# Honesty

- Are the participants in a stable matching algorithm motivated to report their preferences truthfully?

# Honesty for residents in hospital-proposing version

| | 1st | 2nd | 3rd |
|---|---|---|---|
| X | C | A | B |
| Y | A | C | B |
| Z | C | A | B |

Hospitals preferences

| | 1st | 2nd | 3rd |
|---|---|---|---|
| A | X | Y | Z |
| B | X | Y | Z |
| C | Y | X | Z |

Residents preferences

if C reports Y ≻ X

C will end up with a better match from perspective of true preferences.

**Thm:** The G-S alg is truthful for proposing side.

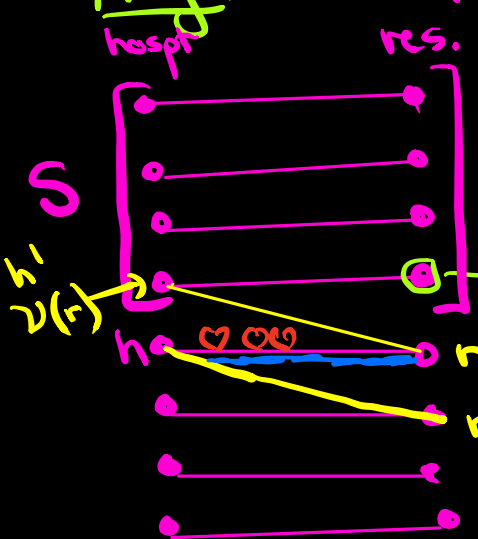
**Lemma:** Suppose $\mu$ is hosp-opt stable matching.

Let $\nu$ be any other matching.

Let $S$ be hospitals that prefer their match in $\nu$ to their match in $\mu$. (G-S)

$\exists (h, r)$ that are unstable in $\nu$ s.t. $h \notin S$

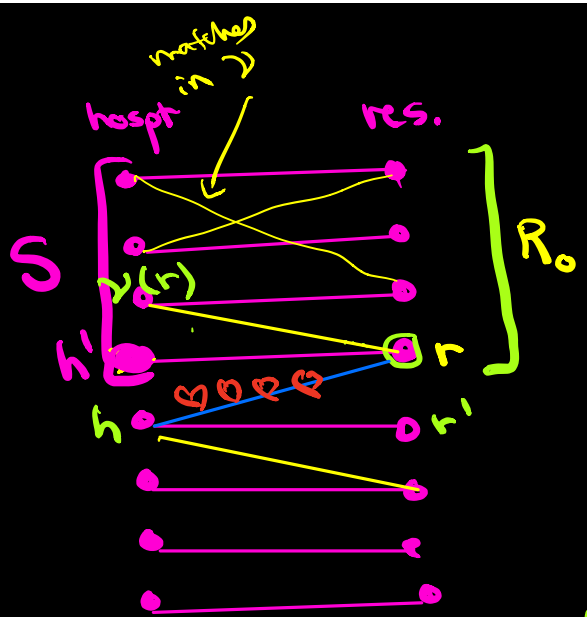

**Proof!** pink is outcome of G-S (hosp opt.)




**Case 1:** $\mu(s) \neq \nu(s)$

Claim: $(h, r)$ is unstable for $\nu$ yellow

- Since $h \notin S$, $h$ doesn't like $r'$ as much as $r$.
- $r$ prefers $h$ to $h'$
  $h'$ proposed to $r$ before $h'$ proposed to $\mu(h')$ & was rejected by $r$.




ν yellow.

**Case 2:** $\mu(S) = \nu(S) = R_0$

During GS execution, each $r \in R_0$ received & rejected a proposal from her match in ν.

Let r be last one in $R_0$ to receive a proposal during GS (from some hospital, say h')

Claim: at that pt, r was tentatively matched to h who she rejected for h'.

h must be outside S

$(h, r)$ is unstable for ν

h likes r at least as much as $\mu(h) = r'$

likes r' at least as much as $\nu(h)$

r likes h at least as much as $\nu(r)$.

because $\nu(r)$ proposed to r before h did which was before h' did ∎

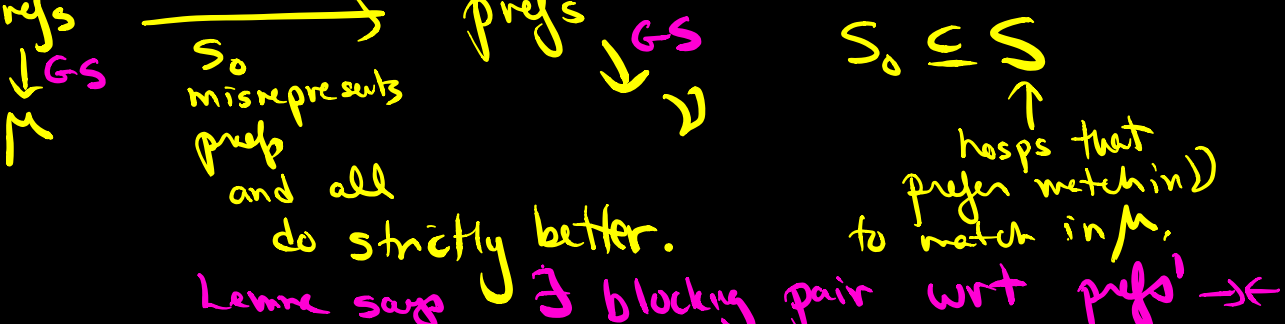
**Thm:** The G-S alg is truthful for proposing side.

**Lemma:** Suppose $\mu$ is hosp-opt stable matching.
Let $\nu$ be any other matching.
Let $S$ be hospitals that prefer their match in $\nu$ to their match in $\mu$. (G-S)
$\exists (h, r)$ that are unstable in $\nu$ s.t. $h \notin S$.

**Corollary** Let $\mu$ be hosp-opt stable matching
Suppose that a set $S_0$ of hospitals misrepresent their preferences. Then there is no stable matching for resulting preferences where all $h \in S_0$ are strictly better off

$\Rightarrow$ truthful for proposers
Just take $S_0$ to be a single hospital.

**Pf**



# Variations on basic problem of matching residents to hospitals

- Variant 1. Some participants declare others as unacceptable.

resident A unwilling to work in Cleveland

- Variant 2. Unequal number of hospitals and residents.
- Variant 3. Hospitals have more than one slot to hire into.

hospital X wants to hire 3 residents

No longer truthful for hospitals

- Def. An assignment of residents to hospitals is unstable if there is a hospital h and resident r such that:
  - h and r are acceptable to each other; and
  - either r is unmatched, or r prefers h to her assigned hospital; and
  - either h does not have all its places filled, or h prefers r to at least one of its assigned residents.

# Used for matching residents to hospitals

- NRMP. (National Resident Matching Program).
  - In USA more than 20,000 doctors and 4,000 hospitals are matched this way.
  - Does stability matter? Roth studied the history of matching mechanisms used in practice, of which there are/were many. The vast majority of matching mechanisms that did not produce stable outcomes did not survive.
  - NRMP used hospital-optimal version until the 90s and then switched to resident-optimal version.

# Bit of history of NRMP

- Medical residencies became widespread around 1900
- Until 1940s decentralized matching.
- Markets were unravelling with offers coming earlier and earlier and quality of matching dropped.
- Started to even offer residencies in their first year of medical school!
- Change called for: medical schools agreed not to release info about students until final year.
- This resulted in hospitals making exploding offers.
- 1952 – centralized "clearinghouse"… settled on algorithm.
- 1962 – Gale Shapley introduced, stability proved.
- 1998 – NMRP introduces matching with couple constraints.
- Stable matching used elsewhere, e.g. Hinge.

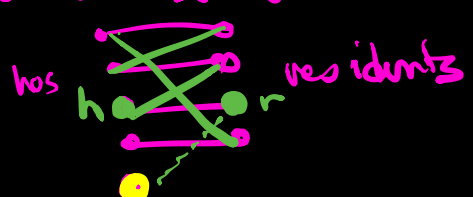
# Used for matching residents to hospitals

- NRMP. (National Resident Matching Program)
- Rural hospital dilemma.
  - Certain hospitals (mainly in rural areas) were unpopular and declared unacceptable by many residents.
  - Rural hospitals were under-subscribed in NRMP matching.
  - How can we find stable matching that benefits "rural hospitals"?
- Rural Hospital Theorem. Rural hospitals get exactly same residents in every stable matching!

n hospitals m residents m=n-1

In any SM, one of hospitals will be Unmatched.

Claim: always same hospital



## Generalized Rural Hospital Thm

$h_1$ has $c_1$ positions

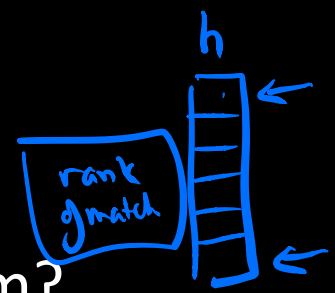
$h_2$ has $c_2$ ...

⋮

Any hospital that doesn't fill all of its slots is assigned precisely the same set of residents in every stable matching

# Rank of match

- Back to n by n case.
- What if preference lists are random?




hospitals residents

hospitals proposing.

$E[\text{rank}]$ of match

"principle of deferred decisions"

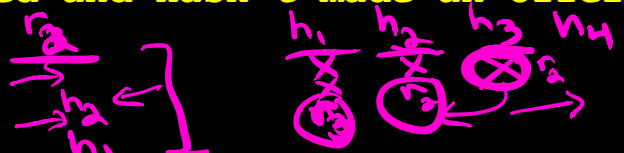
n by n

hosps

# Deferred Acceptance Algorithm
# Gale-Shapley Algorithm [1962]

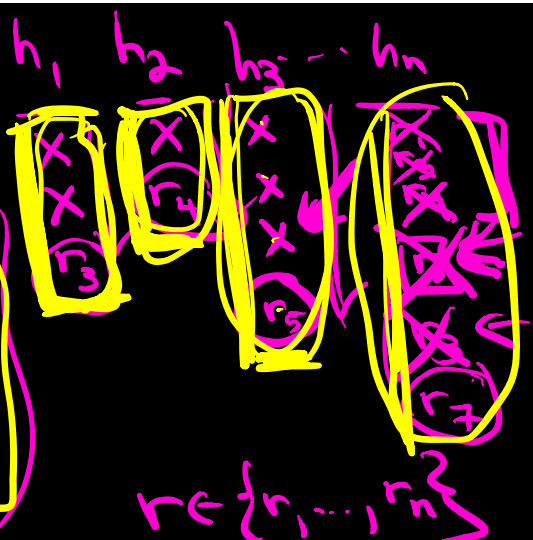
```
Initialize all hospitals and residents to be unmatched

while (some hospital unmatched and hasn't made an offer to
every resident)
{
    Choose such a hospital h
    r = 1st applicant on h's list to whom h has not made an
offer
    if (r is unmatched)
        tentatively match h and r.    (h "proposes" to r.)
    else if (r prefers h to her tentative match h')
        tentatively match h and r, and set h' to be unmatched
    else
        r rejects h (and h remains unmatched)
}
```



When does alg terminate?

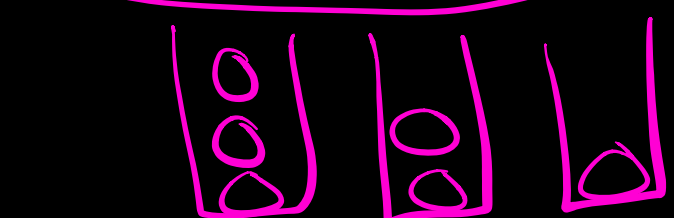
As soon as all $r \in R$ have received a proposal as soon as all have been selected once



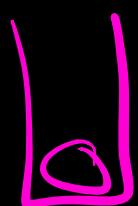

$r \in \{r_1, \ldots, r_n\}$

Coupon collectors problem

Balls in bins




$n \log n$.

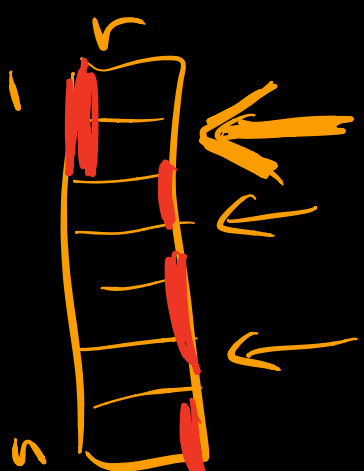



Total # proposals $n \log n$ $\Leftarrow$ $\sum$ ranks
nosp

$\Rightarrow$ exp rank of hospital's match

$\log n$. $\Leftarrow$

$\Rightarrow$ exp rank of resident's match when hospitals doing proposing




$\approx$ $n/\log n$

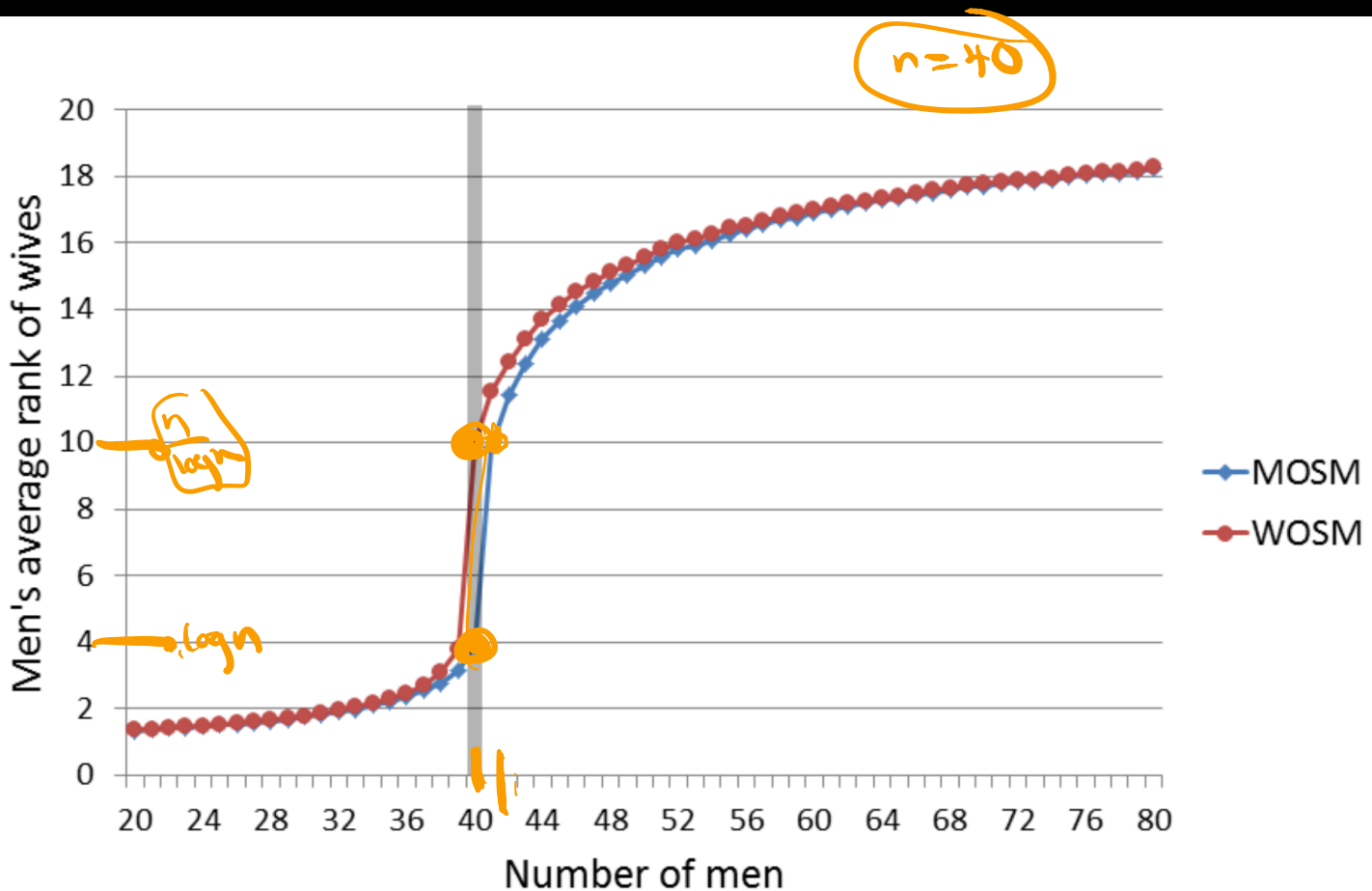
| Exp rank of match | n hosps n res | n+1 hosps n res. |
|---|---|---|
| hospitals | $\log n$ | $n/\log n$ |
| residents | $n/\log n$ | $\log n$ |

hosps proposing

hosp proposing

# Rank of match

- What if preference lists are random, but the number of hospitals and applicants is not equal, e.g. more competition for the applicants?


n=40
n
log n
log n
Men's average rank of wives
0
2
4
6
8
10
12
14
16
18
20
20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80
Number of men
MOSM
WOSM