The Hidden Subgroup Problem

CSE 490Q: Quantum Computation

Shor's Algorithm

- Shor's (1994) breakthrough result was an efficient q. algorithm for **factoring**
  - strongly believed to be classically hard
    - 200+ years of failed attempts to solve it
    - hardness is *assumed* by cryptosystems like RSA
  - not a toy problem, not a black box

- Fastest classical algorithm runs in $O(2^{n/3})$ time
- Shor's algorithm is $O(n^3)$

Post-Shor Algorithms

- Much later work focused on *generalizing* it
  - we will look at some of the generalizations (more than one!)
  - Shor's paper still has insights that may not be fully understood still
- First generalization was the phase estimation algorithm
- Second generalization is…

Hidden Subgroup Problem

Problem: Given a function $f : G \to \{0,1\}^k$ (via an oracle) that is constant on cosets of some subgroup H, find the subgroup H.

- What does all that mean?

Groups

$f(x, y)$

$\boxed{x+y}$

Definition: A group G is a set with…

- a special element $e \in G$
- an operation taking any $x, y \in G$ to some $xy \in G$
- an operation taking any $x \in G$ to some $x^{-1} \in G$

satisfying certain rules…

$ex = x = xe$	for all $x \in G$	("identity")
$x x^{-1} = e = x^{-1} x = e$	for all $x \in G$	("inverses")
$(xy)z = x(yz)$	for all $x, y, z \in G$	("associativity")

Definition: A group G is abelian if $xy = yx$ for all $x, y \in G$.

- (In this case, we often write $x+y$ instead of xy.)

Subgroups

- A subgroup is a subset of a group that is itself a group (with the same operations)

- In particular, $S \subseteq G$ is a subgroup iff
  - $e \in S$
  - if $x \in S$, then $x^{-1} \in S$
  - if $x \in S$ and $y \in S$, then $xy \in S$

- So you can perform group operations on elements of S and you will never see an element outside of S (i.e., from G – S)

Cosets

- If $S \subseteq G$ is a subgroup, then a coset of S is a set of the form

  $$gS = \{ gs : s \in S \} \qquad \text{for some } g \in G$$

  $\lfloor S \rfloor$

- This is not a subgroup (e.g, we do not have $e \in gS$). It is just a subset of G.

- The cosets of S partition, so we can write

  $$G = g_1S \cup g_1S \cup \ldots \cup g_kS$$

  for some choice of $g_1, \ldots, g_k \in G$.

g_1S vs g_2S

gS $\qquad g = ge \in S$

HSP

$f(e_G) = y$

Problem: Given a function $f : G \to \{0,1\}^k$ (via an oracle) that is constant on cosets of some subgroup H, find the subgroup H.

iff

- I.e., we are promised that $f(gh_1) = f(gh_2)$ for all $h_1, h_2 \in H$ and $g \in G$. gH
- Equivalently, $f(g) = f(gh)$ for all $h \in H$ and $g \in G$.

ge

$f(g_1) = f(g_2)$

iff $g_1 H = g_2 H$.

- Outputs need not have any meaning.
- Requires exponential time classically.

Discrete Logarithm

- Shor's paper solved another problem called "discrete logarithm"

Definition: Given $b, d \in \mathbb{Z}_p$, find an $r \in \mathbb{Z}_{p-1}$ such that $b^r \equiv d \pmod{p}$

- If these were real numbers, we would have $r = \log_b(d)$, but this is $\mathbb{Z}_p$.

- Strongly believed to be hard classically
  - e.g., the Diffie-Hellman key exchange protocol uses this assumption

Discrete Logarithm

Definition: Given b, c $\in \mathbb{Z}_p$, find an r $\in \mathbb{Z}_{p-1}$ such that $b^r \equiv d \pmod p$

$b^r d^{-1} \equiv d d^{-1} \equiv 1$

- Solve the HSP over $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ with the following hiding function

(a, b)

$\{0, \ldots p-2\}$

$$f(x, y) = b^x (d^{-1})^y \pmod p$$

- (We can find d^{-1} efficiently by Euclid's algorithm.)

- Can see that $f(r, 1) = b^r (d^{-1})^1 = b^r d^{-1} \equiv 1 \pmod p$

- More generally, $f(kr, k) = b^{kr} (d^{-1})^k = (b^r d^{-1})^k \equiv 1^k = 1 \pmod p$

$b^r (d^{-1})^3 = b \cdots b d^{-1} d^{-1} \cdots$

$= b^2 d^{-1} b^2 d^{-1} b^2 d^{-1}$

$= (b^2 d^{-1})^3$

Discrete Logarithm

Definition: Given $b, c \in \mathbb{Z}_p$, find an $r \in \mathbb{Z}_{p-1}$ such that $b^r \equiv d \pmod p$

- Solve the HSP over $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ with the following hiding function

  $$f(x, y) = b^x (d^{-1})^y \pmod p$$

- We can see that $f(kr, k) = b^{kr} (d^{-1})^k = (b^r d^{-1})^k \equiv 1^k = 1 \pmod p$
- f is constant (1) on the subset $H = \{ (kr, k) : k \in \mathbb{Z}_{p-1} \}$
  - can check that H is a group
  - can check that f is constant on all cosets of H

Discrete Logarithm

Definition: Given b, c $\in \mathbb{Z}_p$, find an r $\in \mathbb{Z}_{p-1}$ such that $b^r \equiv d \pmod p$

- Solve the HSP over $\mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$ with the following hiding function

$$f(x, y) = b^x (d^{-1})^y \pmod p$$

- Solving the HSP will give us a generator (a, k) $\in$ H = { (kr, k) : k $\mathbb{Z}_{p-1}$ }

- The answer is k^{-1} a = k^{-1} kr = r

Solving the HSP for Abelian Groups

- Kitaev solved this problem (essentially) for all abelian groups.
- We will see the modern solution shortly.
- First, we need a bit more background on groups…

Characters

Exponentials

- Working with G = $\mathbb{Z}_N$, consider these functions:

$$\chi_j(k) = e^{2\pi i j k/N} \qquad \text{for } j \in \mathbb{Z}_N$$

- They have the properties that

$$\chi_j(x+y) = e^{2\pi i j(x+y)/N} = e^{2\pi i j x/N + 2\pi i j y/N}$$

$$= e^{2\pi i j x/N} e^{2\pi i j y/N} = \chi_j(x)\chi_j(y)$$

and

$$\chi_j(0) = e^0 = 1$$

Exponentials

- Working with $G = \mathbb{Z}_N$, consider these functions:

$$\chi_j(k) = e^{2\pi i j k / N} \qquad \text{for } j \in \mathbb{Z}_N$$

- They have the properties that
  - $\chi_j(x+y) = \chi_j(x)\,\chi_j(y)$
  - $\chi_j(0) = 1$

Characters

- A function $\chi : G \to \mathbb{C}$ is called a "character of G" if it satisfies
  - $\chi(xy) = \chi(x)\,\chi(y)$
  - $\chi(e) = 1$

- The "irreducible" characters of $\mathbb{Z}_N$ are $\chi_0, \chi_1, \ldots, \chi_{N-1}$

Characters

- A function $\chi : G \to \mathbb{C}$ is called a "character of G" if it satisfies
  - $\chi(xy) = \chi(x)\,\chi(y)$
  - $\chi(e) = 1$

- The "irreducible" characters of $\mathbb{Z}_N$ are $\chi_0, \chi_1, \ldots, \chi_{N-1}$

- The set of irreducible characters is often denoted $\hat{G}$ (or $G\hat{}$)
  - we have $|G\hat{}| = |G|$
  - nearly all the information about the group is also in its characters

Properties of Characters

- It can be shown that, for any two ~~distinct~~ irreducible characters χ and γ, we have

$\chi \cdot \gamma$

$$\frac{1}{|G|}\sum_{g\in G}\overline{\chi(g)}\,\gamma(g) = \begin{cases} 1 & \text{if } \chi=\gamma \\ 0 & \text{otherwise} \end{cases}$$

row orthogonality

and for any ~~distinct~~ elements g, h ∈ G, we have

$$\frac{1}{|G|}\sum_{\chi\in\hat{G}}\overline{\chi(g)}\,\chi(h) = \begin{cases} 1 & \text{if } g=h \\ 0 & \text{otherwise} \end{cases}$$

column orthogonality

- For $\mathbb{Z}_N$, these are calculations involving exponentials that we did already…

Properties of Characters

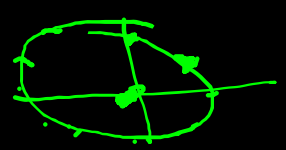
- For $\mathbb{Z}_N$, these are calculations involving exponentials that we did already:

$$\sum_{x=0}^{N-1} \overline{\chi_j(x)}\,\chi_k(x) = \sum_{x=0}^{N-1} e^{-2\pi i j x/N}\, e^{2\pi i k x/N}$$

$$= \sum_{x=0}^{N-1} e^{2\pi i (k-j) x/N} = \sum_{x=0}^{N-1} \Big(\underbrace{e^{2\pi i (k-j)/N}}_{\omega}\Big)^x$$

$$= \omega^0 + \omega^1 + \omega^2 + \cdots + \omega^{N-1} \quad \text{where } \omega = e^{2\pi i (k-j)/N}$$

$$= \begin{cases} N & \text{if } \omega = 1 \quad (k=j) \\ 0 & \text{otherwise} \quad (k \neq j) \end{cases}$$



Properties of Characters

- Since we want

$$\sum_{g \in G} \overline{\chi(g)}\, \gamma(g) = \begin{cases} 1 & \text{if } \chi = \gamma \\ 0 & \text{otherwise} \end{cases}$$

we need to rescale by $N^{-1/2}$ so that we have

$$\chi_j(k) = \frac{1}{\sqrt{N}} e^{2\pi i j k / N}$$

Quantum Fourier Transform

QFT Setup

- We will work in a basis labeled by group elements: $\{ |g\rangle : g \in G \}$
  - need $\log_2(|G|)$ qubits
  - choose any convenient mapping between bits and group elements

- Can also have a basis labeled by characters : $\{ |\chi\rangle : \chi \in G\hat{} \}$
  - exactly the same size
  - choose any convenient mapping of bits to character names

QFT

- The QFT F is a change of basis from group elements to characters:

$$F|g\rangle = \frac{1}{\sqrt{|G|}} \sum_{\chi \in \hat{G}} \chi(g)\, |\chi\rangle$$

- Takes a group element to a vector of its character values

- This is a matrix whose columns are the characters
- Hence, this is unitary by ~~column~~ row orthogonality

$$\frac{1}{\sqrt{|G|}} \begin{array}{c} \\ \chi_1 \\ \vdots \\ \chi_k \end{array} \begin{array}{c} \begin{array}{cccc} g_1 & g_2 & \cdots & g_n \end{array} \\ \begin{bmatrix} \vdots & \vdots & & \vdots \\ 1 & \chi_1(g_2) & & 1 \\ \vdots & \vdots & & \vdots \end{bmatrix} \end{array} =: F$$

$$F^\dagger F \qquad [F F^\dagger]_{1,k} = \chi_1 \cdot \chi_k$$

QFT Example 1

- When $G = \mathbb{Z}_N$, the characters are the functions

$$\chi_j(k) = e^{2\pi i j k/N} \qquad \text{for } j \in \mathbb{Z}_N$$

- So the QFT is

$$F|j\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i j k/N} \overset{|\chi_k\rangle}{|k\rangle}$$

which is exactly our definition from before of "the QFT" with $N = 2^n$.

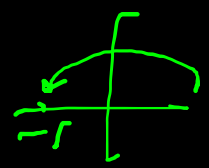
QFT Example 2

- When $G = \mathbb{Z}_N$, the characters are the functions

$$\chi_j(k) = e^{2\pi i j k / N} \qquad \text{for } j \in \mathbb{Z}_N$$

- If N = 2, then the two characters are…

$$\chi_0(k) = e^0 = 1$$



$$\chi_1(k) = e^{2\pi i k / 2} = e^{\pi i k} = (e^{\pi i})^k = (-1)^k$$

QFT Example 2

- When $G = \mathbb{Z}_N$, the characters are the functions

$$\chi_j(k) = e^{2\pi i j k / N} \qquad \text{for } j \in \mathbb{Z}_N$$

- If N = 2, then the two characters are
  - $\chi_0(k) = 1$
  - $\chi_1(k) = (-1)^k$

- So the Fourier transform is…

  i.e., the Hadamard gate!

$$\begin{array}{cc} \chi_0 & \chi_1 \\ \end{array}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

QFT Example 2

- When $G = \mathbb{Z}_2$, the characters are $\chi_0(k) = 1$ and $\chi_1(k) = (-1)^k$

- For a product group $G_1 \times G_2$, the characters are products of the form

$$\chi(g_1, g_2) = \chi_j(g_1)\, \chi_k(g_2) \qquad \text{for some } j, k$$

$\chi_j \in \widehat{G_1}$ $\quad \chi_k \in \widehat{G_2}$

$|G_1||G_2|$

- For $\mathbb{Z}_2 \times \mathbb{Z}_2$, there are four elements and four characters

$$F = \frac{1}{2} \begin{array}{c} (0,0) \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_{00} & \chi_{01} & \chi_{10} & \chi_{11} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} = H \otimes H$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} H & +H \\ H & -H \end{bmatrix}$$

QFT Example 2

- More generally, the QFT for $G = (\mathbb{Z}_2)^n = \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ is $H^{\otimes n}$
- So all of our algorithms using $H^{\otimes n}$ are using QFTs.
- In particular, Deutsch-Josza used the QFT over $(\mathbb{Z}_2)^n$
- Then, Shor used the QFT over $\mathbb{Z}_N$ $\mathbb{Z}_{2^n}$
- Now, we realize that these are just QFTs over different abelian groups

HSP for Abelian Groups

The "Standard" Solution

- For all abelian groups (and some non-abelian ones), we can solve the HSP using the following procedure…

1. Prepare the state

$$\left(\frac{1}{\sqrt{|G|}}\sum_{g\in G}\underbrace{|g\rangle}_{\lg|G|}\right)\otimes\underbrace{|0\rangle}_{k} \qquad \left(=\underline{F^{\dagger}|x_0\rangle}\otimes|0\rangle\right)$$

The "Standard" Solution

1. Prepare the state

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |0^k\rangle$$

2. Apply U_f to the second register giving us

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |f(g)\rangle$$

3. Measure (and discard) the second part of the state…

The "Standard" Solution

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes \underbrace{|f(g)\rangle}$$

$$\underline{|x\rangle} \otimes |0\rangle + \underline{|y\rangle} \otimes |1\rangle$$

3. Measure (and discard) the second part of the state…

- Recall that G can be partitioned by cosets into

$$G = \underbrace{g_1H \cup g_1H \cup \ldots \cup g_kH}$$

for some choice of $g_1, \ldots, g_k \in G$.

- Recall that $f(gh) = f(g)$ for all $h \in H$ since f is constant on cosets.

The "Standard" Solution

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |f(g)\rangle$$

$f(g,h) = f(g_1)$

3. Measure (and discard) the second part of the state…

Since G = $g_1H \cup g_1H \cup \ldots \cup g_kH$, the state above is

$H = \{h_1, \ldots, h_\ell\}$

$$\frac{1}{\sqrt{|G|}} \Big(|g_1 h_1\rangle \otimes |f(g_1 h_1)\rangle + \cdots + |g_1 h_\ell\rangle \otimes |f(g_1 h_\ell)\rangle + \quad g_1 H$$

$$\vdots$$

$$|g_k h_1\rangle \otimes |f(g_k h_1)\rangle + \cdots + |g_k h_\ell\rangle \otimes |f(g_k h_\ell)\rangle \Big) \quad g_k H$$

The "Standard" Solution

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |f(g)\rangle$$

3. Measure (and discard) the second part of the state…

Since $G = g_1H \cup g_1H \cup \ldots \cup g_kH$ and $f(gh) = f(g)$ for all $h \in H$, the state above is

$$\frac{1}{\sqrt{|G|}} \Big(|g_1h_1\rangle \otimes |f(g_1)\rangle + \cdots + |g_1h_\ell\rangle \otimes |f(g_1)\rangle + \\ \vdots \\ |g_kh_1\rangle \otimes |f(g_k)\rangle + \cdots + |g_kh_\ell\rangle \otimes |f(g_k)\rangle \Big)$$

The "Standard" Solution

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |f(g)\rangle$$

3. Measure (and discard) the second part of the state…

Since $G = g_1H \cup g_1H \cup \ldots \cup g_kH$ and $f(gh) = f(g)$ for all $h \in H$, the state above is

$$\frac{1}{\sqrt{|G|}} \Big(\big(|g_1h_1\rangle + \cdots + |g_1h_\ell\rangle \big) \otimes \underline{|f(g_1)\rangle} + \cdots \\ \vdots \\ \big(|g_kh_1\rangle + \cdots + |g_kh_\ell\rangle \big) \otimes \underline{|f(g_k)\rangle} \Big)$$

different

The "Standard" Solution

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |f(g)\rangle$$

3. Measure (and discard) the second part of the state…

$$\frac{1}{\sqrt{|G|}} \Big(\overbrace{\big(|g_1 h_1\rangle + \cdots + |g_1 h_\ell\rangle \big)}^{\ell/|G|} \otimes \underline{|f(g_1)\rangle} \; + \; \vdots \; \big(|g_k h_1\rangle + \cdots + |g_k h_\ell\rangle \big) \otimes \underline{|f(g_k)\rangle} \Big)$$

We will get one of $|f(g_1)\rangle + \ldots + |f(g_k)\rangle$, uniformly at random

The "Standard" Solution

$$\frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle \otimes |f(g)\rangle$$

3. Measure (and discard) the second part of the state…

$$\underbrace{\frac{1}{\sqrt{|G|}} \left(|g_i h_1\rangle + \cdots + |g_i h_\ell\rangle \right)} \otimes \underline{|f(g_i)\rangle}$$

$g_i H$.

for a uniformly random i = 1, …, k

- The function value has no meaning, so we discard it.

The "Standard" Solution

3. Measure (and discard) the second part of the state…

$$\frac{1}{\sqrt{|H|}}\left(|g_i h_1\rangle + \cdots + |g_i h_k\rangle\right)$$

$$\sum_h |g_i h\rangle$$

for a uniformly random i = 1, …, k.

4. Apply the QFT for G to get

$$F\left(\frac{1}{\sqrt{|H|}}\sum_{h\in H}|gh\rangle\right) = \frac{1}{\sqrt{|H|}}\sum_{h\in H}\frac{1}{\sqrt{|G|}}\sum_{\chi\in\hat{G}}\chi(gh)\,|\chi\rangle$$

$$\cancel{= \frac{1}{\sqrt{|G||H|}}\sum_{\chi\in\hat{G}}\chi(g)\left(\sum_{h\in H}\chi(h)\right)|\chi\rangle}$$

The "Standard" Solution

4. Apply the QFT for G to get

$$\chi(gh) = \chi(g)\chi(h)$$

$$F\left(\frac{1}{\sqrt{|H|}}\sum_{h\in H}|gh\rangle\right) = \frac{1}{\sqrt{|H|}}\sum_{h\in H}\frac{1}{\sqrt{|G|}}\sum_{\chi\in\hat{G}}\chi(gh)\,|\chi\rangle$$

$$= \frac{1}{\sqrt{|G||H|}}\sum_{\chi\in\hat{G}}\sum_{h\in H}\chi(g)\chi(h)\,|\chi\rangle$$

$$= \frac{1}{\sqrt{|G||H|}}\sum_{\chi\in\hat{G}}\chi(g)\left(\sum_{h\in H}\chi(h)\right)|\chi\rangle$$

5. Measure this to get some χ. $\in \hat{G}$

The "Standard" Solution

$$\frac{1}{\sqrt{|G||H|}} \sum_{\hat{\chi} \in G} \chi(g) \left(\sum_{h \in H} \chi(h) \right) |\chi\rangle$$

4. Measure this to get some χ.

- The probability of measuring χ is given by

$$\frac{|\chi(g)|^2}{|G||H|} \left| \sum_{h \in H} \chi(h) \right|^2$$

The "Standard" Solution

- The probability of measuring χ is given by

$$\rightarrow \frac{|\chi(g)|^2}{|G||H|} \left| \sum_{h \in H} \chi(h) \right|^2$$

- It is always the case that $\overline{\chi(g)} = \chi(g^{-1})$

$= \overline{\chi(g)}\,\chi(g)$

- So we have $|\chi(g)|^2 = \chi(g^{-1})\,\chi(g) = \chi(g^{-1}g) = \chi(e) = 1$

- Hence, we can simplify this to…

The "Standard" Solution

- The probability of measuring χ is given by

$$\frac{1}{|G||H|}\underbrace{\left|\sum_{h\in H}\chi(h)\right|^2}$$

$\chi(g_1g_2) = \chi(g_1)\chi(g_2)$

$g_1, g_2 \in G$

$H \subseteq G$

- Note that χ is *also* a character of H (since it satisfies $\chi(h_1h_2) = \chi(h_1)\,\chi(h_2)$), so

$$\sum_{h\in H}\chi(h) = \sum_{h\in H} 1\cdot\chi(h) = \sum_{h\in H}\overline{\chi_0(h)}\,\chi(h)$$

$\overline{1} = \overline{\chi_0(h)}$

$\overline{1} = 1$

$$= \begin{cases} 1 & \text{if } \chi \equiv 1 \text{ on } H \\ 0 & \text{otherwise} \end{cases}$$

The "Standard" Solution

- The probability of measuring χ is given by

$$\frac{1}{|G||H|}\left|\sum_{h\in H}\chi(h)\right|^2$$

- Unless $\chi \equiv 1$ on H, this is probability is 0.
- When $\chi \equiv 1$ on H, the probability is

$$\frac{1}{|G||H|}\left|\sum_{h\in H}1\right|^2 = \frac{|H|^{\cancel{2}}}{|G|\cancel{|H|}} = \boxed{\frac{|H|}{|G|}} \quad \frac{|G|}{|H|}.$$

The "Standard" Solution

- The result of this calculation is an element of the set of irreducible characters of G that are **trivial** on H.
- In other words, we have $H \subseteq \mathrm{Ker}(\chi) = \{ g \in G : \chi(g) = 1 \}$
- To solve the problem, we do this multiple times and get $\chi_1, \ldots, \chi_t$.
- H must be contained in all of their kernels, so we have $H \subseteq \mathrm{Ker}(\chi_1) \cap \ldots \cap \mathrm{Ker}(\chi_t)$

Ker(χ1)

Ker(χ2)

The "Standard" Solution

- Perform this calculation multiple times and get $\chi_1, \ldots, \chi_t$.
- H must be contained in all of their kernels, so we have $H \subseteq \underbrace{\mathrm{Ker}(\chi_1) \cap \ldots \cap \mathrm{Ker}(\chi_t)}$

  $\forall \in$
- It can be shown (with more character theory), that each new character that we measure shrinks the size of $\mathrm{Ker}(\chi_1) \cap \ldots \cap \mathrm{Ker}(\chi_t)$ by at least a factor of 2.
  - easy to check for the exponentials:

    $e^{2\pi i jx/N}$
    - we only have $\chi_j(x) = 1$ if xj is a multiple of N
    - this is most likely if N is even and $x = N/2$, in which it holds for half of j's
- Hence, the intersection converges to H after a small number of samples

The "Standard" Solution

- Perform this calculation multiple times and get $\chi_1, \ldots, \chi_t$.
- H must be contained in all of their kernels, so we have $H \subseteq \mathrm{Ker}(\chi_1) \cap \ldots \cap \mathrm{Ker}(\chi_t)$
- The intersection converges to H after a small number of samples
- It remains to show how to perform this set intersection
  - in general, this depends on the details of the group
  - however, for abelian groups, which are just $\mathbb{Z}_{N1} \times \ldots \times \mathbb{Z}_{Nk}$, this turns out to be a straightforward, classical calculation
    - (similar to Gaussian elimination)

HSP

- Solution to the HSP for all abelian groups
  - solves factoring and discrete logarithm as special cases
  - more general and cleaner than prior solutions

- Next time: non-abelian groups
  - will be higher level (and shorter)