Scoring rules

• A different kind of mechanism design problem: how to elicit a good prediction of an uncertain event?
  – Weather forecaster: will it rain tomorrow?
  – Political pundit: will a Democrat or Republican win next election
  – Microsoft employee: will the next version of MS Office ship on time?

• How should we evaluate the quality of a prediction/pay based on the quality of predictions/incentivize the work needed to output the best possible prediction?
Scoring rules

• $X$ finite set of possible outcomes of uncertain event.

• A **scoring rule** is a real-valued function $S(\vec{q},i)$
  – $\vec{q}$ is a probability distribution over $X$ (**a prediction**)
  – $i$ is some outcome in $X$ (**the realized outcome**)

\[
X = \{\text{sun}, \text{rain}, \text{snow}\}
\]
Model for incentives

- Forecaster has a belief $p$, prob distribution over $X$.
- Forecaster will choose prediction $q$ to maximize expected score
Strictly proper scoring rules

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  - $q$ is a probability distribution over $X$ (a prediction)
  - $i$ is some outcome in $X$ (the realized outcome)

- A scoring rule is strictly proper if, no matter what the true belief $p$ of the forecaster is, her unique best response is to report truthfully, i.e. to set $q = p$. 

Strictly proper scoring rules

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**Example:**

$$S(q,i) = q$$

Expected payoff = $p \cdot q + (1-p)(1-q)$

Given $p$ what $q$ maximizes this?
Quadratic scoring rule

\[ S(q_{i}) = q_{i} - \frac{1}{2} \sum_{j \neq i} q_{j}^{2} \]

- \( q_{i} = 1 \) for some \( i \) if \( i \) happens.
- \( q_{i} = 0 \) if \( i \) doesn't happen.
- \( q_{i} = \frac{1}{n} \) no matter what payoff \( \geq \frac{1}{2n} \).
\[ S(q; i) = q_i - \frac{1}{2} \sum_{j \in X} q_j^2 \]

QSR is symmetric prior

\[ E(\text{sum}) = \sum_i p_i q_i - \frac{1}{2} \sum_{i \in X} p_i \sum_{j \in X} q_j^2 \]

is max at \( p = q \)

\[ \frac{d}{dq_k} E(\text{sum}) = p_k - \sum_{i \in X} p_i q_k \]

\[ q_k = \frac{p_k}{\sum_{i \in X} p_i} = 1 \]

\[ aS + b \]

\[ + \frac{1}{2} \]
Logarithmic scoring rule

\[ S(q, i) = \ln q_i \]

add \( \ln |x| \)  \( |x| = n \)

Forecaster can guarantee nonreg exp utility.

\[ q^* = (\frac{1}{n}, \ldots, \frac{1}{n}) \]

\[ E(\text{score}) = \sum_{i=1}^{n} p_i \ln(\frac{1}{q_i}) + \ln(n) = 0 \]

\[ q_i = 0 \quad \text{for} \quad q_i < 0 \]

\[ q_i = \frac{1}{n} \quad \text{for} \quad 3 \]
Logarithmic scoring rule is strictly proper.

- incentivizing honest feedback
- predictor markets
Incentivizing honest feedback

- Example: peer grading, where students grade the assignments of other students.

- How to incentivize accurate grading, without direct verification?
Model

- n players (graders of an assignment, say in MOOC)
- Player i has a “signal” \( S_i \)
- Each player submits a report \( r_i \) to a mechanism.
- Mechanism pays player \( \pi_i (r_1, \ldots, r_n) \)

Assume signals \((s_1, \ldots, s_n)\) drawn from correlated data \( D \).

**Example:**

\[
\begin{array}{c|cc}
\text{bad} & \text{good} \\
\hline
s_2=0 & 0.3 & 0.1 \\
0.4 & 0.6 \\
\end{array}
\]

\[
\begin{align*}
\Pr(s_2=0 | s_1=0) &= \frac{2}{4} \\
\Pr(s_2=1 | s_1=0) &= \frac{1}{4}
\end{align*}
\]
How to choose payment $\Pi_1(\mathcal{F}) \ldots \Pi_n(\mathcal{F})$ to incentivize truthful reporting?
Output Agreement

• For each player $i$
  – Pick a random player $j \neq i$
  – Set payoff $\pi_i$ equal to 1 if they agree, 0 otherwise.
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<table>
<thead>
<tr>
<th></th>
<th>bad $s_1=0$</th>
<th>good $s_1=1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>bad $s_2=0$</td>
<td>0.3</td>
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<td></td>
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<td>0.6</td>
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<tr>
<td>good $s_2=1$</td>
<td>0.1</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Is it a Nash eq to report truthfully?

$$
\begin{array}{c}
\text{Pr}(s_2=x | s_1=x) \\
\end{array}
$$

$$
\begin{array}{c|c|c}
& s_2=0 & s_2=1 \\
\hline
s_1=0 & 0.3 & 0.1 \\
0 & 0.4 & 0.6 \\
1 & 0.1 & 0.5 \\
\end{array}
$$

Mechanism has bad NE: everyone report good
Peer prediction mechanism

- Suppose the distribution $D$ over signals is known to mechanism.
- For each player $i$
  - Pick a random player $j \neq i$
  - Let $D_j(r_i)$ be the distribution of $s_j$ conditioned on $s_i = r_i$
  - Set $i$'s payoff $\pi_i := S(D_j(r_i), r_j)$

\[
\begin{array}{c|cc}
\text{bad} & s_i = 0 & s_i = 1 \\
\hline
s_j = 0 & 0.3 & 0.1 \\
\hline
s_j = 1 & 0.1 & 0.5 \\
\end{array}
\]

\[
D_2(0|n_i = 0) = \frac{3}{4} \\
D_2(1|n_i = 0) = \frac{1}{4}
\]
Problems

- Requires advance knowledge of distribution.
- Other non-truthful and “bad” equilibria.

In experiments:
- Participants coordinate on high-payoff but uninformative equilibria
- Empirically, people give better/truthful reports when paid a fixed reward (indep of their report).
Prediction Markets

• Suppose you’re interested in an uncertain event e.g.,
  – Will Trump be reelected?
  – Will there be a Covid-19 vaccine by the end of 2020?
  – Who will win the next superbowl?

Pred market: stock market for uncertain events like political outcomes
IEM PredictIt.
Prediction markets

- Idea: say want to predict which of two candidates A or B will win election.
- Create two securities a and b:
  - Each share of security a will pay out $1 if A wins.
  - Each share of security b will pay out $1 if B wins.
- Allow people to buy and sell these securities.
- Suppose current price of a is 75 cents (and b is 25 cents) and you believe A will win with probability p.
- What do you do?
Will Trump be the 2020 GOP nominee?
Yes
90¢
NC
105K Shares Traded
You believe that chance that Trump will win is 52%. 

\[ \text{Exp (payoff)} = 10.52 - 0.49 = 0.03 \]
Prediction markets

• Idea: say want to predict which of two candidates A or B will win election.

• Create two securities a and b:
  – Each share of security a will pay out $1 if A wins.
  – Each share of security b will pay out $1 if B wins.

• Allow people to buy and sell these securities.

• Interpret market price as the market’s “belief” that the candidate will win the election.

• Market aggregating beliefs of all participants => “consensus opinion”.
Legality Issues

• IEM, PredictIt circumvent regulation through a no-action letter by CFTC which condones IEM
  – Non-profit and used for research purposes
  – Stakes are small

• Several prediction markets with fictitious currency.

• No real path to establishing legal real-money prediction markets.
Accuracy

• Prediction markets vs polls

• Historically, prediction markets have done pretty well
  – People are better at predicting what other people will do than themselves.

bad in 2016
Basic prediction market (e.g. IEM)

- Use continuous double auctions
  - Trader can submit a buy or sell order any time.

  - An order:
    - Price
    - Max number of shares to be bought/sold.
    - Expiration date.

  - Trades are executed greedily (with nuances).
B

5 shares at $0.5/share.

B

5 shares at $0.6/share.

S ≤ 10 shares $0.7/share.
The Pentagon office that proposed spying electronically on Americans to monitor potential terrorists has a new experiment. It is an online futures trading market, disclosed today by critics, in which anonymous speculators would bet on forecasting terrorist attacks, assassinations and coups.

Traders bullish on a biological attack on Israel or bearish on the chances of a North Korean missile strike would have the opportunity to bet on the likelihood of such events on a new Internet site established by the Defense Advanced Research Projects Agency.

The Pentagon called its latest idea a new way of predicting events and part of its search for the "broader possible set of new ways to prevent terrorist attacks." Two Democratic senators who reported the plan called it morally repugnant and grotesque. The senators
Pentagon kills ‘terror futures market’

Senate urged Defense Dept. to scrap system to predict events

By John W. Schoen
msnbc.com

July 29 — A controversial plan to set up a “futures market” to use market forces to help predict political upheaval in the Middle East has been scrapped. The Pentagon Tuesday agreed to abandon the plan, the Senate Armed Services Committee chairman said, after Senate Democrats Monday blasted the plan as nothing more than state-sponsored “gambling on terrorism.”

SEN. JOHN WARNER, R-Va., said Monday he spoke by phone with the program’s director, “and we mutually agreed that this thing should be stopped.”

Warner announced the decision not long after Senate Democratic Leader Thomas Daschle took to the floor to denounce the program as “an incentive actually to commit acts of terrorism.”

“This is just wrong,” declared Daschle, D-S.D.

- HP ran in 90's.
- Google "goobles"
- Diversity of opinion.
- Independence.
- Decentralized.
- Aggregation.
- Trust.
Another Approach – Market Scoring Rules

• CDAs work well for “thick” markets – lots of traders, but not in
  – “thin” markets – few traders
  – “illiquid” markets -- large “bid-ask spread”

• Different approach: automated market-maker
  – At any time there is a price, and the market is always happy to buy or sell shares at this price.
  – Price evolves as shares are bought and sold.
Automated Market Makers

- Implemented using strictly proper scoring rule that is “shared” by all the players.
- Let $S$ be a strictly proper scoring rule.

\[
\text{Initialize } \ p^0 = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \ \text{state over } X.
\]

At any time $t$,
- any player can update $p^t \rightarrow p^{t+1}$ when outcome $i \in X$ is realized.
- payout to players who $p^{t+1} \rightarrow p^t$ update is

\[
S(p^t, i) - S(p^{t+1}, i) < 0
\]


paid out according to extent to which report improved product.

**Properties:**
- Market maker has bounded financial loss [Specifically for log Sarnoff rule]
- if it runs for $T$ steps, total payoff

\[
S(p^T, i) - S(p^0, i) - i
\]

- $S(p^0, i) = -\ln \left( \frac{1}{n} \right) = \ln(n)$
If players are myopic and each player must once in a while choose their unique best response for each player to update to their true belief.

Suppose betting on is whether 2 fair edge coins will both come up heads.

Suppose Alice knows outcome of coin 1 & she knows that its tails. should report (0,1)

(0.5, 0.5)

Bob knows outcome of 2nd coin toss & it's heads

(1,0) \rightarrow (0,1)
What does this do?

- Player is rewarded according to extent her report improves the prediction.
- Final prediction is last distribution.
- Predictions tend to settle down.