Revenue Maximization

1 seller n bidders

bidder i's value $V_i \sim F$

Revenue maximizing truthful auction:

Thm If all $V_i \sim F$ independently
(regular)

then rev maximizing auction is Vickrey VCG auction with monopoly reserve price r^*

$$r^* = \arg\max_p \; p(1 - F(p))$$

$p \Pr(V \geq p)$

to any "single parameter setting"

$v_1 \to$ item 1

$v_2 \to$ item 2

Auctioneer selling 2 items. single bidder

$V_1 \sim F_1$ $\quad V_2 \sim F_2$

r_1 $\quad r_2$

Example:

$$\boxed{V_i = \begin{cases} 1 & \frac{1}{2} \\ 2 & \frac{1}{2} \end{cases}}$$

$i = 1, 2.$

set price of 1 $\quad [2 \cdot 1 - \varepsilon]$

$E(\text{rev}) = 2 - 2\varepsilon$

$u_i = (V_1 - p_1)^+ + (V_2 - p_2)^+$

set $p_1 = 2 - \varepsilon$ $\quad p_2 = 2 - \varepsilon$

$E(\text{rev}) = \left(\frac{2-\varepsilon}{2}\right) 2$

set price of $3-\varepsilon$ for both items

$$\Pr(\text{buys}) = 1 - \underbrace{\Pr[v_1 = 1 = v_2]}_{\frac{1}{4}} = \frac{3}{4}.$$

$$E(\text{rev}) = 3 \cdot \frac{3}{4} = \frac{9}{4} >$$

$r_{\text{bundle}} = \arg\max_p p \Pr(v_1 + v_2 \geq p)$

$$v_1, v_2 = \begin{cases} 0 & \frac{1}{3} \\ 1 & \frac{1}{3} \\ 2 & \frac{1}{3} \end{cases}$$

p_1, p_2

$\boxed{p_1 = p_2 = 1-\varepsilon}$

$v_1 \backslash v_2$	0	1	2
0	X	$1-\varepsilon$	$1-\varepsilon$
1	$1-\varepsilon$	$2(1-\varepsilon)$	
2	$1-\varepsilon$	"	"

auctioneer. revenue.

$p_1 = 1-\varepsilon$ | $p_2 = 1-\varepsilon$

$$E(\text{rev}) = \frac{4}{9} + \frac{4}{9} \cdot 2 = \frac{12}{9}$$

$v_1 \backslash v_2$	0	1	2
0	X	X	$2-\varepsilon$
1	X	X	$2-\varepsilon$
2	$2-\varepsilon$	$2-\varepsilon$	$4-2\varepsilon$

$\boxed{p_1 = 2-\varepsilon}$ $\boxed{p_2 = 2-\varepsilon}$

$$E(\text{rev}) = \frac{4}{9} \cdot 2 + \frac{1}{9} \cdot 4 = \boxed{\frac{12}{9}}$$

$p_{\text{bundle}} = 2-\varepsilon$

	0	1	2
0	X	X	$2-\varepsilon$
1	X	$2-\varepsilon$	$2-\varepsilon$
2	$2-\varepsilon$	$2-\varepsilon$	$2-\varepsilon$

$$E(\text{rev}) = (2-\varepsilon) \cdot \frac{2}{3} = \frac{4}{3} = \frac{12}{9}$$

$p_{\text{bundle}} = 3-\varepsilon$

0 1 2

	0	1	2
0	X	X	X
1	X	X	$3-\varepsilon$
2	X	$3-\varepsilon$	$3-\varepsilon$

$E(\text{rev}) = 3 \cdot \frac{1}{3} = 1$

		$4-\varepsilon$

$p_{\text{bundle}} = 4$

$E(\text{rev}) = 4 \cdot \frac{1}{9}$

__Better__ Offer buyer choice between any single item at price of $2-\varepsilon$ or bundle of both items at price of $3-2\varepsilon$

	0	1	2
0	X	X	$2-\varepsilon$
1	X	X	$3-2\varepsilon$
2	$2-\varepsilon$	$3-2\varepsilon$	$3-2\varepsilon$

1	2	1 2
$2-\varepsilon$	$2-\varepsilon$	$\boxed{3-2\varepsilon}$

$E(\text{rev}) = 3 \cdot \frac{3}{9} + 2 \cdot \frac{2}{9} = \frac{13}{9}$

F $\Rightarrow$ menu size of optimal auction is infinite.

1 at $2-\varepsilon$
2 at $2-\varepsilon$
both $3-2\varepsilon$

.5 1,2 at price x
.3 1 at y
.2 2 at z

when values are correlated,
possible to design an auction
that extracts "full value"

single item auction! best possible case

$$\boxed{E(\text{rev}) = E(\max(v_1, \dots, v_n))}$$

2 item $U[0,1]$

single item $\frac{5}{12}$ $\frac{2}{3}$

Prior indep auctions.

How do we know F?

Thm Suppose F is regular. single item auctions

Then expected revenue of Vickrey auction with $n+1$ bidders

$\geq$ exp rev of opt auction w/ n bidders

competition is more important than reserve price.

Corollary If $n \geq 2$, exp revenue of Vickrey auction(n)

assuming all values $\sim F$

$\geq \frac{n-1}{n}$ of exp rev of opt auction(n)

Pf

A^*_n opt auction $\hat{n}$ bidder

$\Downarrow \implies \hat{A}_{n-1}$

$\frac{n-1}{n}$ of rev of A^*_n

$\boxed{\hat{A}_{n-1}}$ $\quad \underbrace{(b_1, \dots, b_{n-1})}_{\text{inputs}}$ $\quad b_n$ Achters. $\quad \hat{b}_n \sim F$

Run $A^*_n(b_1, \dots, b_{n-1}, \hat{b}_n)$

if ~~sells~~ sells to $1 \dots n-1$ ✓

if it sells to bidder n do nothing.

$$\text{rev}(\hat{A}_{n-1}) \geq \frac{n-1}{n} \text{rev}(A^*_n)$$

$$\text{rev Vickrey}(n \text{ bidders}) \geq \text{rev OPT}(n-1) \geq \text{rev}(\hat{A}_{n-1})$$

$$\geq \frac{n-1}{n} \text{rev}(A^*_n)$$

Mechanisms for profit maximization

- Research divided into three strands:
  - Bayesian:
    - Agents values assumed to come from publicly known prior distributions.
    - Goal: to do well in expectation
  - Prior-independent
    - There is a prior, but auctioneer doesn't know it.
    - Goal: to do well in expectation.
  - Prior-free
    - What if we don't want to assume a prior?
    - Want to do well in worst case

$\forall \quad \vec{v}=(v_1,\dots,v_n)$

π

Prior–free

- Key questions:
  - How do we design mechanisms for profit maximization that work well without priors?
  - How do we evaluate these mechanisms?

Example: Digital Goods Auction

What does VCG do for digital goods?

- Given
  - Unlimited number of copies of identical items for sale
  - n bidders, bidder i has private value v_i for obtaining one item (and no additional value for more than one)
- Goal: Design truthful auction to maximize profit

Maximizing Profit: A Competitive Analysis Framework

- Goal: truthful profit maximizing basic auction
- There is no auction that is best on every input.
  - How do we evaluate auctions?

\$100

\$100

truthfulness $\Rightarrow$ imposes constraints on what we can achieve.

absolute optimality $\Rightarrow$ relative optimality.

Maximizing Profit: A Competitive Analysis Framework

- Goal: truthful profit maximizing basic auction

- There is no auction that is best on every input.
  - How do we evaluate auctions?

$\vec{v} = (v_1, \ldots, v_n)$

- Competitive analysis
  - Compare auction profit to “profit benchmark” OPT(v).

$= \sum_{i=1} v_i$

Profit Benchmark for Digital Goods Auction

$\vec{v} = (v_1, \dots, v_n)$

Definition:

A truthful auction is c-competitive if for all v its profit is at least OPT($\vec{v}$)/c

2

- Define OPT(v) = optimal fixed price revenue

  $OPT(\vec{v}) = \max_p p \left(\begin{array}{c} \#\text{bidders} \\ \text{with value} \\ \geq p \end{array} \right)$

  - Example: v= ($\cancel{3}$, 2, 2, 1, 1)   OPT($\vec{v}$)= 6

1

3 3

2 6

c