Quantum states:

\[ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]
\[ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]
\[ |+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
\[ |-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]

Quantum gates:

\[ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]
\[ Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]
\[ H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \]
\[ \text{CNOT} = \begin{cases} I \otimes I & \text{If first bit is } |0\rangle \\ I \otimes X & \text{If first bit is } |1\rangle \end{cases} \]
\[ CX = \text{CNOT} \]
\[ CZ = \text{CNOT} \text{ with } Z \text{ replacing } X \]
\[ SWAP = \text{Swaps the order of its two input qubits} \]
\[ CNOT_{1,2} = \text{CNOT} \]
\[ CNOT_{2,1} = SWAP \times CNOT_{1,2} \times SWAP \] (Inverts first bit if second bit is \(|1\rangle\))
Properties:

- The Z gate swaps $|+\rangle$ with $|-\rangle$, so $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$
- The Z gate is just a change of phase on $|0\rangle$ and $|1\rangle$, so $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$
- The X gate swaps $|0\rangle$ with $|1\rangle$, so $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$
- The X gate is just a change of phase on $|+\rangle$ and $|-\rangle$, so $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$
- The Z-basis consists of $|0\rangle$, $|1\rangle$. The X-basis consists of $|+\rangle$, $|-\rangle$. These bases are orthogonal, since the dot products $\langle 0|1\rangle = 0$ and $\langle +|-\rangle = 0$.
- Can define a matrix that transforms one basis to the other:
  $|+\rangle \langle 0| + |-\rangle \langle 1| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$, the Hadamard gate! Because $H^\dagger = H$, it maps between the $Z$ and $X$ bases in both directions.
- CX and CZ are correlated by a change of basis on the second bit:
  $CZ = (I \otimes H)CX(I \otimes H)$ and vice versa
- A gate $G$ is symmetric if $SWAP \times G \times SWAP = G$. Examples of symmetric gates are $CZ$, and $(U \otimes U)$ for any unitary $U$.
  - $(H \otimes H)CNOT_{1,2}(H \otimes H) = CNOT_{2,1}$
  - $(I \otimes X)CNOT(I \otimes X) = CNOT$
  - $(X \otimes I)CNOT(X \otimes I) = (I \otimes X)CNOT = CNOT(I \otimes X)$

Many algebraic relationships exist between these gates: not just coincidence, there’s some deeper connection between them.