Linear Algebra Background

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The following is a list of facts from linear algebra that are assumed to be familiar.

- A Hermitian matrix $H$ (i.e., one satisfying $H^\dagger = H$) has real eigenvalues.
  - If the eigenvector decomposition of $H = V \text{ diag}(w_1, \ldots, w_n)V^\dagger$, then we can see that $H^\dagger = V \text{ diag}(w_1, \ldots, w_n)V^\dagger = V \text{ diag}(\overline{w_1}, \ldots, \overline{w_n})V^\dagger$, so the equation $H^\dagger = H$ tells us that $\overline{w_i} = w_i$, ..., $\overline{w_n} = w_n$, which is possibly only if each $w_i$ is real.

- A regular matrix with real eigenvalues is Hermitian.
  - If the eigenvector decomposition of $H = V \text{ diag}(w_1, \ldots, w_n)V^\dagger$ with each $w_i$ real, then $H^\dagger = V \text{ diag}(\overline{w_1}, \ldots, \overline{w_n})V^\dagger = V \text{ diag}(w_1, \ldots, w_n)V^\dagger = H$.

In summary, the following are equivalent for a regular matrix $H$:

1. $H^\dagger = H$
2. Every eigenvalue $H$ is real.

- A unitary matrix $U$ (i.e., one satisfying $U^\dagger U = I$) does not change the (2-)norm of any vector when multiplied against it.
  - If $x$ is a vector, then $\|Ux\|^2 = (Ux)^\dagger (Ux) = x^\dagger U^\dagger Ux = x^\dagger x = \|x\|^2$.
  - The reverse is also true...

- If a regular matrix $U$ does not change the norm of any vector, then each of its eigenvalues $w_i$ must satisfy $|w_i| = 1$. I.e., it is a number of the form $e^{i\theta}$ for some $\theta \in \mathbb{R}$.
  - Let $v$ be an eigenvector of $U$ with corresponding to eigenvalue $w$. Then $Uv = wv$, so $(Uv)^\dagger (Uv) = v^\dagger w^\dagger wv = |w|^2 \|v\|^2$. Since multiplying $v$ by $U$ does not change its norm, we must have $|w| = 1$.

- If each of $U$’s eigenvalues has absolute value 1, then $U$ is unitary.
  - If $U$ has eigenvector decomposition $U = V \text{ diag}(w_1, \ldots, w_n)V^\dagger$, then by the previous fact, we must have $|w_i| = 1$ for each $i$. But this means that $U^\dagger U = (V \text{ diag}(w_1, \ldots, w_n)V^\dagger)(V \text{ diag}(w_1, \ldots, w_n)V^\dagger)$

    $= V \text{ diag}(w_1, \ldots, w_n)V^\dagger V \text{ diag}(w_1, \ldots, w_n)V^\dagger$

    $= V \text{ diag}(w_1, \ldots, w_n)^\dagger \text{ diag}(w_1, \ldots, w_n)V^\dagger$

    $= V \text{ diag}(|w_1|^2, \ldots, |w_n|^2) V^\dagger$

    $= V \text{ diag}(1, \ldots, 1)V^\dagger$

    $= VIV^\dagger = VV^\dagger = I$

In summary, the following are equivalent for a regular matrix $U$:

1. $U^\dagger U = I$
2. Every eigenvalue $w$ of $U$ satisfies $|w| = 1$.
3. $\|Ux\| = \|x\|$ for every vector $x$.

- The product of unitary matrices is unitary.
  - If $U$ and $V$ are unitary, then $U^\dagger U = I$ and $V^\dagger V = I$, so we have $(UV)^\dagger (UV) = V^\dagger U^\dagger UV = V^\dagger IV = V^\dagger V = I$. 
