Solution for Homework 2

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Problem 1

By Theorem 2.8 from Hartline (its version was also discussed in class), an easy way to demostrate the truthfulness of the given mechanism was to show that

- The allocation rule is monotonic.
- The winners pay the threshold bid.

To show monotonicity, consider player i and fix the bids of others. There always exists the largest number k s.t. if bid $b_i \geq \frac{1000}{k}$ then there is a total of k players bidding $\frac{1000}{k}$ or above. For any $b_i < \frac{1000}{k}$ player i loses, and for any $b_i \geq \frac{1000}{k}$ he wins. Thus, the allocation rule is monotonic.

To show that winners pay the threshold bid, observe from the argument above that the threshold bid for each player is exactly $\frac{1000}{k}$ — bidding less guarantees loss and bidding at least that amount guarantees win.

Many people tried showing directly on a case-by-case basis that truth-telling is a dominant strategy. Proofs of this type tend to be much more complicated and error-prone, and very few of those who chose this approach for this problem got the proof entirely right.

Problem 2

- a) WLOG, assume $v_1 < v_2$. Since only one player can win, an upper bound on the total payoff for the colluding players is v_2. It can be achieved if player 2 wins and pays nothing. For this to happen, player 1 should bid 0 and player 2 can bid anything above 0. (If $v_1 = v_2$, one of the players should bid 0 and the other can bid anything.)
- b) The presence of player 3 changes the strategy slightly. Now player 2 should bid truthfully, i.e. bid v_2, while player 1 should still bid 0. To see that this is a BNE strategy for them, note that given this strategy of players 1 and 2, player 3 will want to bid truthfully and consider two cases. First, suppose $v_3 > v_2$. Then player 2 doesn't want to outbid player 3, who is playing truthfully, so player 2 shouldn't overbid. Her underbidding doesn't change the colluders' total expected payoff in this case. Similarly, Player 1 doesn't want to overbid, and his underbidding doesn't help the colluders' cause either. Second,

suppose $v_3 \le v_2$. Then player 2 wants to outbid player 3. Bidding truthfully achieves that, underbidding may lead to loss, and overbidding doesn't help. At the same time, player 2 also wants to pay as little as possible. For any $v_3 < v_2$, player 2 pays $\max\{b_1, b_3 = v_3\}$, so player 1 bidding more than 0 may lead to a suboptimal payoff for the colluders. Thus, the colluders don't want to deviate from the above strategy.

Problem 3

Again using Theorem 2.8 from Hartline, it is enough to show that the allocation rule of the described auction isn't monotone non-decreasing. It isn't, since if everyone bids truthfully, for a given player, having the second-highest valuation leads to a win, but having a higher valuation may result in outbidding everyone and lead to a loss.

As for Problem 1, several people attempted to give a direct proof. Here it is even harder to do correctly than in Problem 1, since the payment rule isn't known, and without making some assumptions about the payment rule one can't say much about players' payoffs in case they deviate. Unfortunately, everyone who tried a direct proof made some such assumptions, which made the proof invalid.

Additional Problem 1

As a consequence of the Revenue Equivalence Theorem, each player with valuation v_i in the all-pay auction will make the same expected payment as a player with the same valuation in a sealed-bid first- or second-price auction. In either of these auctions for 3 players, player i's expected payment is $\frac{2}{3}v_i^3$ (for n players it's $\frac{n-1}{n}v_i^n$). In an all-pay auction, players pay their bids, so it's natural to *guess* that each player i with valuation v_i bidding $\frac{2}{3}v_i^3$ is a BNE strategy. Note, however — so far this is just a guess; we need to actually prove that this is indeed a BNE.

One way to do it is to show, by Theorem 2.7 from Hartline, that the allocation rule under this strategy is monotone non-decreasing and the the payment for each player obeys the formula $p(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)dz$. To show monotonicity, observe that if everyone bids $\frac{2}{3}v_i^3$, the player with the highest valuation wins. To show the payment identity, WLOG assume we are analyzing the game from the point of view of player 1 and observe that $x_1(v_1) = Pr[\text{player 1 outbids the other two}] = Pr[\frac{2}{3}v_1^3 > \frac{2}{3}v_2^3 \wedge \frac{2}{3}v_1^3 > \frac{2}{3}v_3^3] = Pr[v_1 > v_2 \wedge v_1 > v_3] = Pr[v_1 > v_2]\cdot Pr[v_1 > v_3] = v_1^2$. Therefore, $v_1 x_1(v_1) - \int_0^{v_1} x_1(z)dz = v_1^3 - \int_0^{v_1} z^2 dz = v_1^3 - \frac{v_1^3}{3} = \frac{2}{3}v_1^3$, which is exactly the payment under our guessed strategy.

Another, about equally easy way to show that the guessed strategy is a BNE is to show that the expected utility of each player is maximized if he bids $b_i = \frac{2}{3}v_i^3$, assuming others stick to this strategy. To do so, we need to solve for b_1 that maximizes $E[u_1] = Pr[\text{player 1 outbids the other two}]\cdot v_1 - b_1 = Pr[b_1 > \frac{2}{3}v_2^3] \cdot Pr[b_1 > \frac{2}{3}v_3^3] \cdot v_1 - b_1 = Pr[(\frac{3}{2}b_1)^{\frac{1}{3}} > v_2] \cdot Pr[(\frac{3}{2}b_1)^{\frac{1}{3}} > v_3] \cdot v_1 - b_1 =$

$(\frac{3}{2}b_1)^{\frac{2}{3}}v_1 - b_1$. Differentiating, setting to 0, and solving for b_1 yields $b_1 = \frac{2}{3}v_1^3$, as we guessed.

Additional Problem 2

The solution below assumes we are dealing with a digital-goods setting, since its analysis is slightly harder/more interesting than that of a single-item setting. A fully correct solution for a single-item setting would get full credit though.

- a) To maximize auctioneer's profit, we run the optimal Myerson's mechanism, which maximizes the virtual surplus. That is, we find the virtual surplus of each market m, $\Phi_m = \sum_{i_m} \max\{\phi_m(b_{i_m}), 0\}$, where i_m ranges over players in market m, and b_{i_m} are players' bids in that market, and sell in the market with the bigger Φ_m. In that market, we allocate to everyone with a non-negative virtual valuation, i.e. with $\phi_m(b_i) \geq 0$. We charge each winning player the threshold bid, but the threshold bid here is subtle, since we have two markets, and players of one market are aware of players in the other market. Therefore, we charge each winning player the lowest bid he could place that makes sure that both his market m wins *and* that he wins in that market. I.e., we charge each winner $\max\{\phi_m^{-1}(0), \phi_m^{-1}(\phi_m(b_i) - (\max\{\Phi_1, \Phi_2\} - \min\{\Phi_1, \Phi_2\}))\}$, and charge losers nothing.

- b) To maximize social welfare, we sell to everyone in the market with the greatest sum of bids. Again, each winner is charged his threshold bid, which in this case is the amount he would *have to* bid to ensure his market wins. Letting B_m be the sum of bids in market m and b_i be the bid of player i in the winning market, the payment of each player i in the winning market is $\max\{b_i - (\max\{B_1, B_2\} - \min\{B_1, B_2\}), 0\}$. Players in this losing market don't pay anything.

- c) In the profit-maximization setting, we first compute the virtual valuation function $\phi_m(z)$ for each market. The general form for an exponential distribution is $\phi_m(z) = z - \frac{1}{\lambda}$; thus, for $\lambda = 1$ it is $\phi_m(z) = z - 1$ and for $\lambda = 2$ it is $\phi_m(z) = z - \frac{1}{2}$. Using these formulas and the allocation rule from part a), we get $\Phi_1 = 5.5$ and $\Phi_2 = 5.6$, so we should sell in the $\lambda = 2$ market. There, we allocate to the players who bid 1 and 5.6. We charge the one who bid 1 $\max\{\phi_2^{-1}(0), \phi_2^{-1}(0.5 - 0.1)\} = 0.9$ and the one who bid 5.6 $\max\{\phi_2^{-1}(0), \phi_2^{-1}(5.1 - 0.1)\} = 5.5$. The player who bid 0.3 doesn't get allocated and doesn't pay anything.

  In the social welfare-maximization, the sums of bids is 7.7 and 6.9 respectively, so we allocate to everyone in the $\lambda = 1$ market. The player bidding 0.2 doesn't pay anything, the one bidding 1.5 pays $1.5 - 0.8 = 0.7$, and the one bidding 6 pays $6 - 0.8 = 5.2$.