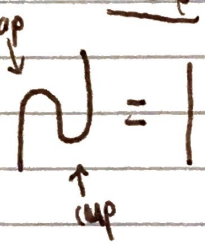
Diagrammatic Reasoning

- Using diagrams to understand quantum processes
  - not same as circuits
  - ignore more details of computation
    - e.g. cup and cap

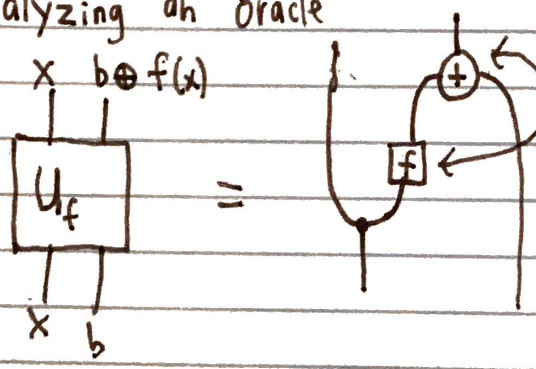
      - Not simple description of any circuit element, but
      - Very strong, useful math properties
      - easy to describe <u>trace</u> of matrix

example:



Analyzing an oracle

$+$ takes $|a, b\rangle$ to $|a+b\rangle$



Not unitary

Describes behavior, not how it's computed

$|x, f(x) \oplus b\rangle$

$|x, f(x), b\rangle$

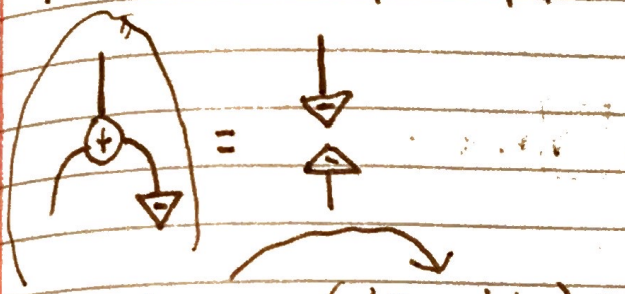
$|x, x, b\rangle$

$|x, b\rangle$

: same result through diagram with more understandable process

Plus operation

"+" has its own special properties including:



Left-hand-side: $(a|0\rangle + b|1\rangle) \otimes (|0\rangle - |1\rangle)$

$= a|00\rangle - a|01\rangle + b|10\rangle - b|11\rangle$

$= a|0\rangle - a|1\rangle + b|1\rangle - b|0\rangle = (a-b)|0\rangle + (-a+b)|1\rangle$

$= (a-b)|0\rangle - (a-b)|1\rangle = (a-b)|-\rangle$

Right-hand-side:

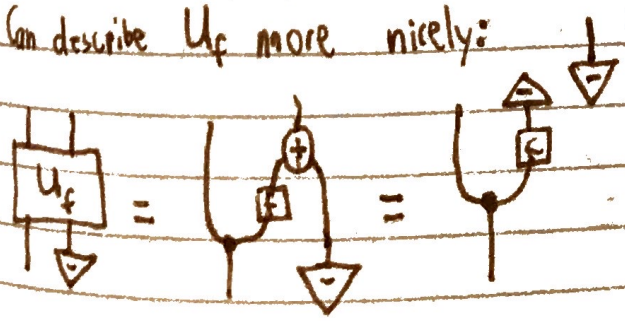
$\underbrace{\langle -|(a|0\rangle + b|1\rangle)}_{\text{inner product}}|-\rangle$

$(\langle 0| - \langle 1|)(a|0\rangle + b|1\rangle)|-\rangle$

$= (a-b)|-\rangle$

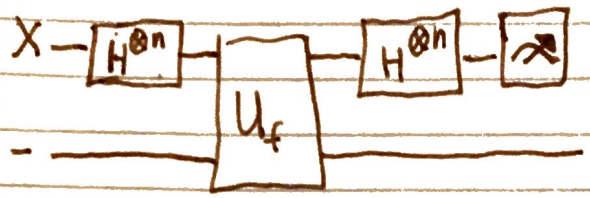
Matches

Can describe U_f more nicely:



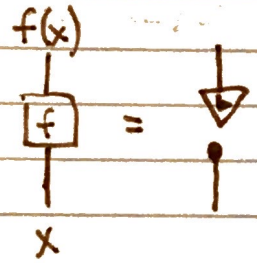
Deutsch-Josza (constant or balanced)

Circuit was:

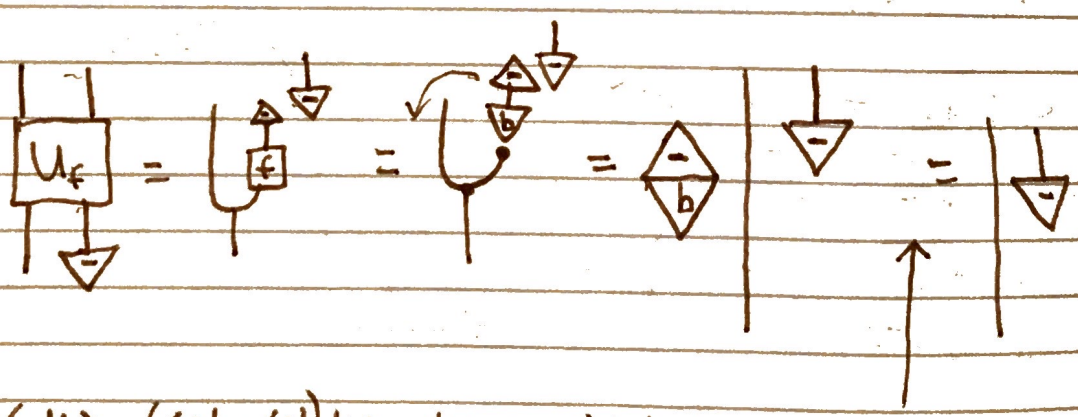


Analysis with diagrams (separately for constant vs. balanced)

Constant:



erases input and replaces with b.



$\langle -|b\rangle = (\langle 0| - \langle 1|)|b\rangle = \pm 1$ is global phase, so no effect

U_f effectively acts as identity when constant

Circuit then just $H^n H^n = I$, so we always measure X

Balanced: half 0s, half 1s

We know U_f is applied to $|\psi\rangle$ (equal superposition of all values)

We then have:

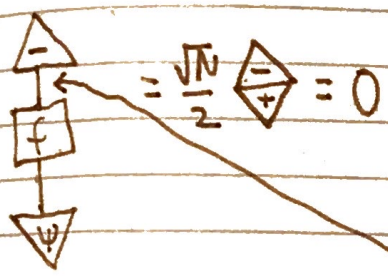
$$f\left(\frac{1}{\sqrt{N}}\sum_{z=0^n}^{1^n}|z\rangle\right)$$

$$N=2^n$$

$$=\frac{1}{\sqrt{N}}\sum_{z=0^n}^{1^n}|f(z)\rangle$$

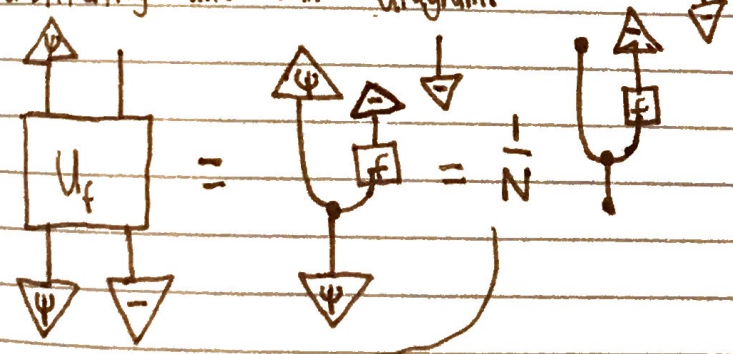
$$=\frac{1}{\sqrt{N}}\left(\frac{N}{2}|0\rangle+\frac{N}{2}|1\rangle\right)=\frac{\sqrt{N}}{2}\left(|0\rangle+|1\rangle\right)$$

$$=\frac{\sqrt{N}}{2}|+\rangle$$



$$=\frac{\sqrt{N}}{2}\quad =0$$

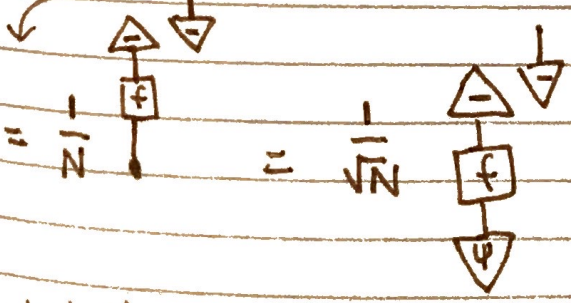
Substituting into main diagram:



$$=\frac{1}{N}$$

$$=\sum_i|i\rangle=\sqrt{N}\,|\psi\rangle$$

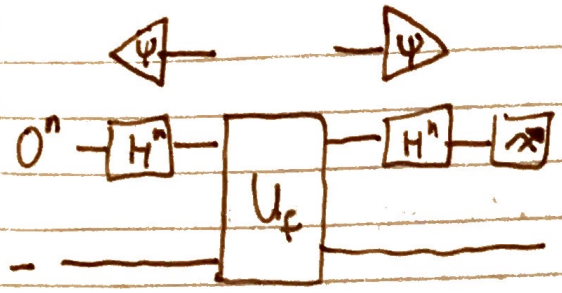
is inner product with denormalized ψ



$$=\frac{1}{N}\qquad=\frac{1}{\sqrt{N}}\qquad=0$$

Input to U_f is ψ when x is $|0\rangle^n$, circuit always tries to measure ψ

Our analysis showed this never succeeds.



Never succeeds with balance

If ψ input,

Never measure 0^n when f is balanced

Always measure 0^n when f is constant

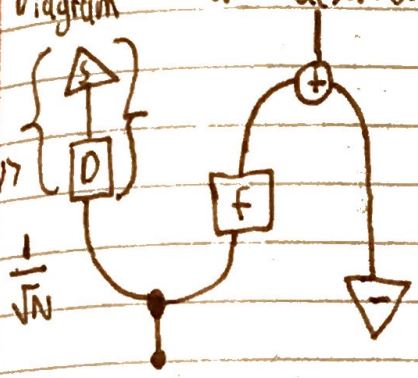
Grover with diagrams

- Consider case when only 1 iteration

Occurs when $\frac{1}{4}$ of all inputs are 1. (Or if only 2 qubits)

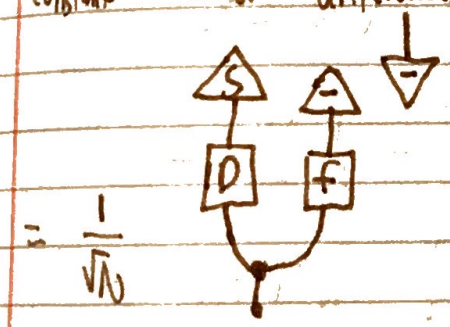
- Algorithm from before generalizes to any constant fraction of 1s.

Diagram below describes measurement result of "s":



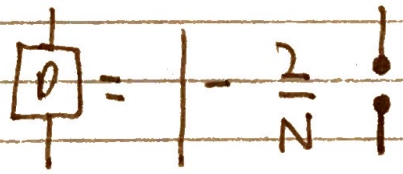
Rest is same as U_f

Constant factor difference between prepare and $|\psi\rangle$



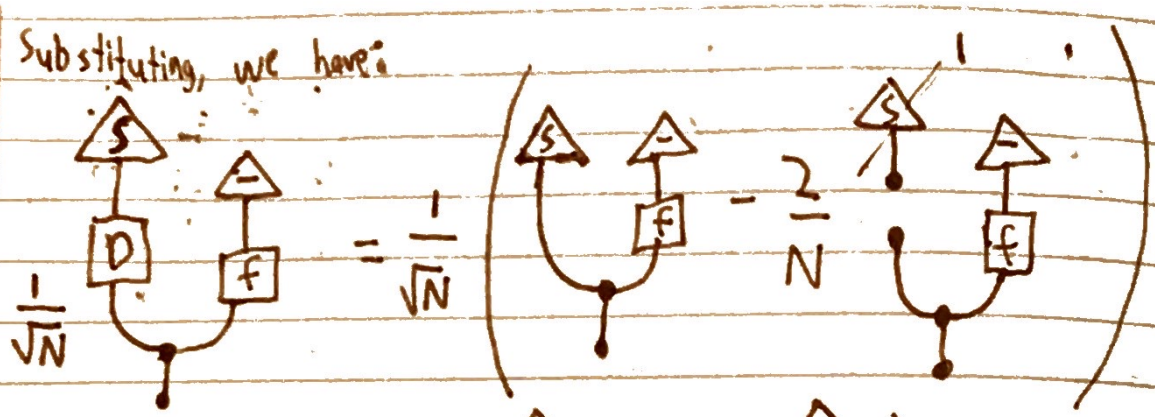
$D = 2|\psi\rangle\langle\psi| - I$

Write diagrammatically



(Can add/subtract diagrams)

Substituting, we have:



Using $\langle s|\psi\rangle = \frac{1}{\sqrt{N}}\left(\boxed{f}_{s} - \frac{2}{N}\boxed{f}\right) = \frac{\#0s}{N} - \frac{\#1s}{N}$

First term is $\langle f(s)|-\rangle = (-1)^{f(s)}$

Last term is 2x fraction of 0s − 1s.

If f has $\frac{1}{4}$ 1s, then we get $2\left(\frac{3}{4} - \frac{1}{4}\right) = 1$

$$\left[\begin{aligned} & f\left(\frac{1}{N}\sum_z |z\rangle\right) \\ &= \frac{1}{N}\sum_z f(z) \\ &= \frac{1}{N}\cdot 0s\,|0\rangle + \frac{1\cdot 1s}{N}\,|1\rangle \end{aligned}\right]$$

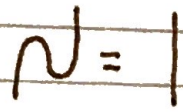
Value in parenthesis is $-1^{f(s)} - 1$

This is 0 if $f(s) = 0$, so measurement never succeeds

I.E. we only measure s such that $f(s) = 1$

Summary/future work:

- Diagrammatic tools can be useful for understanding quantum processes
- Reveal things hidden in linear algebra
  - Eg teleportation:



- Can make more powerful if go further into math weeds
- Not one notation that works perfectly for all situations

- What's missing:
  - Only have equivalences
  - To go further we need inference