Diagrammatic Reasoning

- Using diagrams to understand quantum processes
- Not the same as circuits
- Ignore more details of computation
  - e.g. cup and cap

- Not simple description of any circuit element, but
- Very strong, useful math properties
- Easy to describe trace of matrix

Example:

\[ \text{cup} \quad \text{cap} \]

Analyzing an oracle

+ takes \(|a, b\rangle\) to \(|a+b\rangle\)

\[ x \quad b \otimes f(x) \]

\[ U_f \]

\[ x \quad b \]

Not unitary

Describes behavior, not how it's computed

\[ |x, f(x) \oplus b\rangle \quad : \text{same result through diagram} \]

\[ |x, f(x), b\rangle \quad \text{with more understandable} \]

\[ |x, x, b\rangle \quad \text{process} \]

\[ I, b \]
Plus operation

"+" has its own special properties including:

Left-hand-side:
\[ (a |10\rangle + b |11\rangle) \otimes (|10\rangle - |11\rangle) \]
\[ = a |00\rangle - a |11\rangle + b |10\rangle - b |11\rangle \]
\[ = a |10\rangle - a |11\rangle + b |11\rangle - b |10\rangle = (a-b) |10\rangle + (-a+b) |11\rangle \]
\[ = (a-b) |10\rangle - (a-b) |11\rangle \]

Right-hand-side:
\[ <l (a |10\rangle + b |11\rangle) |1\rangle \]
\[ \text{inner product} \]
\[ <01 - <11| (a |10\rangle + b |11\rangle) |1\rangle \]
\[ = (a-b) |1\rangle \]

Can describe \( U_{\Phi} \) more nicely:
Deutsch–Jozsa (constant or balanced)

Circuit was:

\[
X - H^\otimes n \rightarrow U_f \rightarrow H^\otimes n - \chi
\]

Analysis with diagrams (separately for constant vs. balanced)

Constant:

\[
f(x)
\]

\[
\begin{array}{c}
E = \uparrow \\
\end{array}
\]

erases input and replaces with \( b \).

\[
\begin{array}{c}
x \\
\end{array}
\]

\[
U_f = U_f = U_f = \chi
\]

\[
\langle -1|b\rangle = \langle 01 - 11|b\rangle = \pm 1 \text{ is global phase, so no effect}
\]

\[
U_f \text{ effectively acts as identity when constant}
\]

Circuit then just \( H^\dagger H = I \), so we always measure \( x \).
Balanced: half 0s, half 1s

We know $U_f$ is applied to $|\psi\rangle$ (equal superposition of all values)

We then have:

$$f \left( \frac{1}{\sqrt{N}} \sum_{z=0}^{2^n-1} |z\rangle \right)$$

$$= \frac{1}{\sqrt{N}} \sum_{z=0}^{2^n-1} f(z) |z\rangle$$

$$= \frac{1}{\sqrt{N}} \left( \frac{N}{2} |0\rangle + \frac{N}{2} |1\rangle \right) = \frac{\sqrt{N}}{2} (|0\rangle + |1\rangle)$$

$$= \frac{\sqrt{N}}{2} |+\rangle$$

Substituting into main diagram:

$|\psi\rangle$ is inner product with renormalized $|\psi\rangle$

Impact to $U_f$ is $|\psi\rangle$ when $x$ is $10^n$, circuit always tries to measure $|\psi\rangle$

Our analysis showed this never succeeds.
Never succeeds with balance

If \( \psi \) input:

Never measure \( O^n \) when \( f \) is balanced.
Always measure \( O^n \) when \( f \) is constant.
Grover with diagrams
- Consider case when only 1 iteration
  Occurs when \( \frac{1}{2} \) of all inputs are 1. (Or if only 2 qubits)
  - Algorithm from before generalizes to any constant fraction of 1s.

Diagram below describes measurement result of \( \ket{s} \):

\[ \frac{1}{\sqrt{N}} \]

\[ \text{Rest is same as } U_f \]

Constant factor difference between prepare and \( \ket{\psi} \):

\[ D = 2 \ket{\psi} \bra{\psi} - I \]

Write diagrammatically:

\[ \begin{pmatrix} 1 & - \frac{2}{N} \\ \frac{2}{N} & 1 \end{pmatrix} \]

(can add/subtract diagrams)
Substituting, we have:

\[
\frac{1}{\sqrt{N}} \left( \frac{1}{N} \sum_{s} \langle s | \psi \rangle \right)^{-} - \frac{2}{N} \frac{\#0s}{N} - \frac{\#1s}{N}
\]

Using \( \langle s | \psi \rangle = \frac{1}{\sqrt{N}} \left( \frac{1}{N} \sum_{s} \langle s | \psi \rangle \right)^{-} \):

First term is \( \langle f(s) | - \rangle = (-1)^f(s) \)

Last term is \( 2 \times \) fraction of Os - 1s.

If \( f \) has \( \frac{1}{4} \) ls, then we get \( 2 \left( \frac{3}{4} - \frac{1}{4} \right) = 1 \)

Value in parentheses is \( -1 \) - 1

This is 0 if \( f(s) = 0 \), so measurement never succeeds!

\[
\begin{bmatrix}
\frac{1}{N} \sum_z f(z) \\
\end{bmatrix}
\]

\[
= \frac{1}{N} \sum_z f(z) \\
= \frac{1}{N} \langle 0 | 0 \rangle + \frac{1}{N} \frac{\#1s}{N}
\]

I.e. we only measure s such that \( f(s) \geq 1 \)
Summary/future work:
- Diagrammatic tools can be useful for understanding quantum processes
- Reveal things hidden in linear algebra
  - Eg teleportation: \( N = 1 \)
- Can make more powerful if go further into math weeds
- Not one notation that works perfectly for all situations

What's missing:
- Only have equivalences
- To go further we need inference