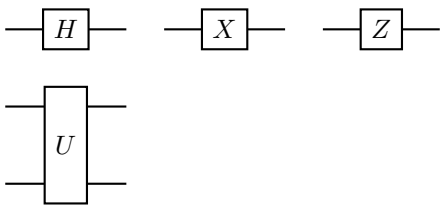
Lecture 10 Notes

1 Quantum Circuit pt. 2

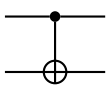
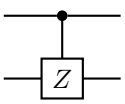
We can draw quantum circuits as graphs:

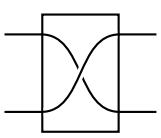
- Qubits becomes wires.
- Gates becomes boxes with labels on them:



Since gates are unitary, number of inputs must be equal to number of outputs.

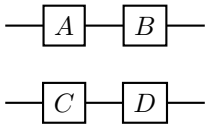
Special gates:

- Measurement: 
- CNOT (CX): 
- CZ: 

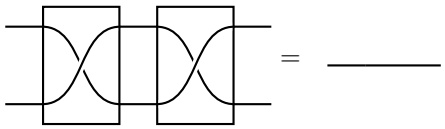
- SWAP: 

Many algebraic equivalences become obvious in circuits:

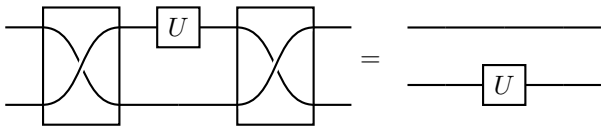
- $(B \otimes D)(A \otimes C) = BA \otimes DC$:



- $\mathrm{SWAP} \cdot \mathrm{SWAP} = I$:

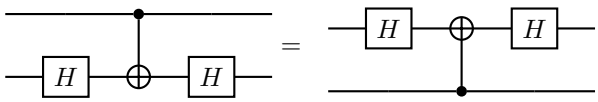


- $\mathrm{SWAP}(U \otimes I)\mathrm{SWAP} = I \otimes U$:



Some algebraic equivalences are still not intuitive in circuits:

- $(I \otimes H)\mathrm{CNOT}_{1,2}(I \otimes H) = (H \otimes I)\mathrm{CNOT}_{2,1}(H \otimes I)$:



2 Quantum Information pt. 1

Cloning classical bits are easy. Cloning qubits are provably impossible:

Proof. Suppose an unitary operation U can clone a qubit $|x\rangle = a|0\rangle + b|1\rangle$. This means

$$\begin{aligned} U(|x\rangle \otimes |0\rangle) &= (|x\rangle \otimes |x\rangle) \\ U(a|00\rangle + b|10\rangle) &= a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle \end{aligned} \tag{1}$$

Suppose U can correctly clone qubit $|0\rangle$ and $|1\rangle$, we know that

$$\begin{aligned} U|00\rangle &= |00\rangle \\ U|10\rangle &= |11\rangle \end{aligned}$$

$$\begin{aligned} U(a|00\rangle + b|10\rangle) &= a \cdot U|00\rangle + b \cdot U|10\rangle \\ &= a|00\rangle + b|11\rangle \end{aligned} \tag{2}$$

By eq 1 and eq 2,

$$\begin{gathered} a^2|00\rangle + ab|01\rangle + ab|10\rangle + b^2|11\rangle = a|00\rangle + b|11\rangle \\ a = 1 \\ b = 0 \\ |x\rangle = |0\rangle \\ \text{or} \\ a = 0 \\ b = 1 \\ |x\rangle = |1\rangle \end{gathered}$$

Therefore, a unitary that correctly clones $|0\rangle$ and $|1\rangle$ can only correctly clone $|0\rangle$ and $|1\rangle$ and not other qubits. □

Similarly, for any orthogonal qubits $|a\rangle$ and $|b\rangle$, we can find a unitary that clones $|a\rangle$ and $|b\rangle$, and that unitary cannot clone other qubits.

For example, CNOT clones $|0\rangle$ and $|1\rangle$; we can make a unitary U that clones $|+\rangle$ and $|-\rangle$ by using H:

$$U = (H \otimes I) \cdot \text{CNOT} \cdot (H \otimes I)$$