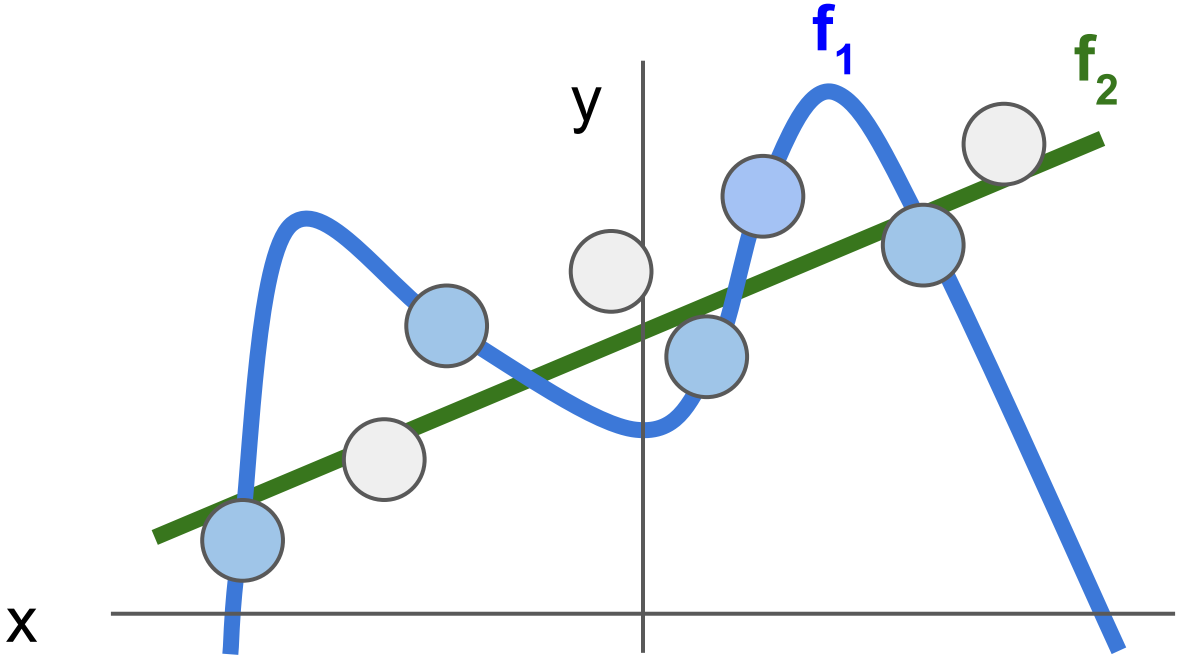
Regularization intuition: toy example training data



Regularization intuition: Prefer Simpler Models



Regularization: Prefer Simpler Models



Regularization pushes against fitting the data *too* well so we don't fit noise in the data

Regularization

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)} + \underbrace{\lambda R(W)}$$

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Occam's Razor: Among multiple competing hypotheses, the simplest is the best, William of Ockham 1285-1347

Regularization

λ = regularization strength (hyperparameter)

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{} + \underbrace{\lambda R(W)}_{}$$

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Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Simple examples

L2 regularization: $R(W) = \sum_k \sum_l W_{k,l}^2$

L1 regularization: $R(W) = \sum_k \sum_l |W_{k,l}|$

Elastic net (L1 + L2): $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$

Regularization

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$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)}_{} + \underbrace{\lambda R(W)}_{}$$

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Elastic net (L1 + L2): $R(W) = \sum_k \sum_l \beta W_{k,l}^2 + |W_{k,l}|$

More complex:

Dropout

Batch normalization, layer norm

Stochastic depth, fractional pooling, etc

Regularization

λ = regularization strength (hyperparameter)

$$L(W) = \underbrace{\frac{1}{N}\sum_{i=1}^{N} L_i(f(x_i, W), y_i)} + \underbrace{\lambda R(W)}$$

Data loss: Model predictions should match training data

Regularization: Prevent the model from doing *too* well on training data

Why regularize?

- Express preferences over weights
- Make the model *simple* so it works on test data
- Improve optimization by adding curvature

Regularization: Expressing Preferences

$$x = [1, 1, 1, 1]$$

$$w_1 = [1, 0, 0, 0]$$

$$w_2 = [0.25, 0.25, 0.25, 0.25]$$

$$w_1^T x = w_2^T x = 1$$

L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Which of w1 or w2 will the L2 regularizer prefer?

Regularization: Expressing Preferences

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$$w_1 = [1, 0, 0, 0]$$

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L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Which of w1 or w2 will the L2 regularizer prefer?

L2 regularization likes to "spread out" the weights

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$$w_1 = [1, 0, 0, 0]$$

$$w_2 = \boxed{[0.25, 0.25, 0.25, 0.25]}$$

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L2 Regularization

$$R(W) = \sum_k \sum_l W_{k,l}^2$$

Which of w1 or w2 will the L2 regularizer prefer?

L2 regularization likes to "spread out" the weights

Which one would L1 regularization prefer?

Softmax classifier

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

cat	**3.2**
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax Function

cat	**3.2**
car	5.1
frog	-1.7

Softmax Classifier (Multinomial Logistic Regression)

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Probabilities must be >= 0

cat	**3.2**		**24.5**
car	5.1	$\xrightarrow{\text{exp}}$	164.0
frog	-1.7		0.18

unnormalized probabilities

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Probabilities must sum to 1

cat	**3.2**	**24.5**	**0.13**
car	5.1	164.0	0.87
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$\xrightarrow{\text{exp}}$ unnormalized probabilities $\xrightarrow{\text{normalize}}$ probabilities

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		exp →	normalize →
cat	**3.2**	**24.5**	**0.13**
car	5.1	164.0	0.87
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	Unnormalized log-probabilities / logits	unnormalized probabilities	probabilities

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Probabilities must sum to 1

$$L_i = -\log P(Y = y_i | X = x_i)$$

cat	**3.2**		**24.5**		**0.13**
car	5.1	$\xrightarrow{\text{exp}}$	164.0	$\xrightarrow{\text{normalize}}$	0.87
frog	-1.7		0.18		0.00
	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities

$\rightarrow$ $L_i = -\log(0.13)$

$= \mathbf{2.04}$

Softmax Classifier (Multinomial Logistic Regression)

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exp

normalize

cat	**3.2**	**24.5**	**0.13**
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	Unnormalized log-probabilities / logits	unnormalized probabilities	probabilities

$\rightarrow L_i = -\log(0.13)$

$= \mathbf{2.04}$

Maximum Likelihood Estimation
Choose weights to maximize the likelihood of the observed data

Softmax Classifier (Multinomial Logistic Regression)

Want to interpret raw classifier scores as **probabilities**

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$$P(Y = k | X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

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cat	**3.2**		**24.5**		**0.13**	→ compare ←	**1.00**
car	5.1	exp →	164.0	normalize →	0.87		0.00
frog	-1.7		0.18		0.00		0.00
	Unnormalized log-probabilities / logits		unnormalized probabilities		probabilities		Correct probs

Softmax Classifier (Multinomial Logistic Regression)

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	Unnormalized log-probabilities / logits	unnormalized probabilities	probabilities		Correct probs

exp

normalize

Kullback–Leibler divergence

$$D_{KL}(\underline{P}\|\underline{Q}) = \sum_y P(y) \log \frac{P(y)}{Q(y)}$$

Softmax Classifier (Multinomial Logistic Regression)

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$$P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}$$

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cat	**3.2**		**24.5**		**0.13**	→ compare ←	**1.00**
car	5.1		164.0		0.87		0.00
frog	-1.7		0.18		0.00		0.00

Cross Entropy

$$H(\underline{P}, \underline{Q}) = H(p) + D_{KL}(P\|Q)$$

Softmax Classifier (Multinomial Logistic Regression)

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Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax Function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i|X = x_i)$$

Putting it all together:

$$L_i = -\log(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}})$$

Softmax Classifier (Multinomial Logistic Regression)

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Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Q1: What is the min/max possible softmax loss L_i?

Softmax Classifier (Multinomial Logistic Regression)

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Want to interpret raw classifier scores as **probabilities**

$$s = f(x_i; W)$$

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Softmax Function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i | X = x_i)$$

Putting it all together:

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

Q1: What is the min/max possible softmax loss L_i?

Q2: At initialization all s_j will be approximately equal; what is the softmax loss L_i, assuming C classes?

Softmax Classifier (Multinomial Logistic Regression)

cat	**3.2**
car	5.1
frog	-1.7

Want to interpret raw classifier scores as **probabilities**

$$\boxed{s = f(x_i; W)}$$

$$\boxed{P(Y = k|X = x_i) = \frac{e^{s_k}}{\sum_j e^{s_j}}}$$

Softmax Function

Maximize probability of correct class

$$L_i = -\log P(Y = y_i|X = x_i)$$

Putting it all together:

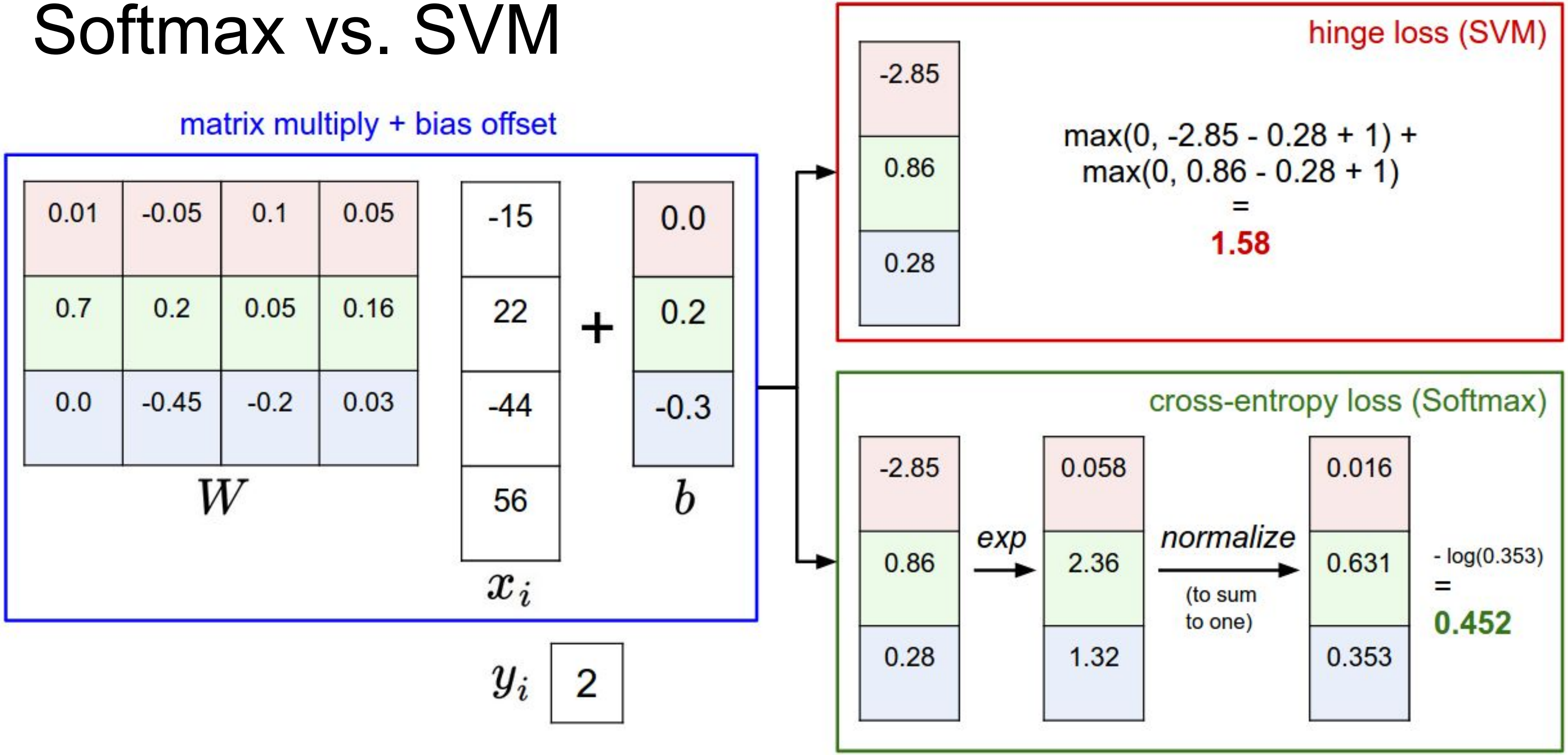
$$L_i = -\log(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}})$$

Q2: At initialization all s will be approximately equal; what is the loss?

A: -log(1/C) = log(C),

If C = 10, then L_i = log(10) ≈ 2.3

Softmax vs. SVM



matrix multiply + bias offset

$$\underset{W}{\begin{bmatrix}0.01 & -0.05 & 0.1 & 0.05\\ 0.7 & 0.2 & 0.05 & 0.16\\ 0.0 & -0.45 & -0.2 & 0.03\end{bmatrix}} \underset{x_i}{\begin{bmatrix}-15\\ 22\\ -44\\ 56\end{bmatrix}} + \underset{b}{\begin{bmatrix}0.0\\ 0.2\\ -0.3\end{bmatrix}}$$

y_i $\boxed{2}$

hinge loss (SVM)

$$\begin{bmatrix}-2.85\\ 0.86\\ 0.28\end{bmatrix}$$

max(0, -2.85 - 0.28 + 1) +
max(0, 0.86 - 0.28 + 1)
=
1.58

cross-entropy loss (Softmax)

$$\begin{bmatrix}-2.85\\ 0.86\\ 0.28\end{bmatrix} \xrightarrow{exp} \begin{bmatrix}0.058\\ 2.36\\ 1.32\end{bmatrix} \xrightarrow[\text{(to sum to one)}]{normalize} \begin{bmatrix}0.016\\ 0.631\\ 0.353\end{bmatrix}$$

- log(0.353)
=
0.452

Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

Softmax vs. SVM

$$L_i = -\log(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}})$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

---

assume scores:
[10, -2, 3]
[10, 9, 9]
[10, -100, -100]
and $\boxed{y_i = 0}$

Q: What is the **SVM loss?**

Softmax vs. SVM

$$L_i = -\log(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}})$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

---

assume scores:
[10, -2, 3]
[10, 9, 9]
[10, -100, -100]
and $\boxed{y_i = 0}$

Q: What is the **SVM loss?**

Q: Is the **Softmax** loss zero for any of them?

Softmax vs. SVM

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

---

assume scores:

[20, -2, 3]

[20, 9, 9]

[20, -100, -100]

and $\boxed{y_i = 0}$

Q: What is the **SVM loss?**

Q: Is the **Softmax** loss zero for any of them?

I doubled the correct class score from 10 -> 20?

Recap

- We have some dataset of (x,y)
- We have a **score function:** $s = f(x; W) \overset{\text{e.g.}}{=} Wx$
- We have a **loss function**:

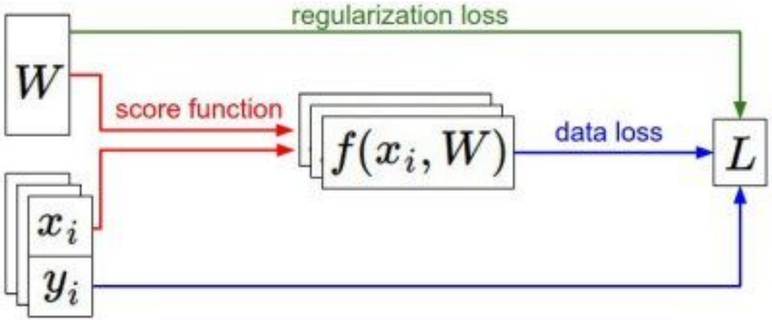
Softmax

$$L_i = -\log(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}})$$

SVM

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^N L_i + R(W)$$ Full loss



Recap

How do we find the best W?

- We have some dataset of (x,y)
- We have a **score function:**
- We have a **loss function**:

$$s = f(x; W) \overset{\text{e.g.}}{=} Wx$$

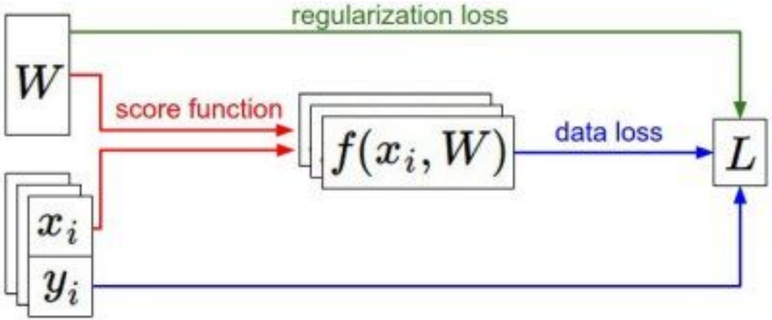
Softmax

$$L_i = -\log\left(\frac{e^{s_{y_i}}}{\sum_j e^{s_j}}\right)$$

SVM

$$L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)$$

$$L = \frac{1}{N} \sum_{i=1}^{N} L_i + R(W)$$ Full loss



Next time:
Optimization & backpropagation