Lyapunov Stability

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TAs: Matthew Rockett, Gilwoo Lee, Matt Schmittle

Recap: PID / Pure Pursuit control

	Pros	Cons
PID Control		
Pure Pursuit		

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Can we get some control law that has formal guarantees?

Table of Controllers

	Control Law	Uses model	Stability Guarantee
PID	$u = K_p e + ...$	No	No
Pure Pursuit	$u = \tan^{-1}\left(\frac{2B\sin\alpha}{L}\right)$	Circular arcs	Yes - with assumptions

Stability:

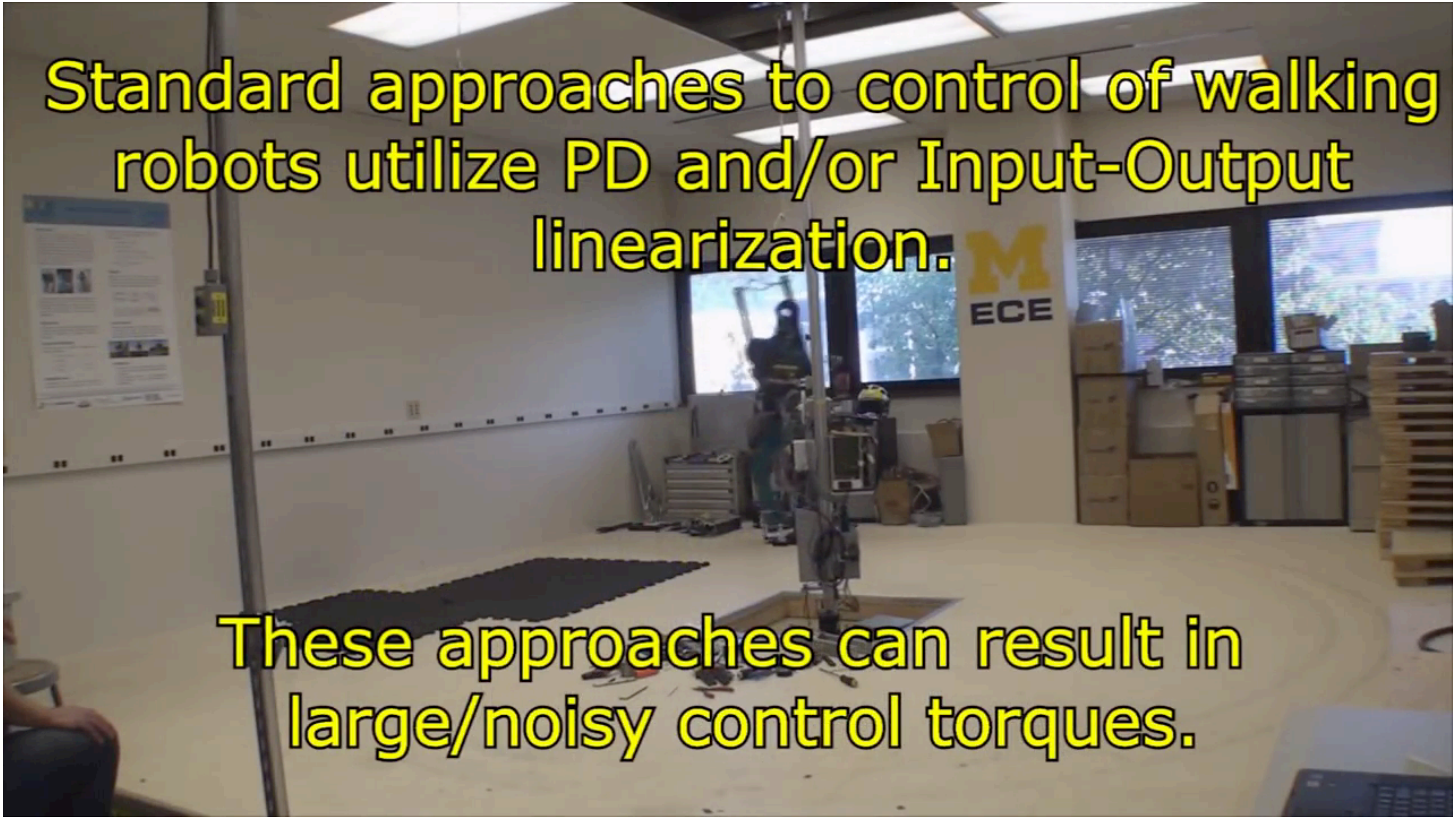
Prove error goes to zero
and stays there

Today's lecture

1. Motivate why underactuated systems are hard to stabilize

2. Lyapunov functions as a tool for stability

Lyapunov control in action



"Rapidly Exponentially Stabilizing Control Lyapunov Functions and Hybrid Zero Dynamics", Ames et al. 2012

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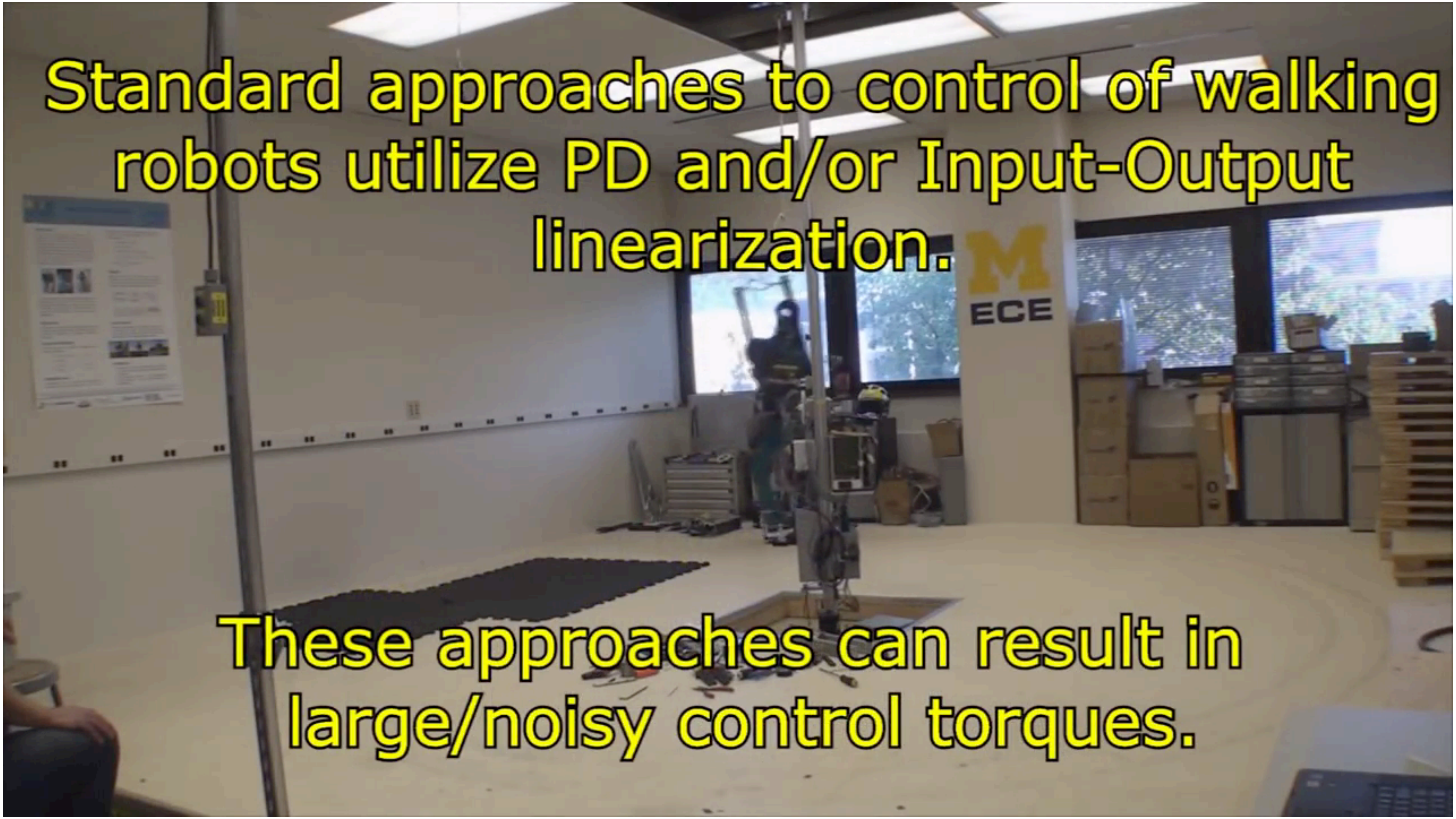

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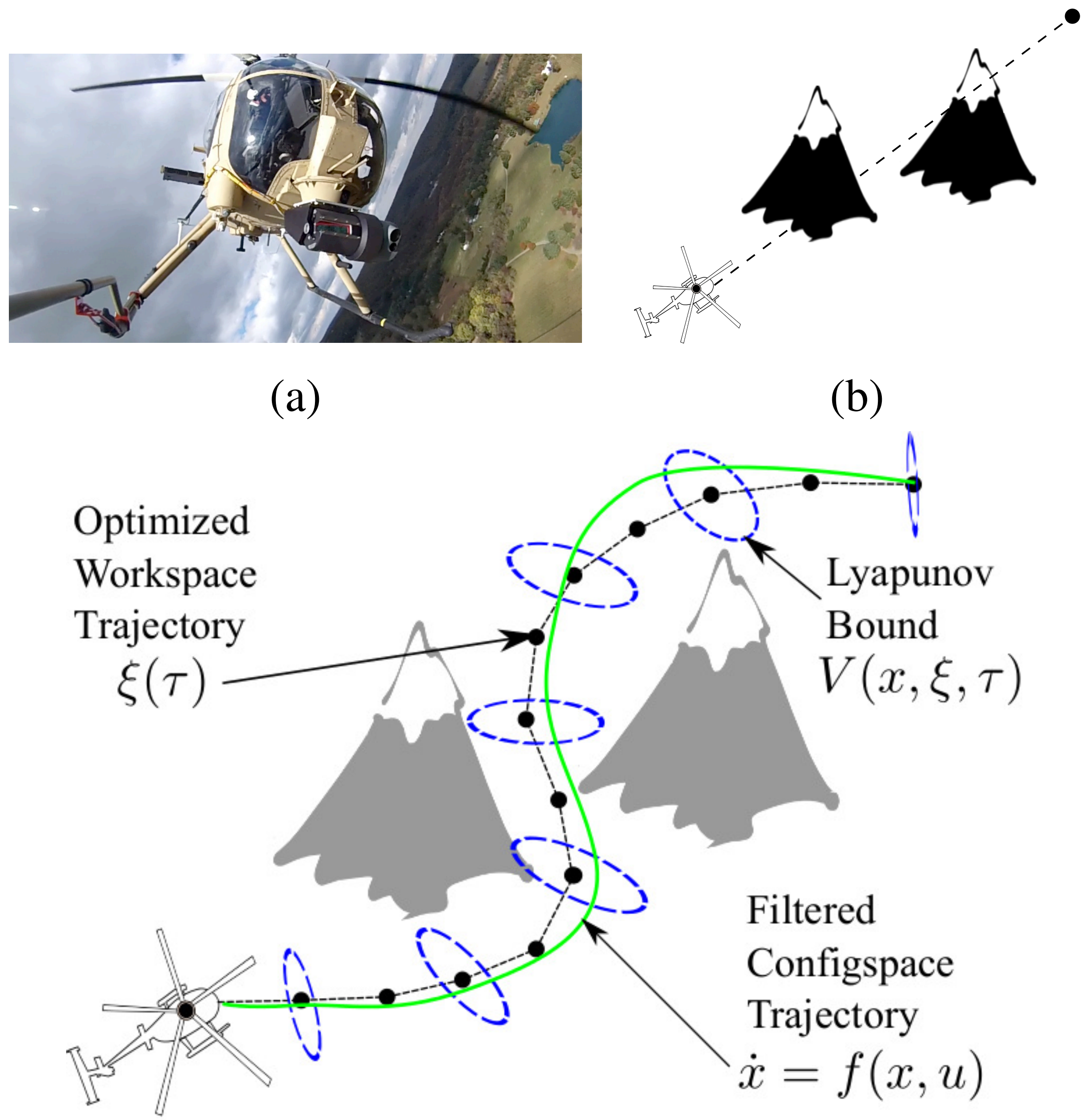


(a) (b)

(c)

"The Dynamics Projection Filter (DPF) - Real-Time Nonlinear Trajectory Optimization Using Projection Operators" Choudhury et al. 2015

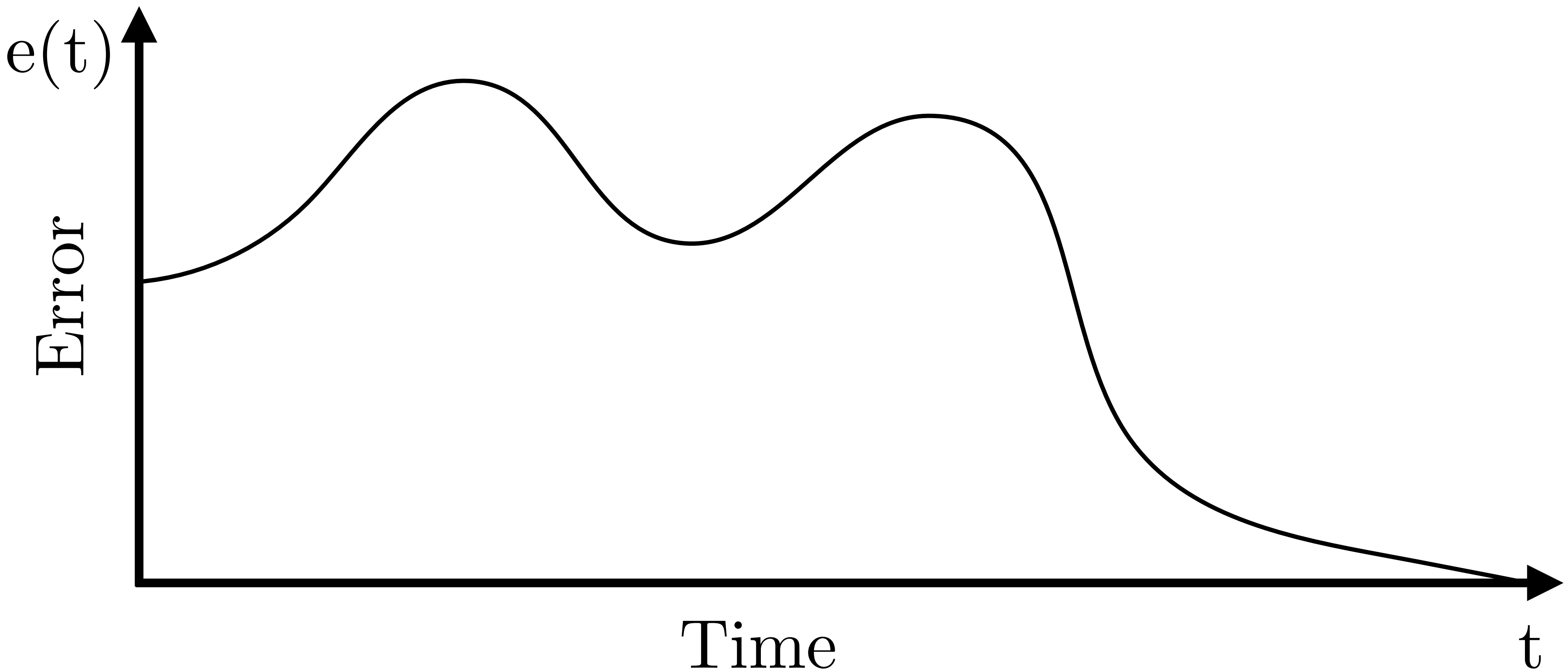
Stability:

Prove error goes to zero

and stays there

What is stability?

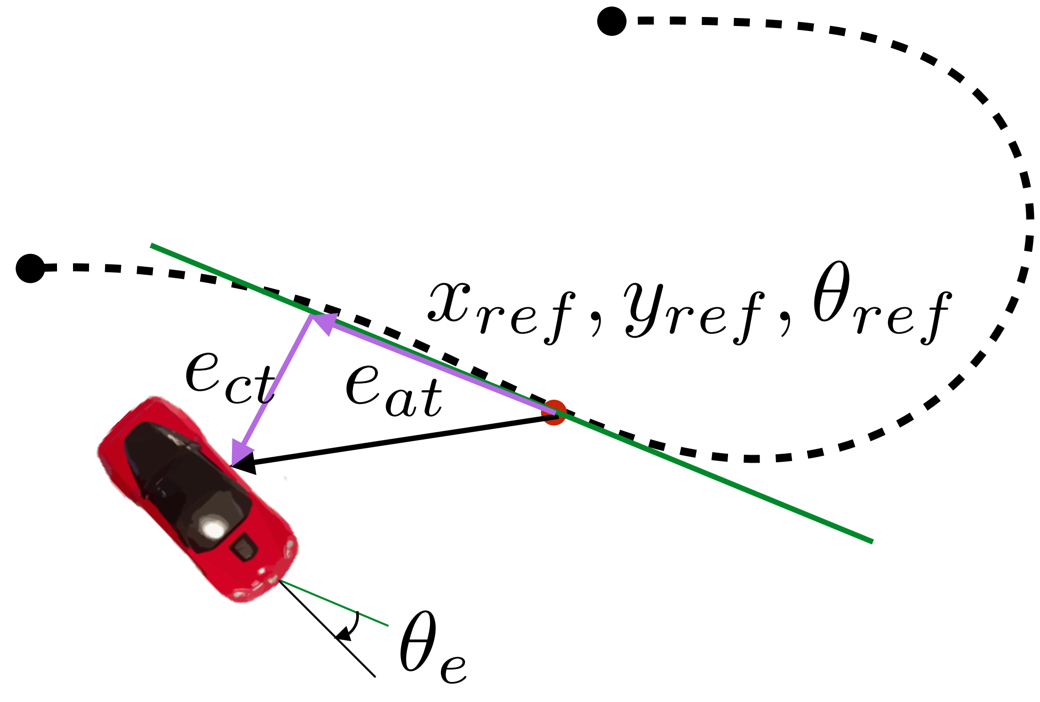
$$\lim_{t \to \infty} e(t) = 0$$



So we want both $e(t) \to 0$ and $\dot{e}(t) \to 0$

Question: Why does the error oscillate?

How does error evolve in time?



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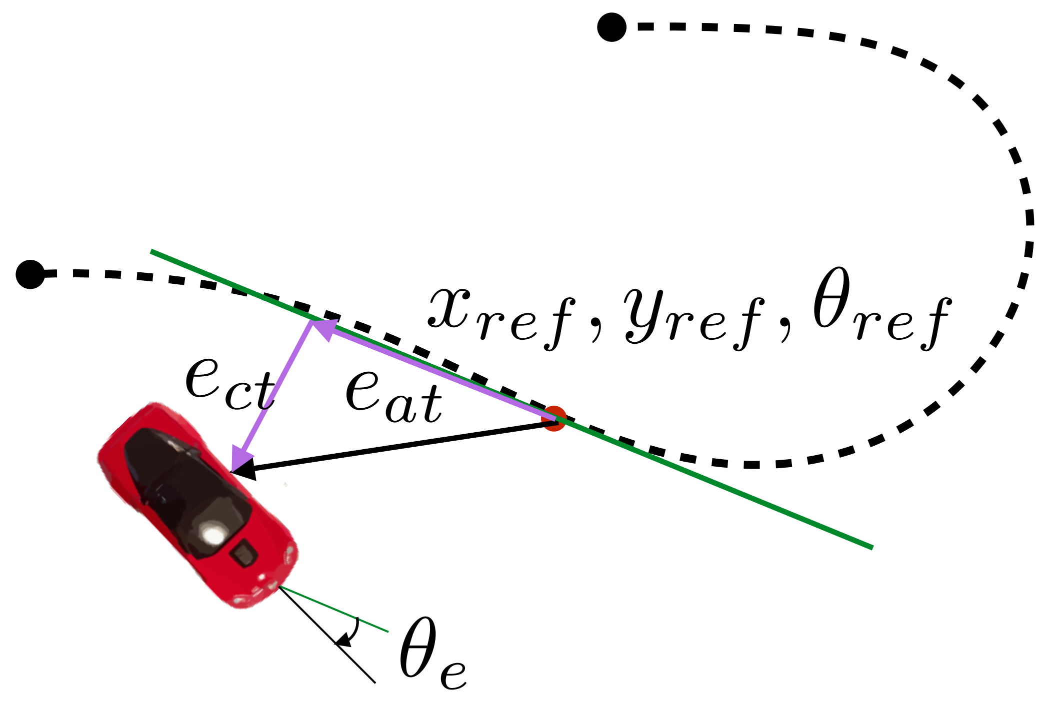

Let's say we were interested in driving both e_{ct} and θ_e to zero

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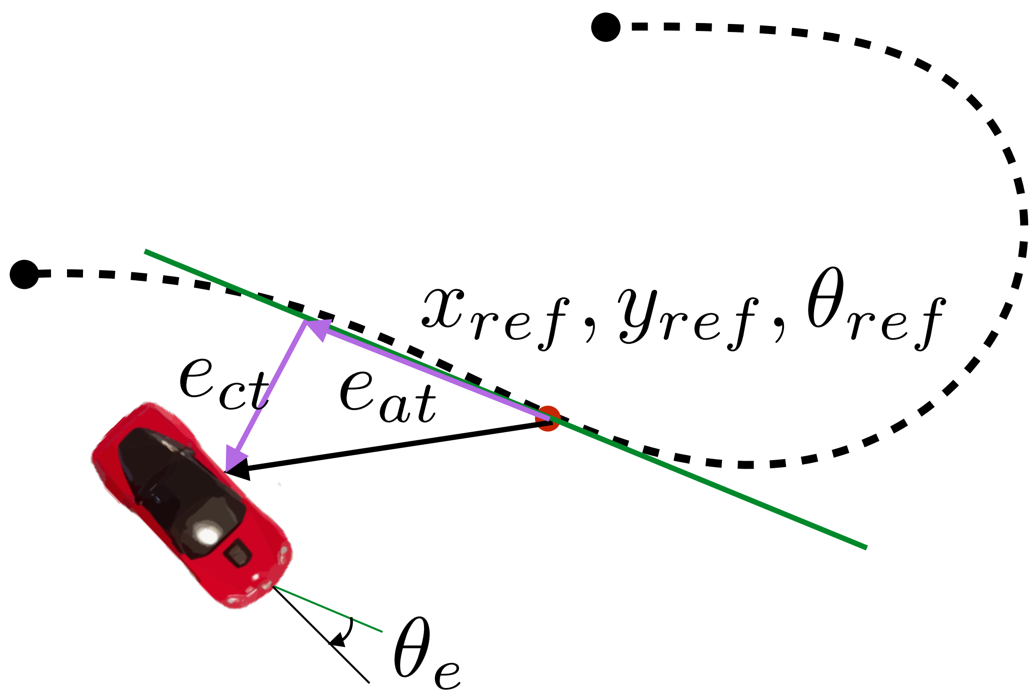

Let's say we were interested in driving both e_{ct} and θ_e to zero

$$e_{ct} = -\sin(\theta_{ref})(x - x_{ref}) + \cos(\theta_{ref})(y - y_{ref}) \qquad \theta_e = \theta - \theta_{ref}$$

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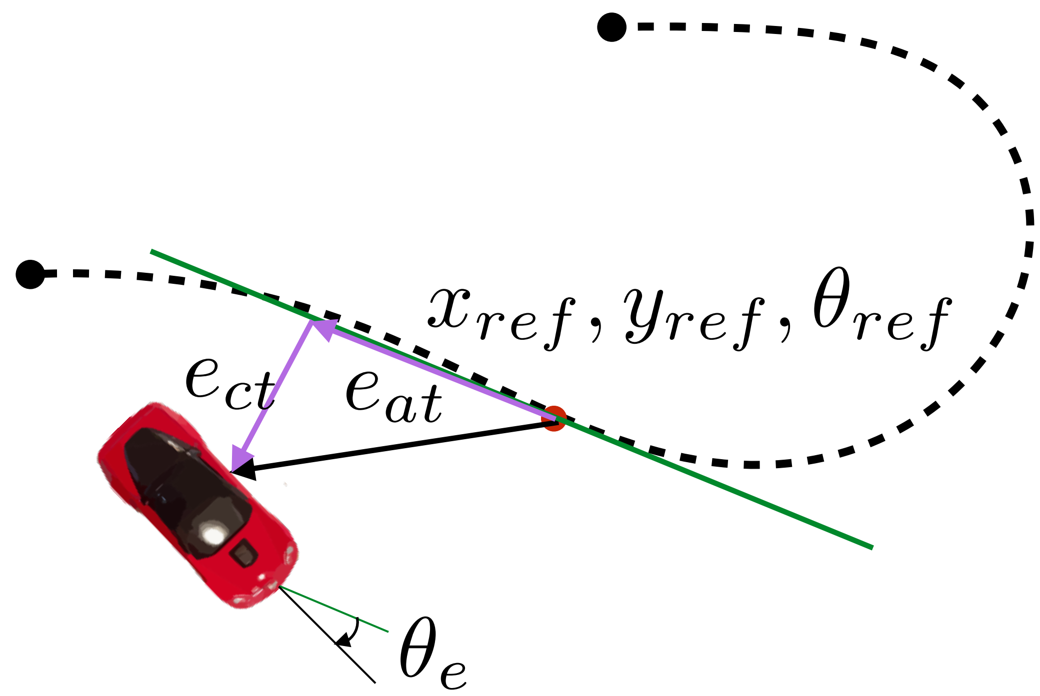

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Notice how our control variable affects all the error terms

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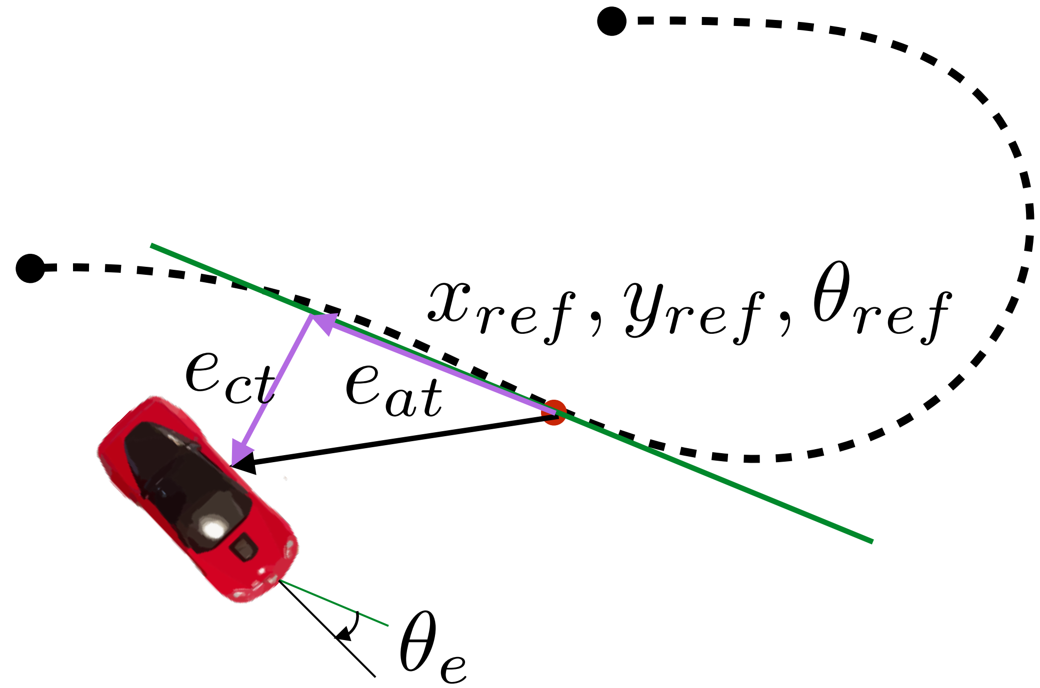

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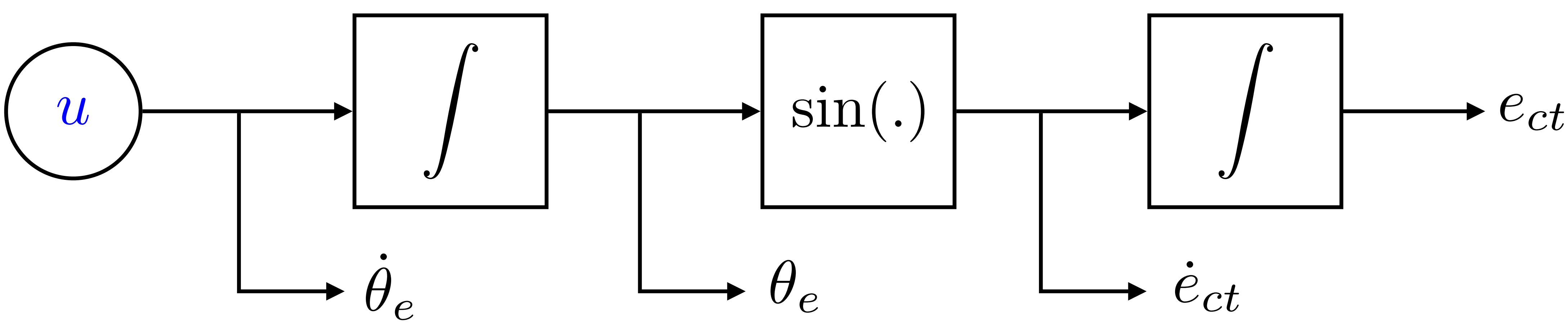

Why is this tricky?

Is it because of non-linearity?

Easy problem: Stability in a linear system

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Stability guaranteed when $k > \dfrac{a}{b}$

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Stability guaranteed when $k > 0 \quad g(x) \neq 0$

Why is this tricky?

Is it because of non-linearity?

BUSTED

Because of underactuated dynamics...

Fundamental problem with underactuated systems

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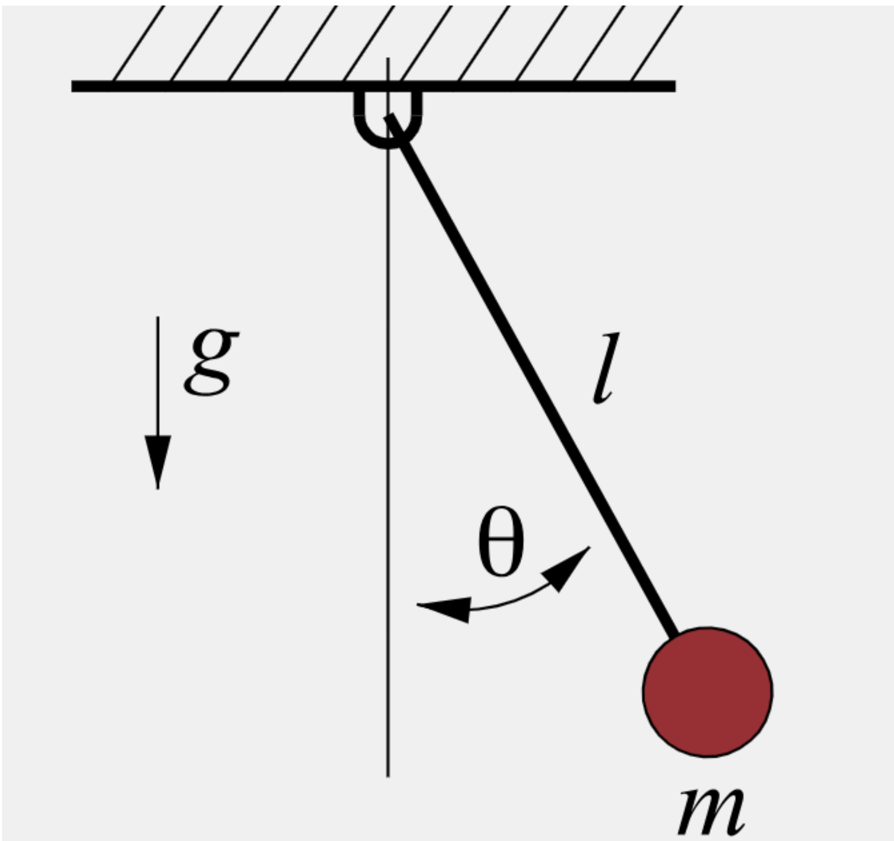
Detour: How do we make a pendulum stable?



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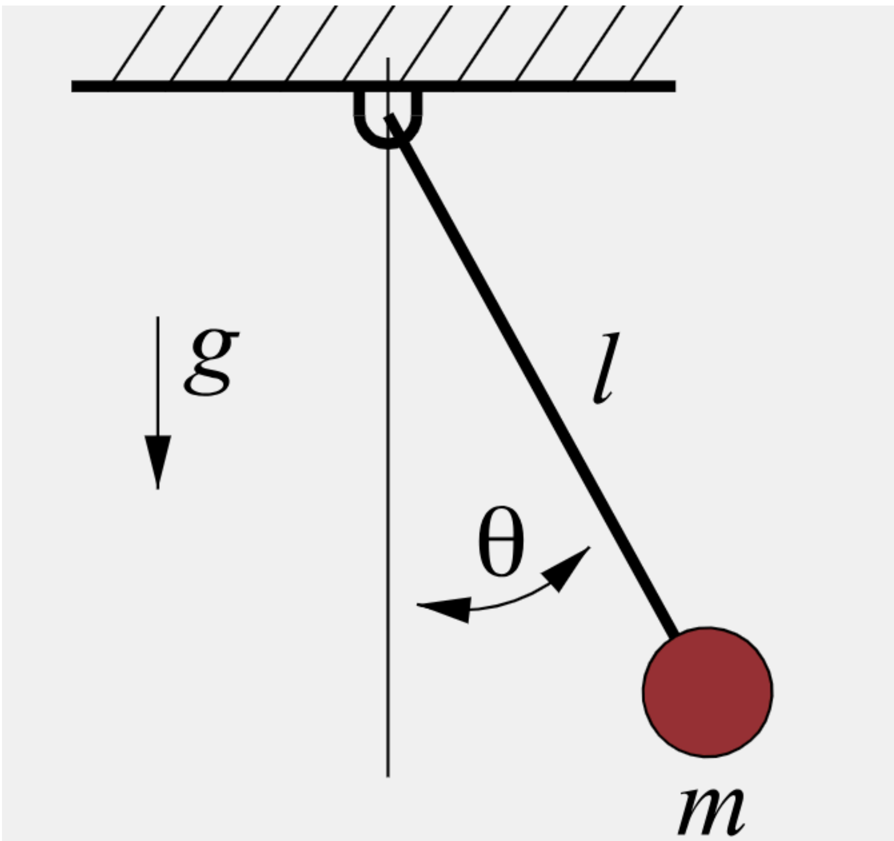

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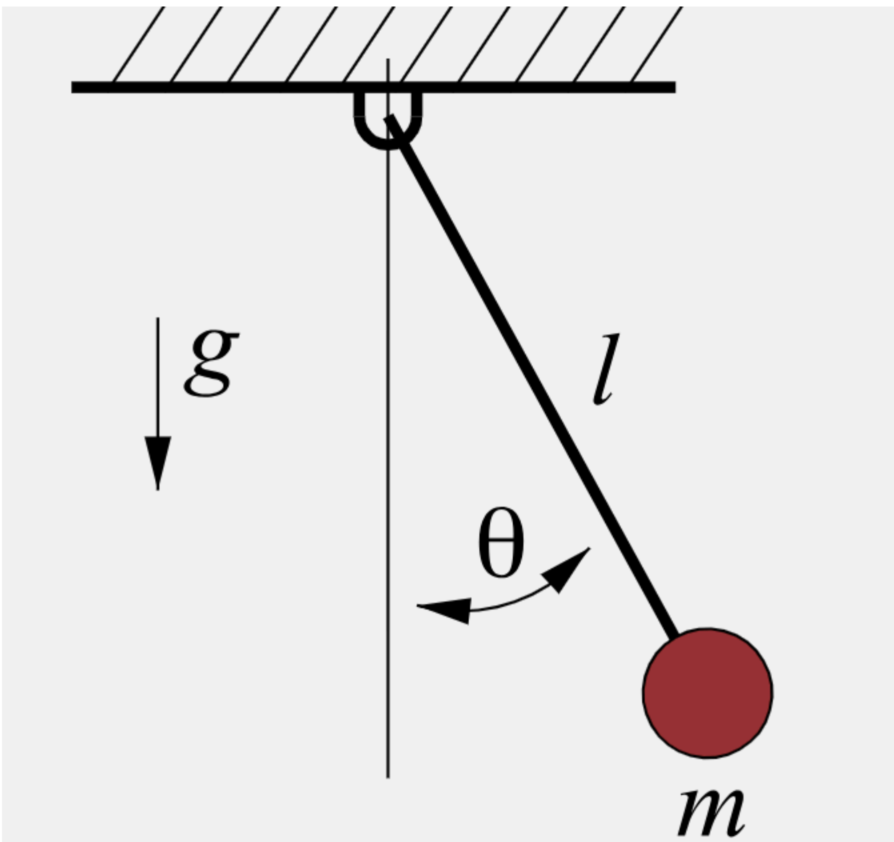

$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

What control law should we use to stabilize the pendulum, i.e.

Choose $u = \pi(\theta, \dot{\theta})$ such that $\theta \to 0$

$\dot{\theta} \to 0$

How does the passive error dynamics behave?

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$$e_1 = \theta - 0 = \theta \qquad\qquad e_2 = \dot{\theta} - 0 = \dot{\theta}$$

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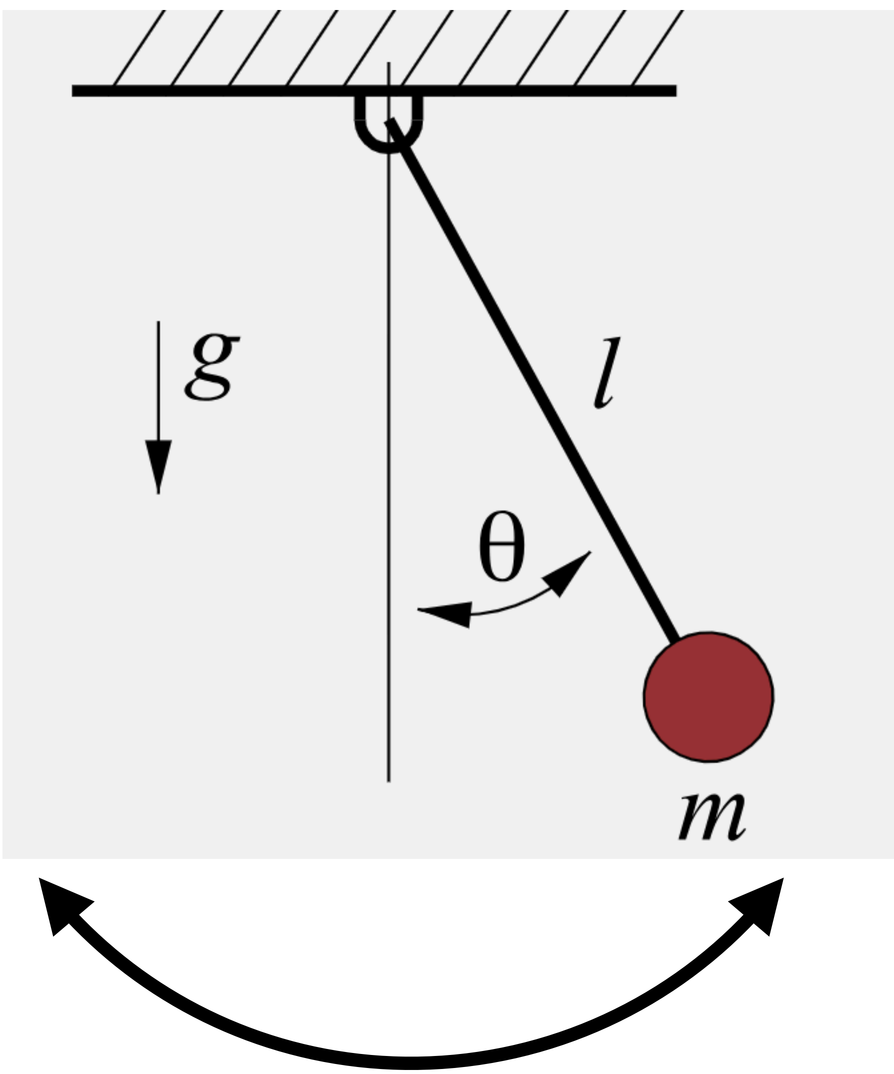
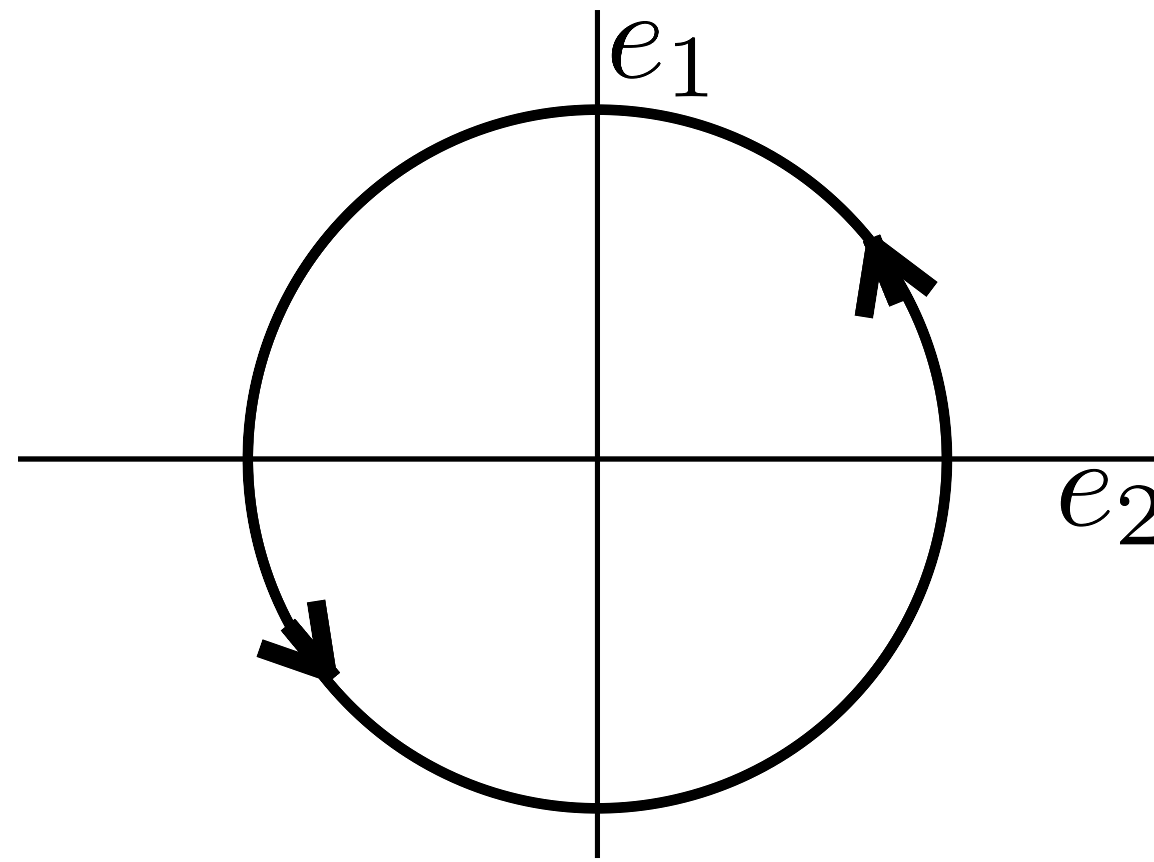
Set u=0. Dynamics is not stable.





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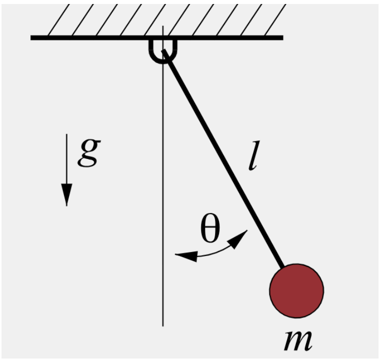

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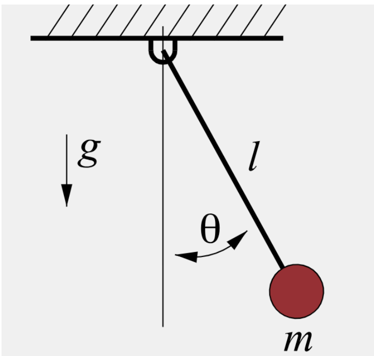

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Lets pick the following law:

$$u = -K\dot{\theta}$$

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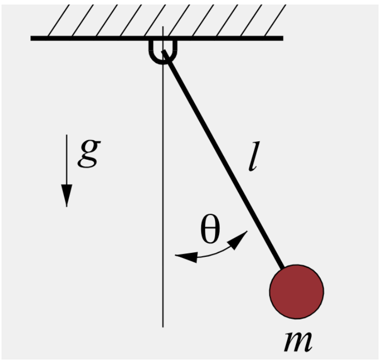

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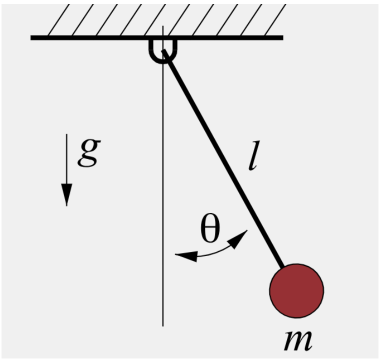

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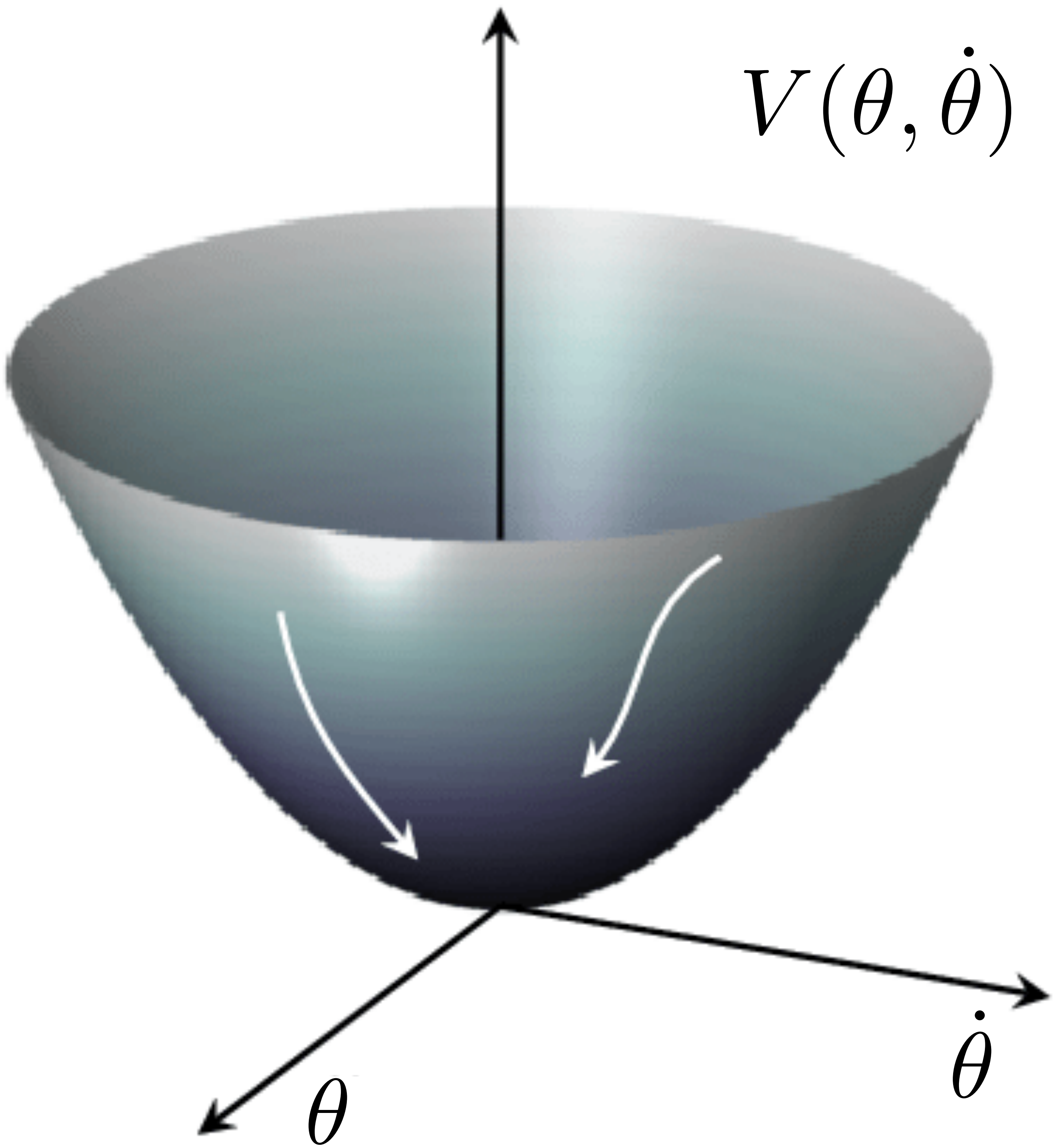
Is this stable? How do we know?

We can simulate the dynamics from different start point and check....

but how many points do we check? what if we miss some points?

Key Idea: Think about energy!



Make energy decay to 0 and stay there

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$$V(\theta, \dot{\theta}) = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

$$> 0$$

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$$\dot{V}(\theta, \dot{\theta}) = m l^2 \dot{\theta}\ddot{\theta} + mgl(\sin\theta)\dot{\theta}$$
$$= \dot{\theta}(u - mgl\sin\theta) + mgl(\sin\theta)\dot{\theta}$$
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Choose a control law $u = -k\dot{\theta}$

$$\dot{V}(\theta, \dot{\theta}) = -k\dot{\theta}^2 < 0$$

Lyapunov function:

A generalization of energy

Lyapunov function for a closed-loop system

1. Construct an energy function that is always positive

$$V(x) > 0, \forall x$$

Energy is only 0 at the origin, i.e. $V(0) = 0$

2. Choose a control law such that this energy always decreases

$$\dot{V}(x) < 0, \forall x$$

Energy rate is 0 at origin, i.e. $\dot{V}(0) = 0$

No matter where you start, energy will decay and you will reach 0!

Let's get provable control for our car!

Dynamics of the car

$$\dot{x} = V \cos\theta$$
$$\dot{y} = V \sin\theta$$
$$\dot{\theta} = \frac{V}{B} \tan u$$

Let's get provable control for our car!

Let's define the following Lyapunov function

$$V(e_{ct}, \theta_e) = \frac{1}{2} k_1 e_{ct}^2 + \frac{1}{2} \theta_e^2 \qquad > 0$$

Compute derivative

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} \dot{e}_{ct} + \theta_e \dot{\theta}_e$$

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin \theta_e + \theta_e \frac{V}{B} \tan u$$

Let's get provable control for our car!

$$\dot{V}(e_{ct}, \theta_e) = k_1 e_{ct} V \sin\theta_e + \theta_e \frac{V}{B} \tan u$$

Trick: Set u intelligently to get this term to always be negative

$$\theta_e \frac{V}{B} \tan u = -k_1 e_{ct} V \sin\theta_e - k_2 \theta_e^2$$

$$\tan u = -\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e$$

$$u = \tan^{-1}\left(-\frac{k_1 e_{ct} B}{\theta_e} \sin\theta_e - \frac{B}{V} k_2 \theta_e\right)$$

(Advanced Reading)

Bank-to-Turn Control for a Small UAV using Backstepping and Parameter Adaptation

Dongwon Jung and Panagiotis Tsiotras

Overcoming simple assumptions

1. Reference point selection logic does not depend on error

2. Feedforward not taken into account

3. More sophisticated heading error

4. How can we handle steering rate, acceleration, jerk, snap constraints?