Revenue Maximization

- Seller in bidders, bidder's value $V_i \sim F$

Revenue maximizing truthful auction:

**Theorem** If all $V_i \sim F$ independently (regular), then the revenue maximizing auction is Vickrey VCG.

To any "single parameter setting" $r^* = \arg \max_{p} p \cdot \prod (1 - F(p))$

**Example:**

- Auction setting: 2 items, single bidder
- $V_1 \sim F_1$, $V_2 \sim F_2$

**Set price of 1:**

- $E(mw) = 2 - 2\epsilon$
- $u_i = (V_i - p)^+$
- $E(mw) = \left(\frac{2 - \epsilon}{2}\right) \cdot 2$

**Set $p_1 = 2 - \epsilon$**

- $p_2 = 2 - \epsilon$
set prob. $g = 3 - \epsilon$ for both items,

$$Pr(v = 3) = 1 - Pr(v = 1 = v_o) = \frac{3}{4}.$$ 

$$E(\nu w) = 3 \cdot \frac{3}{4} = \frac{9}{4} > 1.$$ 

**Bundle Option**: Any pair $Pr(v_i = v_0)$

**Matrix Representation**:

\begin{align*}
&\begin{array}{c|c|c|c}
0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
\end{array} \\
&\begin{array}{c|c|c}
3 & 1 & 2 \\
\hline
1 & 0 & 0 \\
2 & 0 & 0 \\
3 & 0 & 0 \\
\end{array}
\end{align*}

**Revenue Calculation**:

\begin{align*}
&Pr(1-\epsilon) \\
&Pr(2-\epsilon) \\
&E(\nu w) = \frac{4}{9} + \frac{4}{9} - \frac{2}{9}
\end{align*}

\begin{align*}
&Pr(1-\epsilon) \\
&Pr(2-\epsilon) \\
&E(\nu w) = \frac{4}{9} - \frac{1}{9} \cdot \frac{2}{3} \\
&= \frac{4}{9}
\end{align*}

**Bundle Option**: $2 - \epsilon$

$$E(\nu w) = (2 - \epsilon) \frac{2}{3} = \frac{4 - \epsilon}{3}$$

**Bundle Option**: $3 - \epsilon$

$$E(\nu w) = 3 - \epsilon$$
Better. Offer buyer choice between:
- Any single item at price $2-$e
- Bundle of both items at price $3-$e

\[
\begin{array}{c|c|c|c|c|c|c|c}
0 & 1 & 2 & 3-	ext{e} & 3-	ext{e} & 3-	ext{e} & 3-	ext{e} & 3-	ext{e} \\
\hline
0 & X & X & X & X & 0 & 0 & 0 \\
1 & X & X & \text{3-e} & \text{3-e} & 0 & 0 & 0 \\
2 & \text{3-e} & \text{3-e} & \text{3-e} & \text{3-e} & 0 & 0 & 0 \\
\end{array}
\]

\[
E(n) = 3 \cdot \frac{1}{3} = 1
\]

\[
p_{\text{bundle}} = 4
\]

\[
E(n) = 4 \cdot \frac{1}{4} = 1
\]
When values are correlated, possible to design an auction that extracts "full value.

Single item auction: best possible case

\[ E(\text{rev}) = E(\max(V_1, \ldots, V_n)) \]

2 item \([0,1]

single item \[ \frac{5}{12} \]

Prior independence.
How do we know \( F \)?

Then suppose \( F \) is regular.

\[ \text{Then expected revenue of \( n+1 \) Vickrey auction with } \]
\[ \text{bidders} \]
\[ \geq \text{exp revenue of opt auction with } \]
\[ n \text{ bidders} \]

Competition is more important than reserve price.

Corollary: If \( n \geq 2 \), exp revenue of Vickrey auction (n)
\[ \geq \frac{n-1}{n} \text{ exp revenue of opt auction (n)} \]

\[ P^* \]
\[ A^* \text{ opt auction with } n \text{ bidders} \]
\[ \Downarrow \quad \Rightarrow \quad A_{n-1}^* \]

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\[ \hat{A}_{n-1} \quad \text{inputs} \quad b, \ldots, b_{n-1}, b_n \quad \text{to \ Achnor} \]  
Run \( A^n (b, \ldots, b_n) \)

if \( \text{fails to do nothing} \)

\[ \text{reuc}(A^n) \geq \frac{n-1}{n} \text{reuc}(A^n) \]

\[ \text{Vicug (n buddies)} \]

\[ \text{reuc(Vicug (n buddies))} \geq \text{reuc(OPT (n-1))} \geq \text{reuc}(\hat{A}_{n-1}) \]

\[ \geq \frac{n-1}{n} \text{reuc}(A^n) \]
Mechanisms for profit maximization

- Research divided into three strands:
  - **Bayesian:**
    - Agents values assumed to come from publicly known prior distributions.
    - Goal: to do well in expectation
  - **Prior-independent**
    - There is a prior, but auctioneer doesn’t know it.
    - Goal: to do well in expectation.
  - **Prior-free**
    - What if we don’t want to assume a prior?
    - Want to do well in worst case
Prior-free

Key questions:

- How do we design mechanisms for profit maximization that work well without priors?
- How do we evaluate these mechanisms?
Example: Digital Goods Auction

What does VCG do for digital goods?

- Given
  - Unlimited number of copies of identical items for sale
  - \( n \) bidders, bidder \( i \) has private value \( v_i \) for obtaining one item (and no additional value for more than one)
- Goal: Design truthful auction to maximize profit
Maximizing Profit: A Competitive Analysis Framework

- Goal: truthful profit maximizing basic auction
- There is no auction that is best on every input.
  - How do we evaluate auctions?

truthfulness \Rightarrow \text{imposes constraints on what we can achieve.}

absolute optimality \Rightarrow \text{relative optimality.}
Maximizing Profit: A Competitive Analysis Framework

- Goal: truthful profit maximizing basic auction

- There is no auction that is best on every input.
  - How do we evaluate auctions?

- Competitive analysis
  - Compare auction profit to “profit benchmark” $\text{OPT}(v)$. 
Definition:
A truthful auction is $c$-competitive if for all $v$ its profit is at least $\frac{\text{OPT}(v)}{c}$.

Define $\text{OPT}(v) =$ optimal fixed price revenue.

Example: $v = (3, 2, 2, 1, 1)$, $\text{OPT}(v) = 6$