Fair Division

Fair Division

- What is a fair way for 2 people to split a heterogenous, divisible good?

- Why not 50/50?

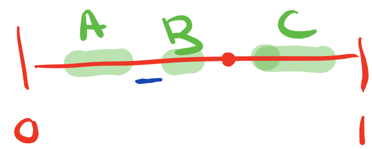
- I Cut, You Choose Protocol
  - Player I told to split the good into two pieces A and B such that $v_I(A) = v_I(B) = \frac{1}{2} v_I(A \cup B)$. ← A∪B whole thing.
  - Player II picks his favorite of A and B. ←
  - Player I takes the other piece. ←

$v_1(\cdot) \neq v_2(\cdot)$

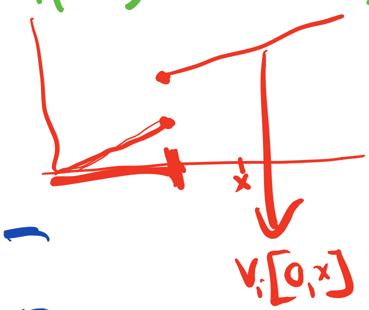
Formal model:

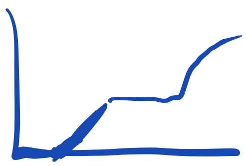
Good is the unit interval $[0,1]$

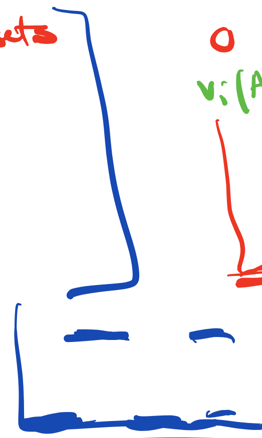
- $v_i(S)$ value player i assigns to subset S of the interval.
- Assumptions about valuation fns:
  - $v_i[0,1] = 1$
  - v_i is additive on disjoint subsets
  - Valuations are "divisible": $\forall\ 0 \le c \le 1,\ \exists\ y$ s.t. $v_i([0,y]) = c$



$v_i(A \cup B) = v_i(A) + v_i(B)$







Are the players incentivized to follow the rules?

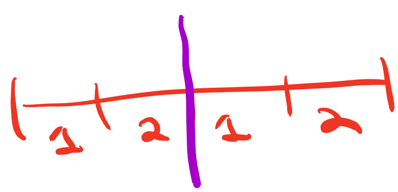
An allocation $A = (A_1, \ldots, A_n)$ $\quad A_i \to$ player i

is Pareto optimal if $\nexists$ no allocation $B = (B_1, \ldots, B_n)$

everyone is at least as happy & at least one strictly happier.

Is I Cut You Choose PO? (assuming player I follows rules)

No.



Defn
An allocatn $A_1, \ldots A_n$ is **proportional** if $\forall i \quad v_i(A_i) \geq \frac{1}{n}$

Defn An allocatn $(A_1, \ldots A_n)$ is **envy-free** (EF) if $\forall i, \forall j \quad v_i(A_i) \geq v_i(A_j)$

v_1	$\frac{1}{3}$	$\frac{2}{3}$	0
v_2	0	$\frac{1}{3}$	$\frac{2}{3}$
v_3	$\frac{2}{3}$	0	$\frac{1}{3}$

If an allocatn is EF, then its also proportional

$$\sum_{j=1}^{n} v_i(A_i) \geq \sum_{j=1}^{n} v_i(A_j)$$

$$n\, v_i(A_i) \geq 1$$

$$v_i(A_i) \geq \frac{1}{n}$$

Moving-knife Algorithm for fair division of a cake among n people

- Move a knife continuously over the cake from left to right until some player yells "Stop!"
- Give that player the piece of cake to the left of the knife.
- Iterate with the other $n-1$ players and the remaining cake.

Claim: when there are k players left over, the leftover cake is worth $\geq \frac{k}{n}$ to each of them.

$\geq \frac{k-1}{k} \frac{k}{n}$ to leftover players

$\frac{1}{k} \frac{k}{n} \geq \frac{1}{n}$