Solution for Problem 3 (Homework 2)

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Suppose we knew that the game has a joint Nash equilibrium strategy $\{\vec{p}, \vec{q}\}$ that assigns probabilities $p_i > 0$ to a subset I_R of row player R's pure strategies and $q_i > 0$ to a subset I_C of column player C's pure strategies. Such a strategy would need to satisfy the following conditions:

- No matter what pure strategy $r \in I_R$ R plays, the expected payoff to R given C's mixed strategy q should be the same value; denote it as V_R. (If this value is not the same for all $r \in I_R$, R has an incentive to deviate by choosing a pure strategy that brings the biggest payoff, i.e. $\{\vec{p}, \vec{q}\}$ isn't a Nash equilibrium.)
- If R decides to play a pure strategy $r \notin I_R$, the expected payoff to R shouldn't be more than V_R. (Otherwise, $\vec{p}$ should assign a non-zero probability to that pure strategy, i.e. the stratety should be in I_R.)
- Similarly, no matter what pure strategy $c \in I_C$ C plays, the expected payoff to C should be the same; denote this value as V_C.
- If C plays a pure strategy $c \notin I_C$, C's expected payoff should be no more than V_C.
- Under $\{\vec{p}, \vec{q}\}$, the probabilities of pure strategies in I_R and in I_C respectively must sum to one and be no less than zero.
- Under $\{\vec{p}, \vec{q}\}$, the probabilities of pure strategies not in I_R and not in I_C respectively must be zero.

These requirements translate into the following system of linear inequalities:

$$\sum_{j=1}^{m} q_j A_{i,j} = V_R \qquad \text{for all } i, \text{ s.t. } s_i \in I_R \tag{1}$$

$$\sum_{j=1}^{m} q_j A_{i,j} \leq V_R \qquad \text{for all } i, \text{ s.t. } s_i \notin I_R \tag{2}$$

$$\sum_{i=1}^{n} p_i A_{i,j} = V_C \qquad \text{for all } j, \text{ s.t. } s_j \in I_C \tag{3}$$

$$\sum_{i=1}^{n} p_i A_{i,j} \leq V_C \qquad \text{for all } j, \text{ s.t. } s_j \notin I_C \tag{4}$$

$$\sum_{i=1}^{n} p_i = 1 \tag{5}$$

$$p_i \geq 0 \qquad \text{for all } i, \text{ s.t. } s_i \in I_R \tag{6}$$

$$p_i = 0 \qquad \text{for all } i, \text{ s.t. } s_i \notin I_R \tag{7}$$

$$\sum_{j=1}^{m} q_j = 1 \tag{8}$$

$$q_j \geq 0 \qquad \text{for all } j, \text{ s.t. } s_j \in I_C \tag{9}$$

$$q_j = 0 \qquad \text{for all } j, \text{ s.t. } s_j \notin I_C \tag{10}$$

To find a Nash equilibrium for given I_R and I_C, we need to find *any* $\{\vec{p}, \vec{q}\}$ satisfying the above system of inequalities. In other words, we need to solve the corresponding LP feasibility problem. LP feasibility problems are just a type of LP problems — in the latter one is interested in an optimal solution to a system of inequalities, while in the former one is interested in any solution to it. Therefore, if a solution exists it can be found with an LP solver in finite time.

Of course, we don't actually know I_R and I_C for which a Nash equilibrium exists. Therefore, we need to iterate over all possible pairs of action sets and for each to try to find a Nash equilibrium strategy as described above. Once such a strategy is found, the algorithm will halt. There are a total of 2^{n+m} possible action set pairs, so the whole algorithm runs in a finite, although large, amount of time.