Linear Algebra Background

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The following is a list of facts from linear algebra that are assumed to be familiar.

- A Hermitian matrix H (i.e., one satisfying $H^\dagger = H$) has real eigenvalues.
  - If the eigenvector decomposition of $H = V \operatorname{diag}(w_1, \dots, w_n) V^\dagger$, then we can see that $H^\dagger = V \operatorname{diag}(w_1, \dots, w_n)^\dagger V^\dagger = V \operatorname{diag}(\overline{w_1}, \dots, \overline{w_n}) V^\dagger$, so the equation $H^\dagger = H$ tells us that $\overline{w_1} = w_1$, …, $\overline{w_n} = w_n$, which is possibly only if each w_i is real.
- A regular matrix with real eigenvalues is Hermitian.
  - If the eigenvector decomposition of $H = V \operatorname{diag}(w_1, \dots, w_n) V^\dagger$ with each w_i real, then $H^\dagger = V \operatorname{diag}(\overline{w_1}, \dots, \overline{w_n}) V^\dagger = V \operatorname{diag}(w_1, \dots, w_n) V^\dagger = H$.
- In summary, the following are equivalent for a regular matrix H:
  1. $H^\dagger = H$
  2. Every eigenvalue H is real.
- A unitary matrix U (i.e., one satisfying $U^\dagger U = I$) does not change the (2-)norm of any vector when multiplied against it.
  - If $\mathbf{x}$ is a vector, then $\|U\mathbf{x}\|^2 = (U\mathbf{x})^\dagger (U\mathbf{x}) = \mathbf{x}^\dagger U^\dagger U \mathbf{x} = \mathbf{x}^\dagger \mathbf{x} = \|\mathbf{x}\|^2$.
  - The reverse is also true.…
- If a regular matrix U does not change the norm of any vector, then each of its eigenvalues w must satisfy $|w| = 1$. I.e., it is a number of the form $e^{i\theta}$ for some $\theta \in \mathbb{R}$.
  - Let $\mathbf{v}$ be an eigenvector of U with corresponding to eigenvalue w. Then $U\mathbf{v} = w\mathbf{v}$, so $(U\mathbf{v})^\dagger (U\mathbf{v}) = \mathbf{v}^\dagger w^\dagger w \mathbf{v} = |w|^2 \|\mathbf{v}\|^2$. Since multiplying $\mathbf{v}$ by U does not change its norm, we must have $|w| = 1$.
- If each of U's eigenvalues has absolute value 1, then U is unitary.
  - If U has eigenvector decomposition $U = V \operatorname{diag}(w_1, \dots, w_n) V^\dagger$, then by the previous fact, we must have $|w_i| = 1$ for each i. But this means that $U^\dagger U =$

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\begin{aligned}
&(V \operatorname{diag}(w_1, \dots, w_n) V^\dagger)^\dagger (V \operatorname{diag}(w_1, \dots, w_n) V^\dagger) \\
&= V \operatorname{diag}(w_1, \dots, w_n)^\dagger V^\dagger V \operatorname{diag}(w_1, \dots, w_n) V^\dagger \\
&= V \operatorname{diag}(w_1, \dots, w_n)^\dagger \operatorname{diag}(w_1, \dots, w_n) V^\dagger \\
&= V \operatorname{diag}(|w_1|^2, \dots, |w_n|^2) V^\dagger \\
&= V \operatorname{diag}(1, \dots, 1) V^\dagger \\
&= V I V^\dagger = V V^\dagger = I
\end{aligned}
$$

- In summary, the following are equivalent for a regular matrix U:
  1. $U^\dagger U = I$
  2. Every eigenvalue w of U satisfies $|w| = 1$.
  3. $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for every vector $\mathbf{x}$.
- The product of unitary matrices is unitary.
  - If U and V are unitary, then $U^\dagger U = I$ and $V^\dagger V = I$, so we have $(UV)^\dagger (UV) = V^\dagger U^\dagger U V = V^\dagger I V = V^\dagger V = I$.