Game Theory Basics

- Game theory is designed to model
  - How rational (payoff-maximizing) ``agents” will behave
  - When individual outcomes are determined by collective behavior.
  - Rules of a game specify agent payoffs as a function of actions taken by different agents.

N, $\underset{100}{S}$, $\underset{100}{T}$, $\underset{10}{W}$, A_s, Q_u, $\{S_{s,u}\}_{u \in U}$

Q_u ↑ qualities of all universities

only 10 universities.

You are $\underline{s}$

$A_{\tilde{s}} \sim U[0, S]$ $\quad \forall \tilde{s}$

$S_{\tilde{s},u} \sim U[0, W]$

$Q_u \sim U[0, T]$

$$\sum_u \left(\text{payoff to you from match to } u\right) \cdot \Pr\left(\text{you get matched to } u^{(s)}\right)$$

- Solve parts (a) & (b) first
- what to do if only get to apply to 1 school.
- is your strategy an approx best response to everyone else using that strategy?
- what might you want to do if A_s particularly high or particularly low?
- if everybody used your strategy, how many students on avg would apply to each university?

Prob Fact: $X \sim U[0, A]$ $\quad Y \sim U[0, B]$ $\quad A \le B$

$$\Pr(\underline{X+Y} \le a) = \begin{cases} \dfrac{a^2}{2AB} & 0 \le a \le A \\ \dfrac{a - \frac{A}{2}}{B} & A \le a \le B \\ 1 - \dfrac{(A+B-a)^2}{2AB} & B \le a \le A+B \end{cases}$$

$X+Y$ ↓ $A_s + S_{s,u}$

Let’s play the median game

- In a **private** message to Aditya Saraf, write down
  - An integer between 0 and 100 (inclusive).

- Later in the lecture, the person (or people) whose selected number is closest to 2/3 of the median of all the numbers (rounded down) wins the game!

- E.g., if the numbers are 3, 4, 5, 38, 60, 70, 70, 90, 100

Prisoners Dilemma

normal form game.

II

I

	stay silent	confess/betray
stay silent	$(-1,-1)$	$(-10,0)$
confess/betray	$(0,-10)$	$(-8,-8)$

Definition 2.2. A strategy s^* for player i is a **best response** to the strategies s_{-i} of others if it maximizes i's utility/payoff. That is

$$u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$$

for all $s \in S_i$.

Definition 2.3. Strategy s_i (strictly) dominates strategy s_i' if no matter what other players are doing, i's payoff playing s_i is at least as good (strictly better) than i's payoff playing s_i'.

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \quad \forall s_{-i} \forall s_i' \in S_i \setminus s_i.$$

If strategy s_i (strictly) dominates all strategies in S_i, then it is a (strictly) dominant strategy.

$\overset{n}{S}_i$: strategy set for player i

$u_i(\vec{s})$

$\vec{s} = (\underbrace{s_1, s_2,}_{\vec{s}_{-i}} s_i, \underbrace{s_n}_{})$

$\vec{s}_{-i}$

Betraying strictly dominated staying silent in PD.

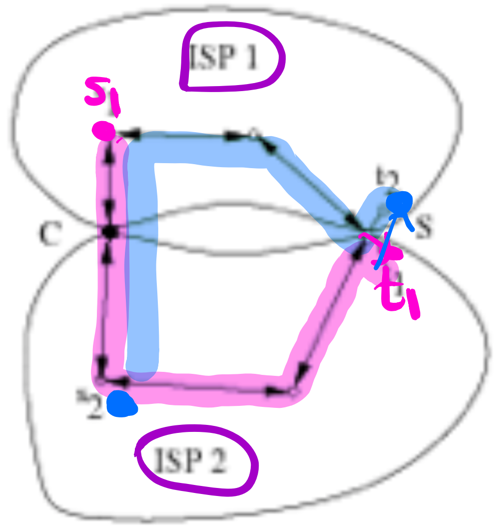
Definition 2.4. A strategy profile $(s_1, \ldots, s_n)$ is a **dominant strategy equilibrium** if for each player i, s_i is a dominant strategy.

short. long

	stay silent	confess/betray
stay silent	$(-1, -1)$	$(-10, 0)$
confess/betray	$(0, -10)$	$(-8, -8)$

ISP routing

cost has to do w/ how much of their own network's capacity is being used.



P2P networks. files shared between peers.

issues: free riding.

users who download files from others but don't upload files

File transfer game

B

A

	upload	don't upload
upload	$(2, 2)$	$(-1, 3)$
don't upload	$(3, -1)$	$(0, 0)$

- these #s don't reflect reality.

each of A & B possesses a file desired by other

simultaneously decide whether or not to upload file.

benefit of receiving file 3

cost of uploading 1

dominant strategy for both not to upload.

Pollution game

n countries

each faces choice — passing legislation to exercise pollution control or not

each country that pollutes adds 1 to cost of all countries.

pollution control costs 3.

Suppose k of other countries polluting. $n-k-1$ aren't.

pollute → $k+1$

don't pollute → $k+3$

if none polluted ⟹ all incur cost of n.

3 ⟹

Eliminating dominated strategies.

Startup Game

Q: whether or not to enter a certain new market.

		Startup	
		Enter	Stay out
Microsoft	Enter	(2, −2)	(4, 0)
	Stay out	(0, 4)	(0, 0)

Staying out is dominated for MS

(Enter, Stay Out)

Definition 2.8. A strategy profile $(s_1, \ldots, s_n)$ is a Nash equilibrium if for every i, s_i is a best response to s_{-i}.

Iterated Deletion of Dominated Strategies

- Deletion of a dominated strategy: find a player i and a strategy $b \in S_i$ such that $a \in S_i$ weakly dominates strategy b. Delete strategy b from S_i.

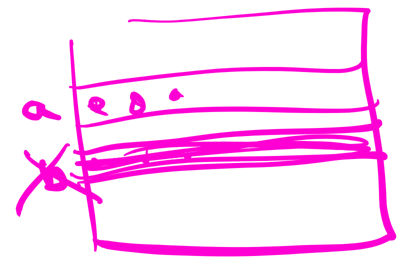
a weakly dominate b.

$\forall s_{-i}$
$u_i(a, s_{-i}) \geq u_i(b, s_{-i})$
and $\exists\ \tilde{s}_{-i}$
$u_i(a, \tilde{s}_{-i}) > u_i(b, \tilde{s}_{-i})$

[strictly dominates if all strict ineqs]

- Update definition of what's dominated (assuming b will never be played).

- Iterate until no weakly dominated strategy remains.



If each player has only a single remaining strategy, we say that the game is *solvable* by iterated deletion of dominated strategies, and we say that iterated deletion of dominated strategies predicts that each player will play their only remaining response.

outcome may depend on order of deletion.

Back to the median game

- Submit
  - An integer between 0 and 100 (inclusive).
- After we collect all the submissions, the person (or people) whose selected number is closest to 2/3 of the median of all the numbers (rounded down) win the game.

Iterated Deletion of Dominated Strategies

- Deletion of a dominated strategy: find a player i and a strategy $b \in S_i$ such that $a \in S_i$ weakly dominates strategy b. Delete strategy b from S_i.
- Update definition of what's dominated (assuming b will never be played).
- Iterate until no weakly dominated strategy remains.

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remove 100

med ≤ 100

$\lfloor \frac{2}{3} \text{med} \rfloor \leq 66$

elim 67 ... 100

new median ≤ 66

$\lfloor \frac{2}{3} \text{med} \rfloor \leq 44$

elim 45 ... 66

NE

0 best response to every submitting 0.

Coordination Game.

B

A

	party	stay home
party	2, 2	−1, 0
stay home	0, −1	0, 0

No dominant strategy

Definition 2.2. A strategy s^* for player i is a **best response** to the strategies s_{-i} of others if it maximizes i's utility/payoff. That is

$$u_i(s^*, s_{-i}) \geq u_i(s, s_{-i})$$

for all $s \in S_i$.

Definition 2.8. A strategy profile $(s_1, \ldots, s_n)$ is a Nash equilibrium if for every i, s_i is a best response to s_{-i}.

(Party, Party) (SH, SH) both NE.

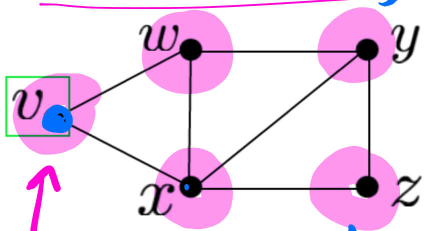
Network coordination game

used to model adoption of new tech

social networking app.
spread of popularity of product.

$k_w = 1$ $\quad$ $k_y = 2$

$k_v = 1$



$k_x = 2$ $\quad$ $k_z = 2$

$$\text{payoff to } v = \begin{cases} 2 & \geq k_v \text{ friends use it \& she uses it} \\ -1 & < k_v \text{ friends use it \& she uses it.} \\ 0 & \text{doesn't use it.} \end{cases}$$

Nash eq: nobody uses it.
all use it.

used to study how ideas & products can go viral.

How to "seed" a product.

mixed strategies

Parking game.

π inspector

I parking		q	$1-q$	
		don't inspect	inspect	
p	legal	$(0, 0)$	$(0, -1)$	$\rightarrow 0$
$1-p$	illegal	$(10, -10)$	$(-90, -6)$	$\rightarrow 10q - 90(1-q)$
		$0 \cdot p - 10(1-p)$	$-1 \cdot p - 6(1-p)$	

When do I prefer to inspect?

$$p < \frac{4}{5}$$

inspector prefers not inspect.

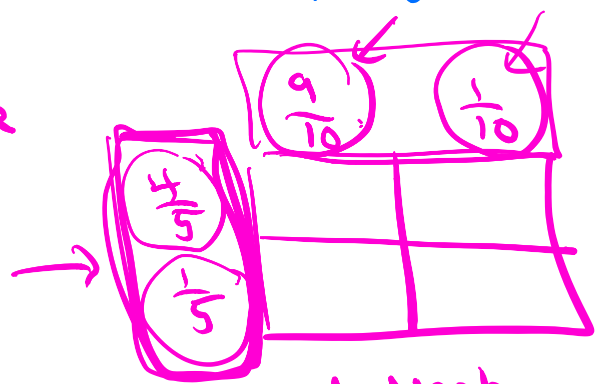
$$p > \frac{4}{5}$$

Illegal is better response

$$10q - 90(1-q) > 0$$

$$100q > 90$$

$$q > \frac{9}{10}$$



mixed Nash equilibrium

Definition 4.1. A **mixed strategy** is a probability distribution over pure strategies.

S_i

$x_i(s)$ = Prob player i plays strategy s $\in S_i$

$\sum_{s \in S_i} x_i(s) = 1$

$x_i(s) \geq 0$

Definition 4.2. A strategy profile $(x_1, \ldots, x_n)$ where each x_i is possibly a mixed strategy $x_i : S_i \to$ probability distribution and $\sum_{s \in S_i} x_i(s) = 1$ is a (mixed) Nash equilibrium if for each i,

$$\boxed{\sum_{s \in S_i, s_{-i} \in S_{-i}} x_i(s) \operatorname{Prob}(s_{-i}) u_i(s, s_{-i})} \geq \boxed{\sum_{s_{-i} \in S_{-i}} \operatorname{Prob}(s_{-i}) u_i(s_i, s_{-i}).}$$

$\forall s_i \in S_i$