CSE 490q Lecture Notes: Quantum Circuits II

October 19th, 2020

Quantum states:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$|+\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$|-\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Quantum gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$CNOT = \begin{cases} I \otimes I & \text{If first bit is } |0\rangle \\ I \otimes X & \text{If first bit is } |1\rangle \end{cases}$$
$$CX = CNOT$$
$$CZ = CNOT \text{ with Z replacing X}$$
$$SWAP = \text{Swaps the order of its two input qubits}$$
$$CNOT_{1,2} = CNOT$$
$$CNOT_{2,1} = SWAP \times CNOT_{1,2} \times SWAP \qquad \text{(Inverts first bit if second bit is } |1\rangle\text{)}$$

Properties:

- The Z gate swaps $|+\rangle$ with $|-\rangle$, so $Z|+\rangle = |-\rangle$ and $Z|-\rangle = |+\rangle$
- The Z gate is just a change of phase on $|0\rangle$ and $|1\rangle$, so $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$
- The X gate swaps $|0\rangle$ with $|1\rangle$, so $X|0\rangle = |1\rangle$ and $X|1\rangle = |0\rangle$
- The X gate is just a change of phase on $|+\rangle$ and $|-\rangle$, so $X|+\rangle = |+\rangle$ and $X|-\rangle = -|-\rangle$
- The Z-basis consists of $|0\rangle$, $|1\rangle$. The X-basis consists of $|+\rangle$, $|-\rangle$. These bases are orthoganol, since the dot products $\langle 0||1\rangle = 0$ and $\langle +||-\rangle = 0$.
- Can define a matrix that transforms one basis to the other:
  $|+\rangle\langle 0| + |-\rangle\langle 1| = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$, the Hadamard gate! Because $H^\dagger = H$, it maps between the Z and X bases in both directions.
- CX and CZ are correlated by a change of basis on the second bit:
  $CZ = (I \otimes H)CX(I \otimes H)$ and vice versa
- A gate G is symmetric if $SWAP \times G \times SWAP = G$. Examples of symmetric gates are CZ, and $(U \otimes U)$ for any unitary U.
- $(H \otimes H)CNOT_{1,2}(H \otimes H) = CNOT_{2,1}$
- $(I \otimes X)CNOT(I \otimes X) = CNOT$
- $(X \otimes I)CNOT(X \otimes I) = (I \otimes X)CNOT = CNOT(I \otimes X)$

Many algebraic relationships exist between these gates: not just coincidence, there's some deeper connection between them.