Fair Division

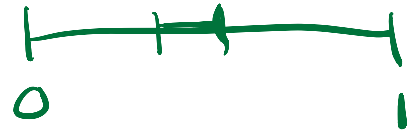
Fair Division

- What is a fair way for 2 people to split a heterogenous, divisible good?

- Why not 50/50?

- I Cut, You Choose Protocol
  - Player I told to split the good into two pieces A and B such that $v(A) = v(B) = \frac{1}{2} v(A \cup B)$.
  - Player II picks his favorite of A and B.
  - Player I takes the other piece.

Formal Model



- The good is the unit interval [0,1].
- $v_i(S)$ is the value that i assigns to the subset S of the cake. (Subset will be a finite union of disjoint intervals.)
- Assumptions about the valuation function
  - v_i is normalized with $v_i[0,1] = 1$.
  - v_i is additive on disjoint subsets.. So if A, B are disjoint, then $v_i(A) + v_i(B) = v_i(A \cup B)$.
  - Valuations are "divisible". For every $c \in [0,1]$ and X, there is a Y in X such that $V_i(Y) = cV_i(X)$.

Moving-knife Algorithm for fair division of a cake among n people

- Move a knife continuously over the cake from left to right until some player yells "Stop!"
- Give that player the piece of cake to the left of the knife.
- Iterate with the other $n-1$ players and the remaining cake.

Allocation $(A_1, A_2, \ldots, A_n)$ is proportional if $\forall i$ $v_i(A_i) \geq \frac{1}{n}$

A_i - piece player i gets

envy-free if $\forall i, j$ $v_i(A_i) \geq v_i(A_j)$

Standard model: access to V_i's is thru 2 types of queries:

Eval$_i$ (x, y) $\longrightarrow$ player i returns $V_i([x,y])$

Cut$_i$ (x, α) $\longrightarrow$ returns y s.t. $V_i([x,y]) = \alpha$.

n cut queries $\longrightarrow$ first alloc.

$O(n^2)$ cut queries.

How to measure "complexity" of a division protocol?

want: running time as fn of n

Standard model:

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How many queries does a protocol take?

I Cut You Choose 2 queries

With divide & conquer, can get proportional alloc. using $O(n \log n)$ queries.

The Cake:			
value to player I	$\frac{1}{3}$	0	$\frac{2}{3}$
value to player II	0	$\frac{1}{2}$	$\frac{1}{2}$
value to player III	0	0	1

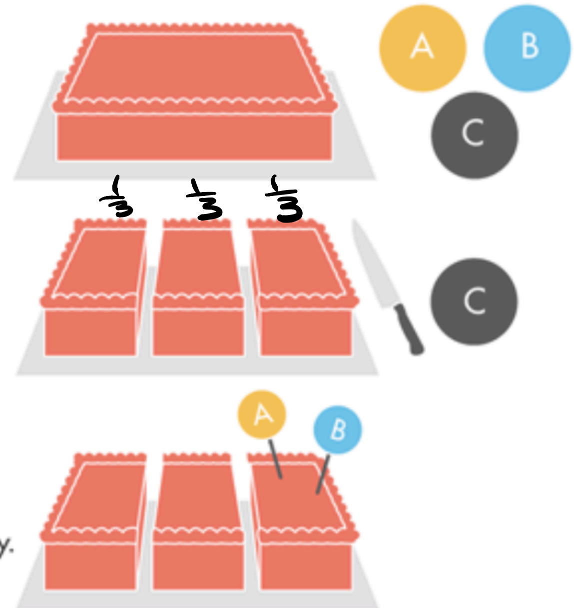
FIGURE 11.3. This figure shows an example of how the Moving-knife Algorithm might evolve with three players. The knife moves from left to right. Player I takes the first piece, then II, then III. In the end, player I is envious of player III.

CAKE CUTTING FOR THREE

1. Alice, Bob and Charlie want to share a cake so that none of them envies other pieces.

2. Charlie cuts the cake into three pieces that are equally valuable from his perspective.

3. Alice and Bob identify their first choices. If they identify the same choice, things get tricky.



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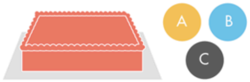

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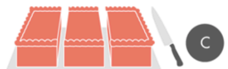

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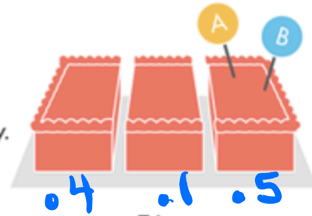
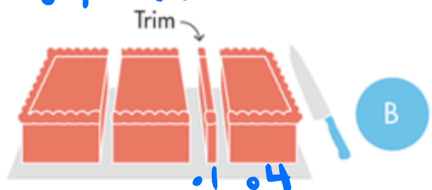


(4) Bob trims his preferred piece to match his second most preferred piece.



(5) Putting the trim to one side they choose in this order: Alice first*, Bob second and Charlie last.

It is envy free

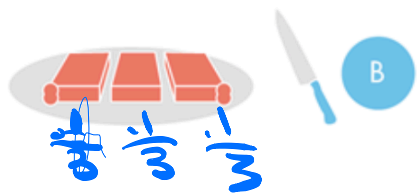
...for Alice, because she got first choice.

...for Bob, because his second choice was equally valuable.

...for Charlie, because the three original slices were equal to him.

*If Alice doesn't choose the trimmed piece, then Bob must take it. Alice and Bob then trade places for the rest of the process.

(6) To divvy up the trimmed slice, first Bob cuts the trim into three pieces that are equally valuable from his perspective.



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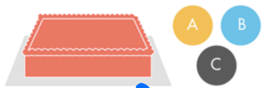

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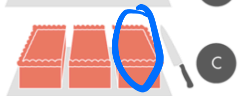

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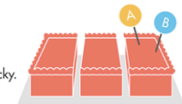

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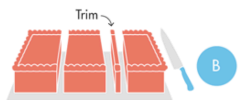

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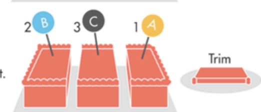

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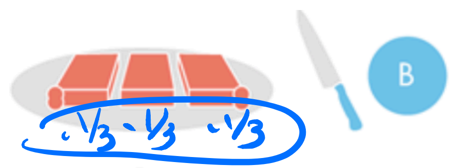
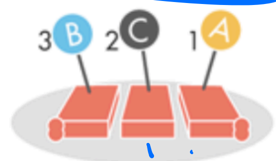


(7) Now they choose a portion of trim in this order: Alice first, Charlie second and Bob last



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CAKE CUTTING FOR THREE

5 queries

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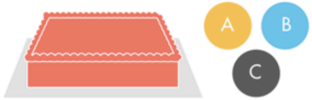

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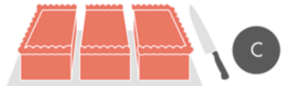

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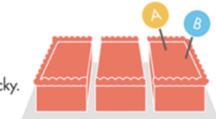

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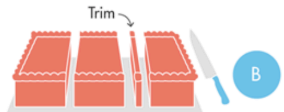

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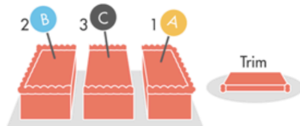

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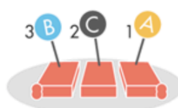

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1960 $n=3$ Selfridge & Conway

1995 Brams & Taylor

$n>3$ even $n=4$

no upper bound on # cuts.

$\forall T$, $\exists$ instance for which protocol used $\geq T$ queries

2016 Aziz McKenzie

$n=4$ ≤ 203 cuts.

protocol for any n

$n^{n^{n^{n^{n^{n}}}}}$ cuts.

best lower bounds $\Omega(n^2)$

Truthfulness

Det mech is truthful if no player can ↑ their utility for outcome by misresponding to any query.

Randomized mech is truthful if no player can ↑ their expected utility by misresponding.

Thm No deterministic mech for proportional cake cutting is truthful.

(not even approx)

Thm ∀ val fns ∃ perfect partition $(B_1, \dots, B_n)$

s.t. $V_i(B_j) = \frac{1}{n}$ $\forall i, j$

nonconstructive

$\Rightarrow$ truthful in exp mech.

compute perfect partition, assign n bundles at random to players.

$\Rightarrow$ truthful in exp.

$\forall$ partition $(C_1, \ldots, C_n)$, if bundles are assigned at random

Exp utility $= \frac{1}{n}$.

$$\frac{1}{n}\sum_{j=1}^{n} v_i(C_j) = 1/n$$

Fair division with indivisible goods.

$v_i(j)$ = value of player i for item j

$j=1$ $j=m$

				
	8	7	20	5
	9	11	12	8
	9	10	18	3

Why EF up to 1 good.

1 2 k n
1 2 k n
1 2 k n
1 2 k

Why doesn't i envy k

(Allocation $(A_1, \ldots, A_n)$ is EF up to one good

Envy-freeness up to one good. if

$\forall i, \forall k$ $\exists$ good $g \in A_k$ s.t.

2 players

$$v_i(A_i) \geq v_i(A_k \setminus g)$$

Maximin share of player i as follows

i partitions goods into n groups $B_1, \ldots, B_n$

takes $\min_j v_i(B_j)$

$$\boxed{MMS_i = \max_{(B_1, \ldots, B_n)} \min_j v_i(B_j)}$$

inputs where no partition that guarantees each agent i $\boxed{MMS}_i$

can guarantee $\geq \frac{3}{4} MMS_i$

				
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Auctions

- Ancient mechanism for buying and selling goods.
- In modern times, used for many economics transactions.
- In the age of the Internet, we can buy and sell goods and services via auctions online, e.g. using eBay
- Companies. like Google and Microsoft use ad auctions to sell advertisement slots that will appear alongside your search results.
- All major search engines and social networking sites (e.g. Facebook) make most of their money from running real-time auctions used to sell online advertisements.
- Consequence: auctions are a major driver of modern Internet economy.

- Why use an auction as opposed to simply fixing prices?

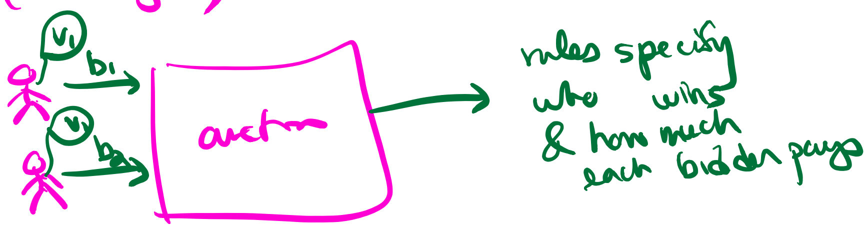
Why might a sell user an auction as opposed to fixing a price?

- Because sellers don’t know how much buyers value their goods and don’t want to risk setting prices that are too low (leaving money on the table) or too high (nobody buys).
- Auction is technique for dynamically setting prices.
- In Internet settings, where the participants in the auction are computer programs or individuals who don’t know each other, price-setting is particularly difficult and this is what motivates auctions.

Single item auctions.

seller w/ single good.

n (strategic) bidders



Each bidder i has a private value v_i for the item being sold.

3 standard types

1st price auction: highest bidder wins & pays what they bid

2nd price auction: highest bidder wins & pays the bid of 2nd highest bidder

All pay auction: highest bidder wins & everyone pays what they bid

bidder utility in auction $= \begin{cases} v_i - p_i & \text{if they win (\& have to pay } p_i) \\ -p_i & \text{if they lose (have to pay } p_i) \end{cases}$

eBay auction = 2nd price auction

submit "sealed bids"

Vickrey auction

Lemma In a second-price auction, it is a dominant strategy to bid truthfully (i.e. set $b_i = v_i$)
(dominant strategy incentive compatible – DSIC)

Proof Fix bids b_{-i}, let $B = \max_{j \neq i} b_j$

utility of $i \leq \max(v_i - B, 0) \Longleftarrow$

bidding truthfully achieves this

$\boxed{\text{Truthful}}$ cond – agents don't need to strategize.

2^{nd} price auction guarantees that item ends up in the hands of the player who values it the most

social welfare, surplus, efficiency

= sum of utilities of all participants (including auctioneer)

$$\boxed{\sum_i \left(v_i \mathbb{1}_{i \text{ won}} - p_i\right) + \sum_{i=1}^{n} p_i}$$

auctioneer utility.

$$= \sum_i v_i \mathbb{1}_{i \text{ won}}$$

Individually rational (IR)

no regret for having participated

$u_i \geq 0$