CSE 490Q

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Computing with Linear Algebra (Pt 1)

→ Quantum computing uses linear algebra to describe quantum states and operations
  → Use of linear algebra is <u>not</u> unique to QC, other forms of computation utilize linear algebra (Machine Learning for example)
  → We will create this connection by showing how linear algebra is used to formulate randomness
→ the "weirdness" of QC comes from linear algebra, not from anything about Quantum Mechanics
→ linear algebra notation differs in Physics

A Random Coin

Consider coin C where $\underbrace{P(C=H) = 1/2,\ P(C=T) = \tfrac{1}{2}}$

These equations ~~can~~ represent the state of the coin. $\Longrightarrow \begin{bmatrix} P(C=H) = 1/2 \\ P(C=T) = 1/2 \end{bmatrix}$

Consider a dice D, for $D \rightarrow \begin{bmatrix} P(D=1) = 1/6 \\ P(D=2) = 1/4 \\ \vdots \end{bmatrix}$

* We would like one compact item that describes what we know
→ We will do this by using vectors where each component represents a predefined probability.

ex:

$\vec{C} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$, $\vec{C}^* = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix} \begin{matrix} H \\ T \end{matrix}$, $\vec{D} = \begin{bmatrix} 1/6 \\ 1/4 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$

↑ state of coin C

$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}\vec{e}_1 + \frac{1}{2}\vec{e}_2$, where $\vec{e}_1, \vec{e}_2$ are eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively.

For notation simplicity,
let $\vec{e}_H \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_T \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Longrightarrow \boxed{\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \frac{1}{2}(\vec{e}_H + \vec{e}_T)}$

↑ can clean up with physics notation.

Dirac Notation

$\vec{v} = |v\rangle$ ⟶ to write a dot product of vectors $\vec{u}, \vec{v} \rightarrow \langle u|v\rangle$

↑ ket

$\vec{v}^T = \langle v|$ ← bra

bra ket

<u>braket → bracket</u>
bracket notation.

Ket: $|v\rangle \rightarrow$ column vector $\begin{bmatrix} \\ \\ \end{bmatrix}$ ↑ $n \times 1$

bra: $\langle v| \rightarrow$ row vector $[\ \ \ \ \ \ ]$ ↑ $1 \times n$

$\boxed{\langle v||w\rangle = \langle v|w\rangle = \vec{v}^T\vec{w}} \boxed{= \vec{v} \cdot \vec{w}}$

$\boxed{|w\rangle\langle v| = \vec{w}\vec{v}^T \longrightarrow \text{matrix } n \times n}$

$\begin{bmatrix} \uparrow \\ n \\ \downarrow \end{bmatrix} [\leftarrow n \rightarrow]$

Now with New notation, back to our coin

$C \longrightarrow \boxed{\frac{1}{2}(|h\rangle + |t\rangle)}$

In dirac notation we can ~~see~~ have a vector "1" = $|1\rangle$ where previously we can't.

States of a Random Bit

State: $|\psi\rangle = a|0\rangle + b|1\rangle$ with $a, b \geq 0$ and $a+b=1 \Rightarrow |\psi\rangle$ Must be Normalized!

Operations on a Random Bit

* we would like to have a way to change from one state to another, in linear algebra we naturally use matrices to do this. In Quantum Mechanics we use Operators which act on states to change that state.

$X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Swap operator

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

$F_1 \equiv \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ Force state to $|1\rangle$ operator

$F_1|1\rangle = |1\rangle$
$F_1|0\rangle = |1\rangle$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

let $|v\rangle = 0.6|0\rangle + 0.4|1\rangle = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

$X|v\rangle = 0.6\,X|0\rangle + 0.4\,X|1\rangle = 0.6|1\rangle + 0.4|0\rangle = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$

Consider the following:

$G|1\rangle = |1\rangle$
$G|0\rangle = \frac{1}{2}(|0\rangle + |1\rangle)$

$\Longrightarrow G = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$ G operator

* are all matrices valid operators? or only some operators are?
→ Not all are, your final state after operation must have $a, b \geq 0$ and $a+b=1$.

For an operator to be valid, it must take all eigenstates to a new, normalized, valid state.

For arbitrary operator A,

$A|0\rangle = a|0\rangle + b|1\rangle$
$A|0\rangle = c|0\rangle + d|1\rangle$

$\Rightarrow A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

, A is only a valid operator if and only if $a, b, c, d \geq 0$ and $a+b=1$, $c+d=1$.

If an operator satisfies these restrains than such a matrix is called a "Stochastic" matrix

Stochastic Matrices

$|v\rangle = s|0\rangle + t|1\rangle$

$A|v\rangle = sA|0\rangle + tA|1\rangle = sa|0\rangle + sb|1\rangle + tc|0\rangle + td|1\rangle = (sa+tc)|0\rangle + (sb+td)|1\rangle$

* because $a, b, c, d, s, t \geq 0 \Rightarrow sa+tc \geq 0$, $sb+td \geq 0$. ✓

$sa + tc + sb + td = s\cancel{(a+b)}^{1} + t\cancel{(c+d)}^{1} = \cancel{s+t}^{1} = 1$ ✓

Any operator is a valid operator if the matrix associated with that matrix is a STOCHASTIC MATRIX.

States are vectors of non-negative numbers that sum to 1.

Dirac Notation is a type-checking notation that simplifies operations.