# Natural Language Processing (CSE 490U): Language Models

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- Sometimes true: $p(X = x, Y = y) = p(X = x) \cdot p(Y = y)$
- The difference between *true* and *estimated* probability distributions

# Language Models: Definitions

- $\mathcal{V}$ is a finite set of (discrete) symbols (☺ "words" or possibly characters); $V = |\mathcal{V}|$
- $\mathcal{V}^\dagger$ is the (infinite) set of sequences of symbols from $\mathcal{V}$ whose final symbol is ⯃
- $p : \mathcal{V}^\dagger \to \mathbb{R}$, such that:
  - For any $\boldsymbol{x} \in \mathcal{V}^\dagger$, $p(\boldsymbol{x}) \geq 0$
  - $\displaystyle\sum_{\boldsymbol{x} \in \mathcal{V}^\dagger} p(\boldsymbol{X} = \boldsymbol{x}) = 1$

  (I.e., $p$ is a proper probability distribution.)

Language modeling: estimate $p$ from examples, $\boldsymbol{x}_{1:n} = \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \rangle$.

# Immediate Objections

1. Why would we want to do this?
2. Are the nonnegativity and sum-to-one constraints really necessary?
3. Is "finite $\mathcal{V}$" realistic?

# Motivation: Noisy Channel Models

A pattern for modeling a pair of random variables, $X$ and $Y$:

$$\boxed{\text{source}} \longrightarrow Y \longrightarrow \boxed{\text{channel}} \longrightarrow X$$

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- $Y$ is the plaintext, the true message, the missing information, the output
- $X$ is the ciphertext, the garbled message, the observable evidence, the input
- Decoding: select $y$ given $X = x$.

$$\begin{aligned} y^* &= \operatorname*{argmax}_y p(y \mid x) \\ &= \operatorname*{argmax}_y \frac{p(x \mid y) \cdot p(y)}{p(x)} \\ &= \operatorname*{argmax}_y \underbrace{p(x \mid y)}_{\text{channel model}} \cdot \underbrace{p(y)}_{\text{source model}} \end{aligned}$$

# Noisy Channel Example: Speech Recognition

$$\boxed{\text{source}} \longrightarrow \text{sequence in } \mathcal{V}^{\dagger} \longrightarrow \boxed{\text{channel}} \longrightarrow \text{acoustics}$$

- Acoustic model defines $p(\text{sounds} \mid \boldsymbol{x})$ (channel)
- Language model defines $p(\boldsymbol{x})$ (source)

# Noisy Channel Example: Speech Recognition

Credit: Luke Zettlemoyer

| word sequence | $\log p(\text{acoustics} \mid \text{word sequence})$ |
|---|---|
| the station signs are in deep in english | -14732 |
| the stations signs are in deep in english | -14735 |
| the station signs are in deep into english | -14739 |
| the station 's signs are in deep in english | -14740 |
| the station signs are in deep in the english | -14741 |
| the station signs are indeed in english | -14757 |
| the station 's signs are indeed in english | -14760 |
| the station signs are indians in english | -14790 |
| the station signs are indian in english | -14799 |
| the stations signs are indians in english | -14807 |
| the stations signs are indians and english | -14815 |

# Noisy Channel Example: Machine Translation

*Also knowing nothing official about, but having guessed and inferred considerable about, the powerful new mechanized methods in cryptography—methods which I believe succeed even when one does not know what language has been coded—one naturally wonders if the problem of translation could conceivably be treated as a problem in cryptography. When I look at an article in Russian, I say: "This is really written in English, but it has been coded in some strange symbols. I will now proceed to decode."*

Warren Weaver, 1955

# Noisy Channel Examples

- Speech recognition
- Machine translation
- Optical character recognition
- Spelling and grammar correction

# Immediate Objections

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# Evaluation: Perplexity

Intuitively, language models should assign high probability to real language they have not seen before.

For out-of-sample ("held-out" or "test") data $\bar{\boldsymbol{x}}_{1:m}$:

- Probability of $\bar{\boldsymbol{x}}_{1:m}$ is $\prod_{i=1}^{m} p(\bar{\boldsymbol{x}}_i)$

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- Average log-probability per word of $\bar{\boldsymbol{x}}_{1:m}$ is

$$l = \frac{1}{M} \sum_{i=1}^{m} \log_2 p(\bar{\boldsymbol{x}}_i)$$

if $M = \sum_{i=1}^{m} |\bar{\boldsymbol{x}}_i|$ (total number of words in the corpus)

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- Perplexity (relative to $\bar{\boldsymbol{x}}_{1:m}$) is $2^{-l}$

Lower is better.

# Understanding Perplexity

$$2^{-\frac{1}{M}\sum_{i=1}^{m}\log_2 p(\bar{\boldsymbol{x}}_i)}$$

It's a branching factor!

- Assign probability of 1 to the test data $\Rightarrow$ perplexity $= 1$
- Assign probability of $\frac{1}{|\mathcal{V}|}$ to every word $\Rightarrow$ perplexity $= |\mathcal{V}|$
- Assign probability of 0 to *anything* $\Rightarrow$ perplexity $= \infty$
  - This motivates a stricter constraint than we had before:
    - For any $\boldsymbol{x} \in \mathcal{V}^\dagger$, $p(\boldsymbol{x}) > 0$

# Perplexity

- Perplexity on conventionally accepted test sets is often reported in papers.
- Generally, I won't discuss perplexity numbers much, because:
  - Perplexity is only an intermediate measure of performance.
  - Understanding the models is more important than remembering how well they perform on particular train/test sets.
- If you're curious, look up numbers in the literature; always take them with a grain of salt!

# Immediate Objections

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3. Is "finite $\mathcal{V}$" realistic?

# Is “finite $\mathcal{V}$” realistic?

No

# Is "finite $\mathcal{V}$" realistic?

No
no
n0
-no
notta
Nº
/no
//no
(no
|no

# The Language Modeling Problem

Input: $\boldsymbol{x}_{1:n}$ ("training data")
Output: $p : \mathcal{V}^\dagger \to \mathbb{R}^+$
☺ $p$ should be a "useful" measure of plausibility (not grammaticality).

# A Trivial Language Model

$$p(\boldsymbol{x}) = \frac{|\{i \mid \boldsymbol{x}_i = \boldsymbol{x}\}|}{n} = \frac{c_{\boldsymbol{x}_{1:n}}(\boldsymbol{x})}{n}$$

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What if $\boldsymbol{x}$ is not in the training data?

# Using the Chain Rule

$$
p(\boldsymbol{X} = \boldsymbol{x}) = \left( \begin{array}{l} p(X_1 = x_1) \\ \cdot\ p(X_2 = x_2 \mid X_1 = x_1) \\ \cdot\ p(X_3 = x_3 \mid \boldsymbol{X}_{1:2} = \boldsymbol{x}_{1:2}) \\ \vdots \\ \cdot\ p(X_\ell = \textcolor{red}{\bigcirc} \mid \boldsymbol{X}_{1:\ell-1} = \boldsymbol{x}_{1:\ell-1}) \end{array} \right)
$$

$$
= \prod_{j=1}^{\ell} p(X_j = x_j \mid \boldsymbol{X}_{1:j-1} = \boldsymbol{x}_{1:j-1})
$$

# Unigram Model

$$p(\boldsymbol{X} = \boldsymbol{x}) = \prod_{j=1}^{\ell} p(X_j = x_j \mid \boldsymbol{X}_{1:j-1} = \boldsymbol{x}_{1:j-1})$$

$$\stackrel{\text{assumption}}{=} \prod_{j=1}^{\ell} p_{\boldsymbol{\theta}}(X_j = x_j) = \prod_{j=1}^{\ell} \theta_{x_j} \approx \prod_{j=1}^{\ell} \hat{\theta}_{x_j}$$

Maximum likelihood estimate:

$$\forall v \in \mathcal{V}, \hat{\theta}_v = \frac{|\{i, j \mid [\boldsymbol{x}_i]_j = v\}|}{N}$$

$$= \frac{c_{\boldsymbol{x}_{1:n}}(v)}{N}$$

where $N = \sum_{i=1}^{n} |\boldsymbol{x}_i|$.
Also known as "relative frequency estimation."

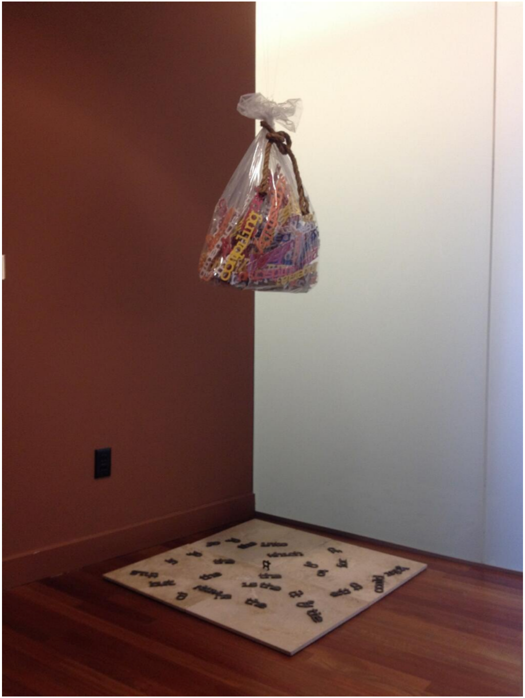
# Responses to Some of Your Questions

I speak roughly 1.3 languages.

Homeworks are mostly programming assignments. They are public, but other than maybe some commentary, solutions won't be public.

Interested in research?

- Faculty doing NLP at UW: `http://nlp.washington.edu`
- Summer internship application form: `https://goo.gl/forms/mwirJD7utUMimVH92`



# Unigram Model

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Maximum likelihood estimate:

$$\forall v \in \mathcal{V}, \hat{\theta}_v = \frac{|\{i, j \mid [\boldsymbol{x}_i]_j = v\}|}{N} = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{N}$$

where $N = \sum_{i=1}^{n} |\boldsymbol{x}_i|$.
Also known as "relative frequency estimation."

# Unigram Models: Assessment

*Pros:*

- Easy to understand
- Cheap
- Good enough for information retrieval (maybe)

*Cons:*

- "Bag of words" assumption is linguistically inaccurate
  - $p(\text{the the the the}) \gg p(\text{I want ice cream})$
- Data sparseness; high variance in the estimator
- "Out of vocabulary" problem

# Markov Models $\equiv$ n-gram Models

$$p(\boldsymbol{X} = \boldsymbol{x}) = \prod_{j=1}^{\ell} p(X_j = x_j \mid \boldsymbol{X}_{1:j-1} = \boldsymbol{x}_{1:j-1})$$

$$\overset{\text{assumption}}{=} \prod_{j=1}^{\ell} p_{\boldsymbol{\theta}}(X_j = x_j \mid \boldsymbol{X}_{j-\mathsf{n}+1:j-1} = \boldsymbol{x}_{j-\mathsf{n}+1:j-1})$$

$(\mathsf{n}-1)$th-order Markov assumption $\equiv$ n-gram model

- Unigram model is the $\mathsf{n} = 1$ case
- For a long time, trigram models ($\mathsf{n} = 3$) were widely used
- 5-gram models ($\mathsf{n} = 5$) are not uncommon now in MT

# Estimating n-Gram Models

| | unigram | bigram | trigram |
|---|---|---|---|
| $p_{\boldsymbol{\theta}}(\boldsymbol{x}) =$ | $\prod_{j=1}^{\ell} \theta_{x_j}$ | $\prod_{j=1}^{\ell} \theta_{x_j \mid x_{j-1}}$ | $\prod_{j=1}^{\ell} \theta_{x_j \mid x_{j-2} x_{j-1}}$ |
| Parameters: | $\theta_v$ <br> $\forall v \in \mathcal{V}$ | $\theta_{v \mid v'}$ <br> $\forall v \in \mathcal{V}, v' \in \mathcal{V} \cup \{\bigcirc\}$ | $\theta_{v \mid v'' v'}$ <br> $\forall v \in \mathcal{V}, v', v'' \in \mathcal{V} \cup \{\bigcirc\}$ |
| MLE: | $\frac{c(v)}{N}$ | $\frac{c(v'v)}{c(v')}$ | $\frac{c(v''v'v)}{c(v''v')}$ |

General case:

$$\prod_{j=1}^{\ell} \theta_{x_j \mid \boldsymbol{x}_{j-\mathsf{n}+1:j-1}} \qquad \theta_{v \mid \boldsymbol{h}}, \ \forall v \in \mathcal{V}, \boldsymbol{h} \in (\mathcal{V} \cup \{\bigcirc\})^{\mathsf{n}-1} \qquad \frac{c(\boldsymbol{h}v)}{c(\boldsymbol{h})}$$

# The Problem with MLE

- The curse of dimensionality: the number of parameters grows exponentially in n
- Data sparseness: most n-grams will never be observed, even if they are linguistically plausible
- No one actually uses the MLE!

# Smoothing

A few years ago, I'd have spent a whole lecture on this! ☹

- Simple method: add $\lambda > 0$ to every count (including zero-counts) before normalizing
- What makes it hard: ensuring that each $\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}$
  - Otherwise, perplexity calculations break
- Longstanding champion: modified Kneser-Ney smoothing (Chen and Goodman, 1998)
- Stupid backoff: reasonable, easy solution when you don't care about perplexity (Brants et al., 2007)

# Interpolation

If $p$ and $q$ are both language models, then so is

$$\alpha p + (1 - \alpha) q$$

for any $\alpha \in [0, 1]$.

- This idea underlies many smoothing methods
- Often a new model $q$ only beats a reigning champion $p$ when interpolated with it
- How to pick the "hyperparameter" $\alpha$?

# Algorithms To Know

- Score a sentence $\boldsymbol{x}$
- Train from a corpus $\boldsymbol{x}_{1:n}$
- Sample a sentence given $\boldsymbol{\theta}$

# n-gram Models: Assessment

*Pros:*

- Easy to understand
- Cheap (with modern hardware; Lin and Dyer, 2010)
- Good enough for machine translation, speech recognition, . . .

*Cons:*

- Markov assumption is linguistically inaccurate
  - (But not as bad as unigram models!)
- Data sparseness; high variance in the estimator
- "Out of vocabulary" problem

# Dealing with Out-of-Vocabulary Terms

- Define a special OOV or "unknown" symbol UNK. Transform some (or all) rare words in the training data to UNK.
  - ☹ You cannot fairly compare two language models that apply different UNK treatments!
- Build a language model at the *character* level.

# To-Do List

- Collins (2011); Jurafsky and Martin (2016)

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Extras

# Relative Frequency Estimation is the MLE

(Unigram Model)

The maximum likelihood estimation problem:

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n})$$

# Relative Frequency Estimation is the MLE

(Unigram Model)

Logarithm is a monotonic function.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n}) = \exp \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n})$$

# Relative Frequency Estimation is the MLE

(Unigram Model)

Each sequence is an independent sample from the model.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log p_{\boldsymbol{\theta}}(\boldsymbol{x}_{1:n}) = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\boldsymbol{x}_i)$$

# Relative Frequency Estimation is the MLE

(Unigram Model)

Plug in the form of the unigram model.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(\boldsymbol{x}_i) = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} \prod_{j=1}^{\ell_i} \theta_{[\boldsymbol{x}_i]_j}$$

# Relative Frequency Estimation is the MLE
## (Unigram Model)

Log of product equals sum of logs.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \log \prod_{i=1}^{n} \prod_{j=1}^{\ell_i} \theta_{[\boldsymbol{x}_i]_j} = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \log \theta_{[\boldsymbol{x}_i]_j}$$

# Relative Frequency Estimation is the MLE
(Unigram Model)

Convert from tokens to types.

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{i=1}^{n} \sum_{j=1}^{\ell_i} \log \theta_{[\boldsymbol{x}_i]_j} = \max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v$$

# Relative Frequency Estimation is the MLE

(Unigram Model)

Convert to a minimization problem (for consistency with textbooks).

$$\max_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v = \min_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v$$

# Relative Frequency Estimation is the MLE
## (Unigram Model)

Lagrange multiplier to convert to a less constrained problem.

$$
\begin{aligned}
& \min_{\boldsymbol{\theta} \in \triangle^{|\mathcal{V}|}} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v \\
= & \max_{\mu \geq 0} \min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left(1 - \sum_{v \in \mathcal{V}} \theta_v\right) \\
= & \min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left(1 - \sum_{v \in \mathcal{V}} \theta_v\right)
\end{aligned}
$$

Intuitively, if $\sum_{v \in \mathcal{V}} \theta_v$ gets too big, $\mu$ will push toward $+\infty$.

For more about Lagrange multipliers, see Dan Klein's tutorial (reference at the end of these slides).

# Relative Frequency Estimation is the MLE

(Unigram Model)

Use first-order conditions to solve for $\boldsymbol{\theta}$ in terms of $\mu$.

$$\min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left( 1 - \sum_{v \in \mathcal{V}} \theta_v \right)$$

$$\text{fixing } \mu \text{, for all } v \text{, set: } 0 = \frac{\partial}{\partial \theta_v}$$

$$= \frac{-c_{\boldsymbol{x}_{1:n}}(v)}{\theta_v} + \mu$$

$$\theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}$$

# Relative Frequency Estimation is the MLE

## (Unigram Model)

Plug in for each $\theta_v$.

$$\min_{\boldsymbol{\theta} \in \mathbb{R}_{\geq 0}^{|\mathcal{V}|}} \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \theta_v - \mu \left( 1 - \sum_{v \in \mathcal{V}} \theta_v \right)$$

$$= \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} - \mu \left( 1 - \sum_{v \in \mathcal{V}} \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} \right)$$

Remember: $\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}}$

# Relative Frequency Estimation is the MLE

(Unigram Model)

Rearrange terms ($a \log \frac{a}{b} = a \log a - a \log b$ and $N = \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v)$).

$$\max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} - \mu \left( 1 - \sum_{v \in \mathcal{V}} \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu} \right)$$
$$= \max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log \mu - \mu + N$$

Remember: $\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}}$

# Relative Frequency Estimation is the MLE

## (Unigram Model)

Use first-order conditions to solve for $\mu$.

$$\max_{\mu \geq 0} - \sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log \mu - \mu + N$$

$$\begin{aligned} \text{set: } 0 &= \frac{\partial}{\partial \mu} \\ &= \frac{N}{\mu} - 1 \\ \mu &= N \end{aligned}$$

Remember: $\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}}$

# Relative Frequency Estimation is the MLE

(Unigram Model)

Plug in for $\mu$.

$$\max_{\mu \geq 0} -\sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log \mu - \mu + N$$

$$= -\sum_{v \in \mathcal{V}} c_{\boldsymbol{x}_{1:n}}(v) \log c_{\boldsymbol{x}_{1:n}}(v) + N \log N$$

$$\boxed{\forall v \in \mathcal{V}, \theta_v = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{\mu}} = \frac{c_{\boldsymbol{x}_{1:n}}(v)}{N}$$

... and that's the relative frequency estimate!

# Language Models as (Weighted) Finite-State Automata

(Deterministic) finite-state automaton:

- Set of $k$ states $\mathcal{S}$
  - Initial state $s_0 \in \mathcal{S}$
  - Final states $\mathcal{F} \subseteq \mathcal{S}$
- Alphabet $\Sigma$
- Transitions $\delta : \mathcal{S} \times \Sigma \rightarrow \mathcal{S}$

A length $\ell$ string $\boldsymbol{x}$ is in the language of the automaton iff there is a path $\langle s_0, \ldots, s_\ell \rangle$ such that $s_\ell \in \mathcal{F}$ and

$$\bigwedge_{i=1}^{\ell} [[s_i = \delta(s_{i-1}, x_i)]]$$

# Language Models as (Weighted) Finite-State Automata

(Deterministic) finite-state automaton:

- Set of $k$ states $\mathcal{S}$ — histories
  - Initial state $s_0 \in \mathcal{S}$ — ◯
  - Final states $\mathcal{F} \subseteq \mathcal{S}$ — histories ending in ⯃
- Alphabet $\Sigma$ — $\mathcal{V}$
- Transitions $\delta : \mathcal{S} \times \Sigma \rightarrow \mathcal{S} \quad \times \mathbb{R}_{>0}$

  A **weighted** FSA defines a weight for every transition; e.g., $w(\boldsymbol{h}, v, \delta(\boldsymbol{h}, v)) = \theta_{v|\boldsymbol{h}}$

A length $\ell$ string $\boldsymbol{x}$ is in the language of the automaton iff there is a path $\langle s_0, \ldots, s_\ell \rangle$ such that $s_\ell \in \mathcal{F}$ and

$$\bigwedge_{i=1}^{\ell} [[s_i = \delta(s_{i-1}, x_i)]]$$

The score of the string is the product of transition weights.

$$score(\boldsymbol{x}) \prod_{i=1}^{\ell} w(\boldsymbol{h}_i, x_i, \delta(\boldsymbol{h}_i, x_i))$$

# Class-Based Language Models

Brown et al. (1992)

Suppose we have a hard clustering of $\mathcal{V}$, $\mathsf{cl} : \mathcal{V} \rightarrow \{1, \ldots, k\}$, where $k \ll |\mathcal{V}|$.

| | n-gram | class-based | |
|---|---|---|---|
| $p_{\boldsymbol{\theta}}(\boldsymbol{x}) =$ | $\prod_{j=1}^{\ell} \theta_{x_j \mid \boldsymbol{x}_{j-\mathsf{n}+1:j-1}}$ | $\prod_{j=1}^{\ell} \theta_{x_j \mid \mathsf{cl}(x_j)} \gamma_{\mathsf{cl}(x_j) \mid \mathsf{cl}(x_{j-1})}$ | |
| Parameters: | $\theta_{v \mid \boldsymbol{h}}$ <br> $\forall v \in \mathcal{V}, \boldsymbol{h} \in (\mathcal{V} \cup \{\bigcirc\})^{\mathsf{n}-1}$ | $\theta_{v \mid \mathsf{cl}(v)}$ <br> $\forall v \in \mathcal{V}$ | $\gamma_{i \mid j}$ <br> $\forall i, j \in \{1, \ldots, k\}$ |
| MLE: | $\frac{c(\boldsymbol{h}v)}{c(\boldsymbol{h})}$ | $\frac{c(v)}{c(\mathsf{cl}(v))}$ | $\frac{c(j)}{c(ji)}$ |