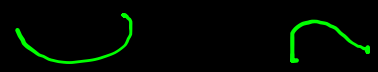
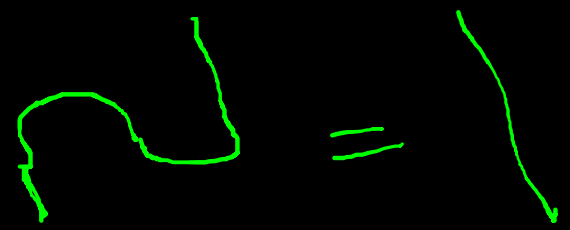
Diagrammatic Reasoning (pt. 2)

CSE 490Q: Quantum Computation

Diagrammatic Reasoning

- Using diagrams to analyze and understand quantum processes
  - not the same as circuits
  - ignore more details of the computation

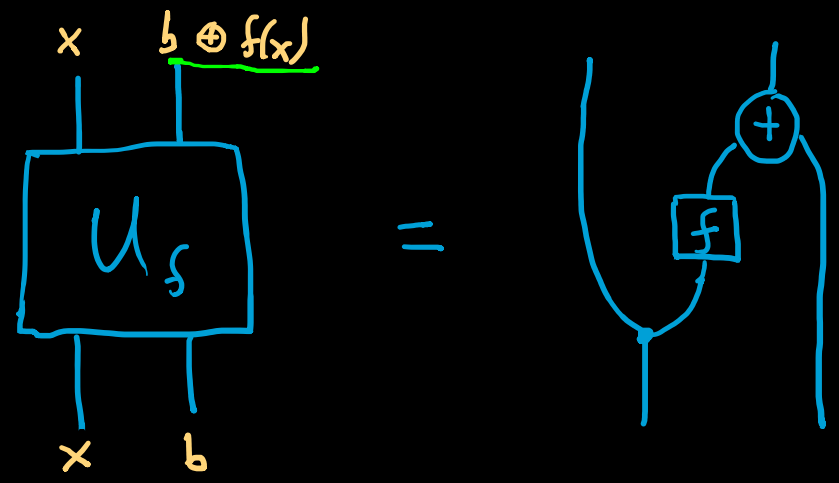
- Cup and cap 
  - not simple descriptions of any circuit element
  - very strong, useful mathematical properties
  - make it easy to describe the trace of a matrix

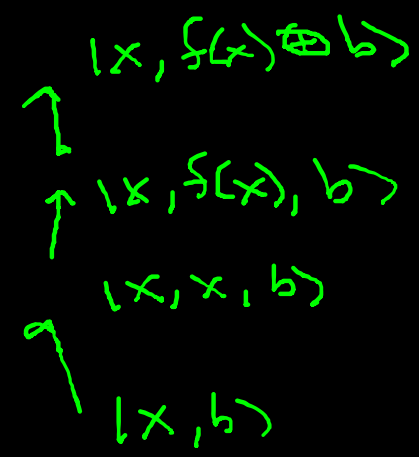


- More examples today

Analyzing an Oracle

- We can describe the Uf gate more nicely as follows:



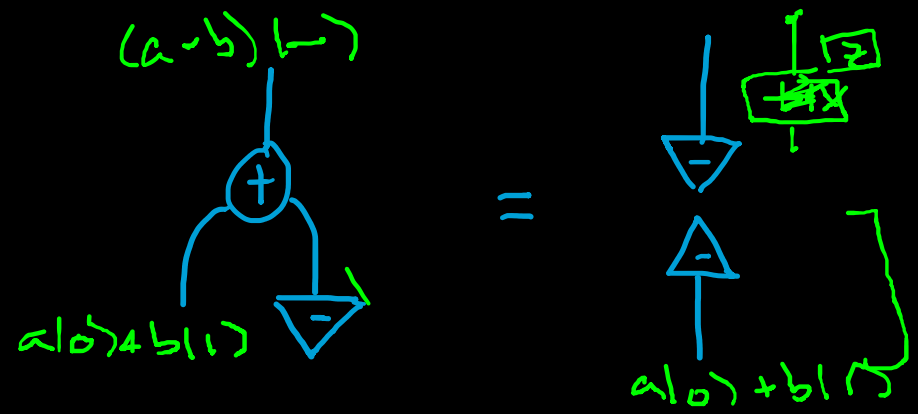


where "+" takes |a,b> to |a+b> x f(x)

- Note: this describes behavior not how it is computed
  - the "+" gate is not even unitary!

Plus Operation

- "+" has its own special properties including:



$$\langle -|(a|0\rangle + b|1\rangle)|-\rangle$$

$$(\langle 0| - \langle 1|)(a|0\rangle + b|1\rangle)$$

$$= a - b$$

LHS

$$(a|0\rangle + b|1\rangle) \otimes (|0\rangle - |1\rangle)$$

$$= a|00\rangle - a|01\rangle + b|10\rangle - b|11\rangle$$

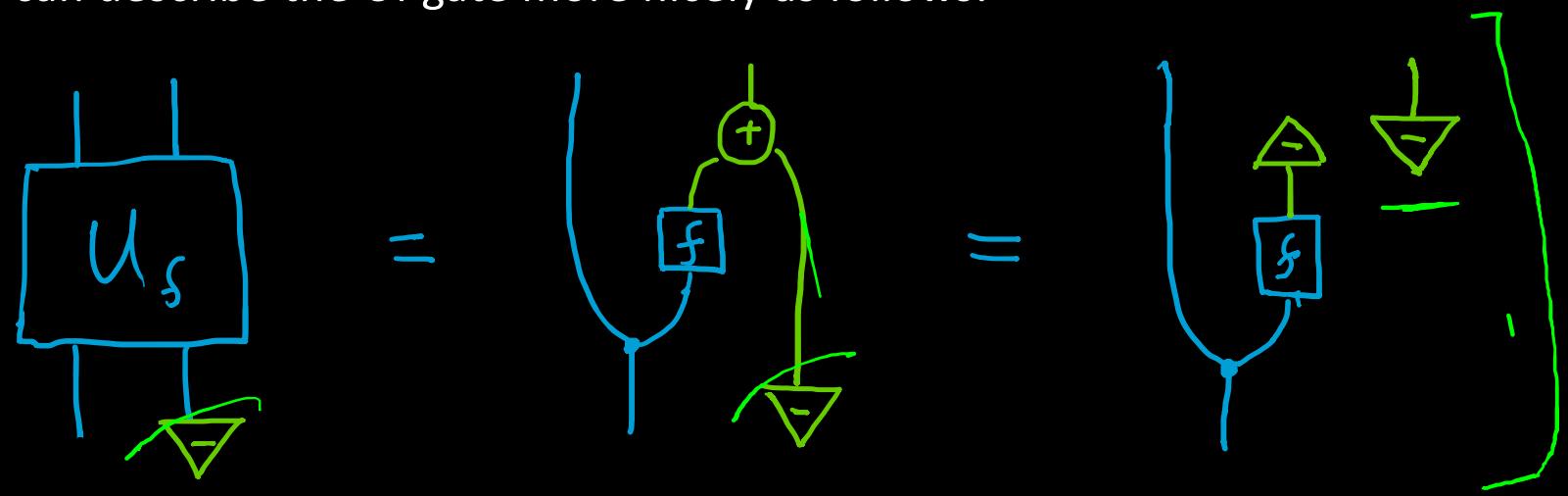
$$\xrightarrow{A^+} = a|0\rangle - a|1\rangle + b|1\rangle - b|0\rangle = (a-b)|0\rangle + (-a+b)|1\rangle$$

$$= (a-b)|0\rangle - (a-b)|1\rangle$$

$$= (a-b)(|0\rangle - |1\rangle) = (a-b)|-\rangle$$

Analyzing an Oracle

- We can describe the Uf gate more nicely as follows:



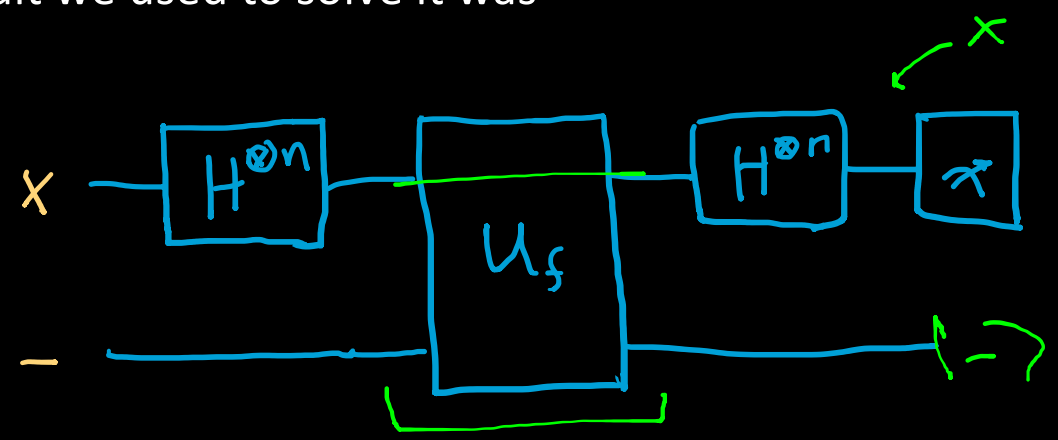
where ”+” takes |a,b> to |a+b>

- Note: this describes behavior not how it is computed
  - the “+” gate is not even unitary!

Deutsch-Josza

Problem: Determine whether f is *constant* or *balanced*

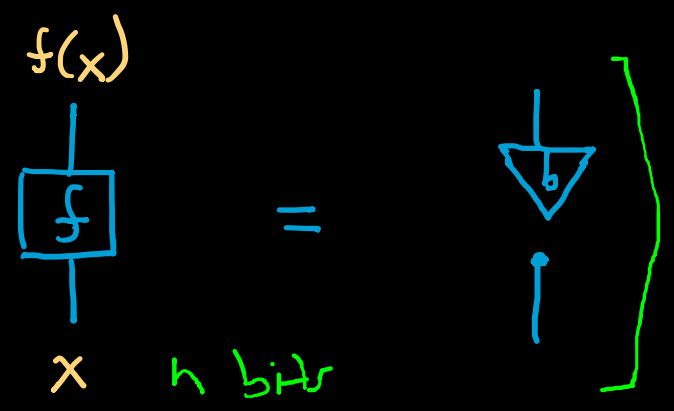
- The circuit we used to solve it was



- Let's do the analysis this time using diagrams…

Deutsch-Josza with Diagrams

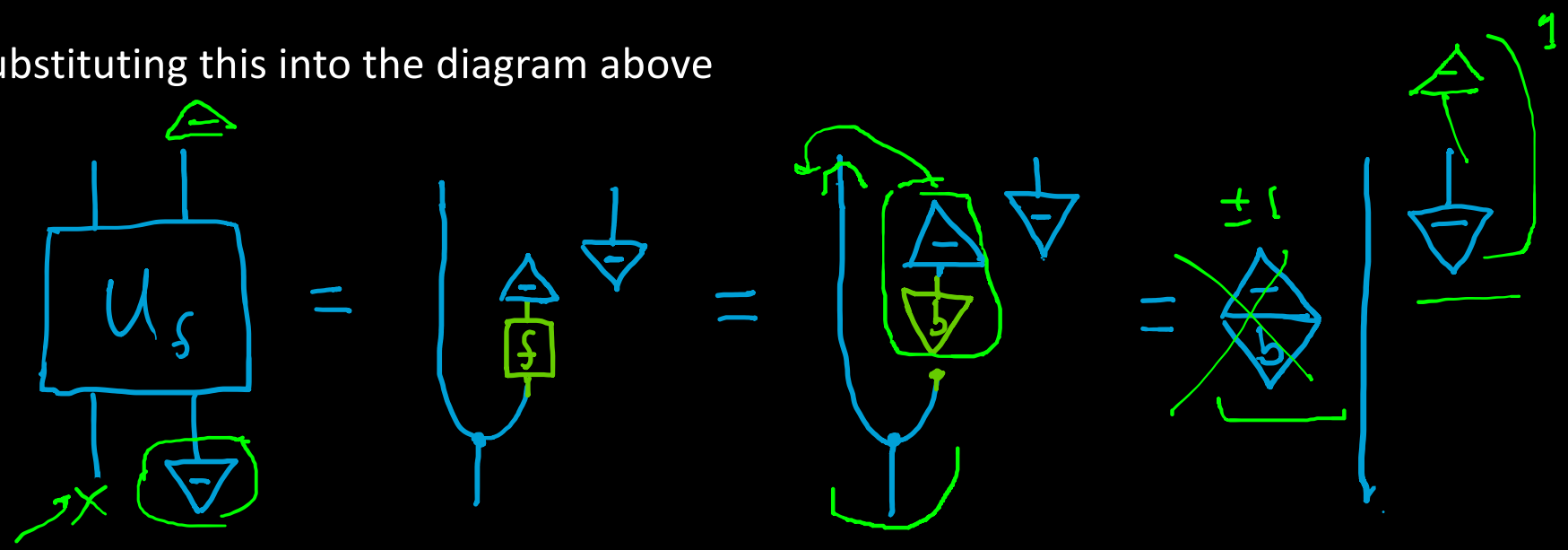
- Suppose that f is constant (f(x) = b for all x, where b = 0 or 1)



erases the input and replaces it with b

Deutsch-Josza with Diagrams

- Substituting this into the diagram above



- <-|b> = (<0| - <1|)|b> = ±1 is a global phase, so it has no effect
- The circuit is then just $H^{\otimes n}$ $H^{\otimes n}$ = I, so we always measure $|0\rangle^{n}$

Deutsch-Josza with Diagrams

- Suppose that f is balanced (half 0s, half 1s)

$i(\langle 0| - \langle 1|)(|0\rangle + |1\rangle)$

- This time, we will use the fact that we know U_f is applied to $|\psi\rangle$, an equal superposition over al possible values

- Then, we have

$N := 2^n$

$$f\left(\frac{1}{\sqrt{N}}\sum_{z=0}^{1^n} |z\rangle\right)$$

$$= \frac{1}{\sqrt{N}}\sum_{z=0}^{1^n} |f(z)\rangle$$

$$= \frac{1}{\sqrt{N}}\left(\frac{N}{2}|0\rangle + \frac{N}{2}|1\rangle\right) = \frac{\sqrt{N}}{2}(|0\rangle + |1\rangle) = \frac{\sqrt{N}}{2}|+\rangle$$

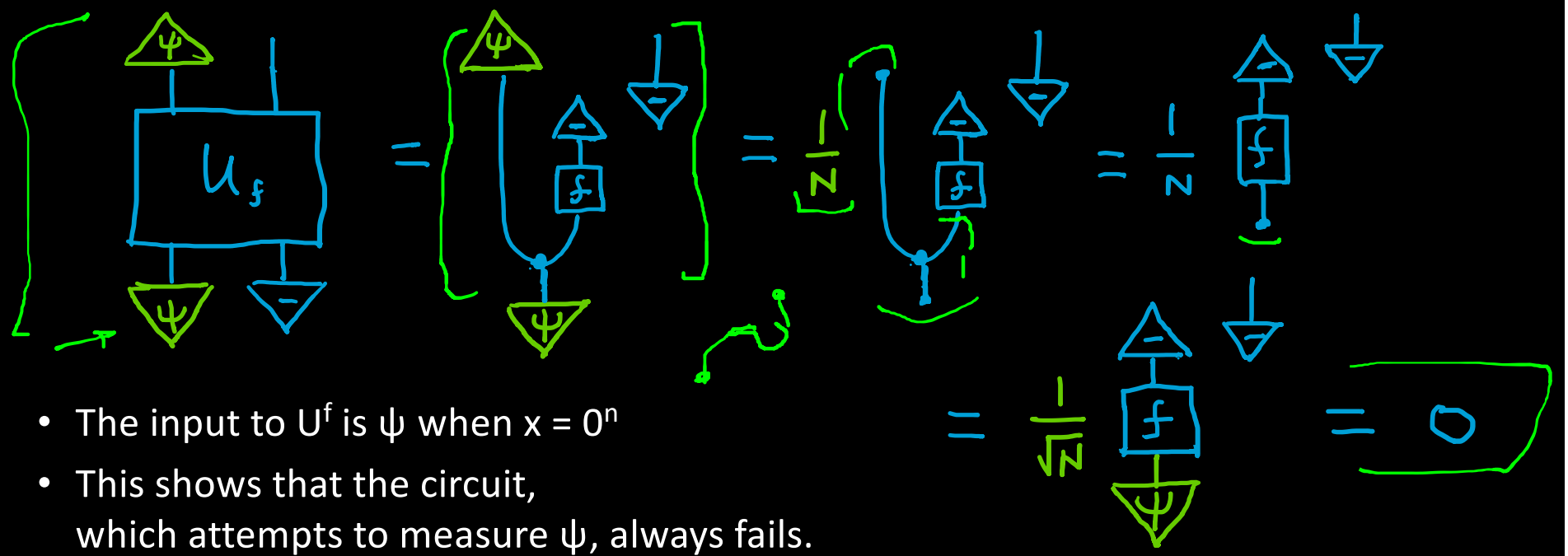
$\frac{\sqrt{N}}{2}|+\rangle$

$= 0$

$\frac{\sqrt{N}}{2}$

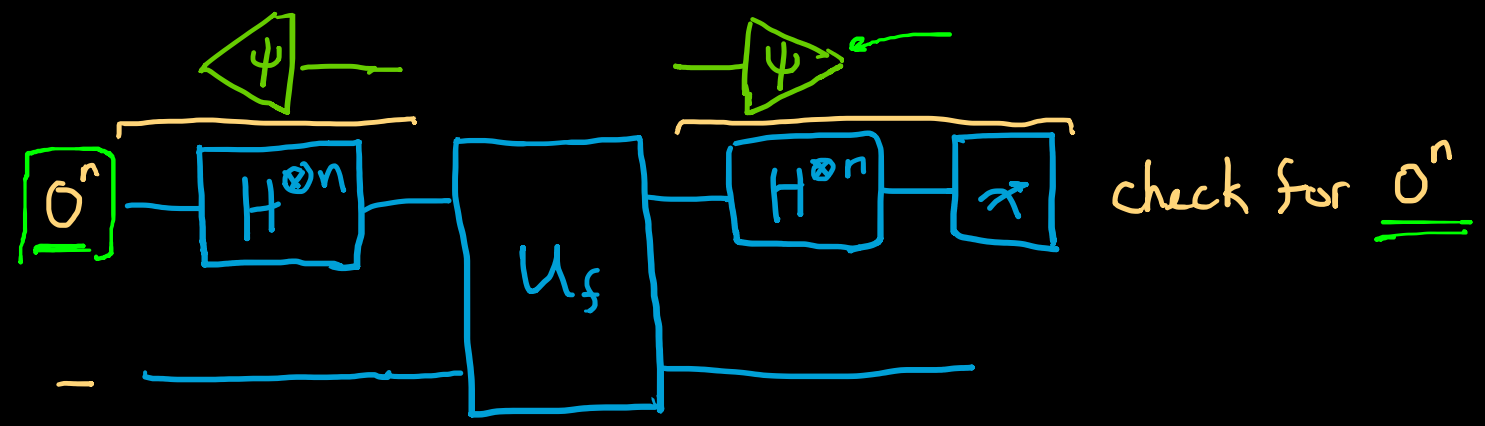
Deutsch-Josza with Diagrams

- Substituting this into the diagram above



- The input to U^f is ψ when $x = 0^n$
- This shows that the circuit,
  which attempts to measure ψ, always fails.
- I.e., we never get that measurement outcome (0^n)

Deutsch-Josza with Diagrams

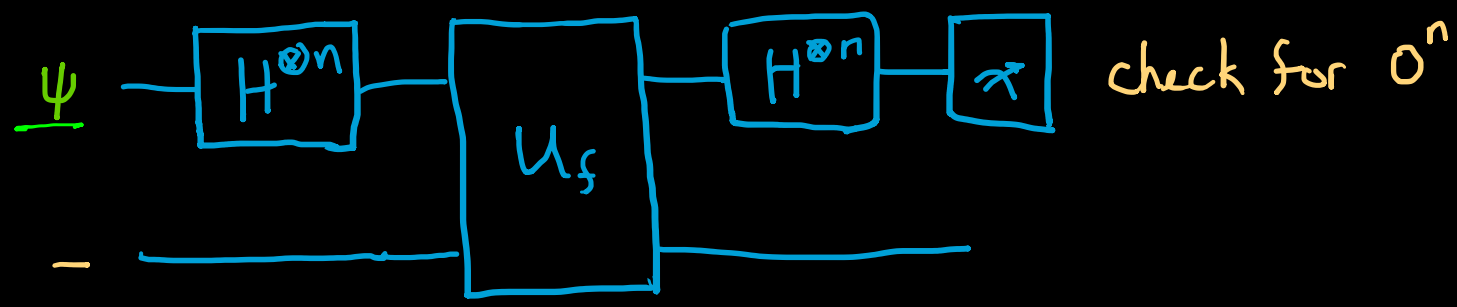


- The input to U_f is ψ when $x = 0^n$.
- The circuit always tries to measure ψ.

balanced.

- Our analysis showed that this never succeeds.

Deutsch-Josza with Diagrams



- If we use an input of ψ, then
  - we never measure 0^n when f is balanced
  - we always measure 0^n when f is constant

Grover with Diagrams

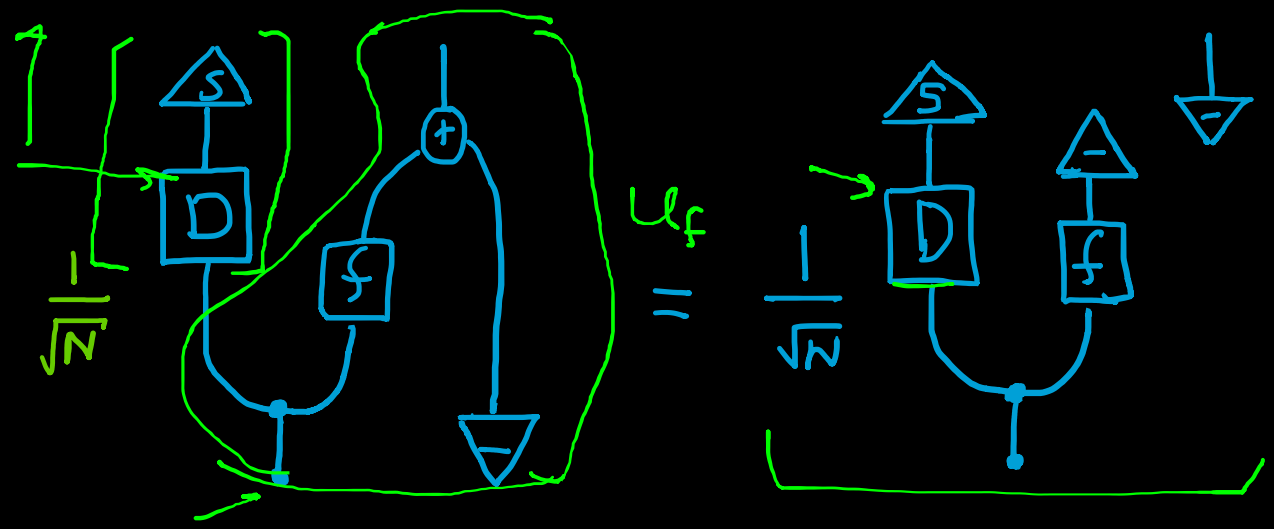
- We will consider the case when Grover only requires 1 iteration.

$n=2$ $f(x) = 1$

- This occurs when ¼ of all the inputs are 1. (Or if we have only 2 qubits.)
  - the algorithm from before generalizes to any constant fraction of 1s
  - (redo the analysis… only the number of iterations changes)

Grover with Diagrams

- This diagram describes a measurement result of ”s”

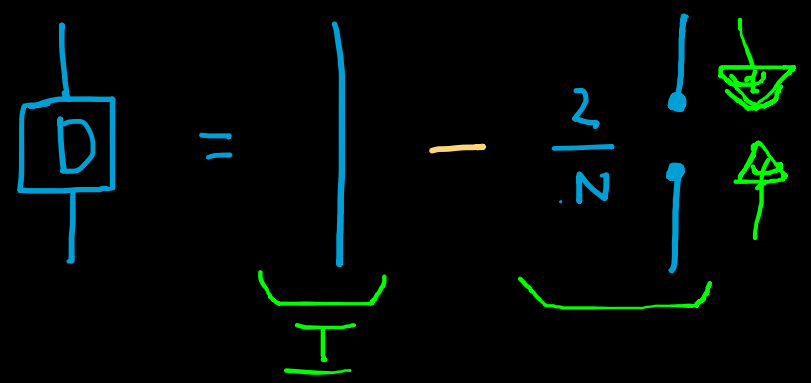


(constant factor is the scale difference ψ between and “prepare”)

$$D = 2|\psi\rangle\langle\psi| - I$$

Grover with Diagrams

- D operation is “yellow part”, given by $2|\psi\rangle\langle\psi| - I$

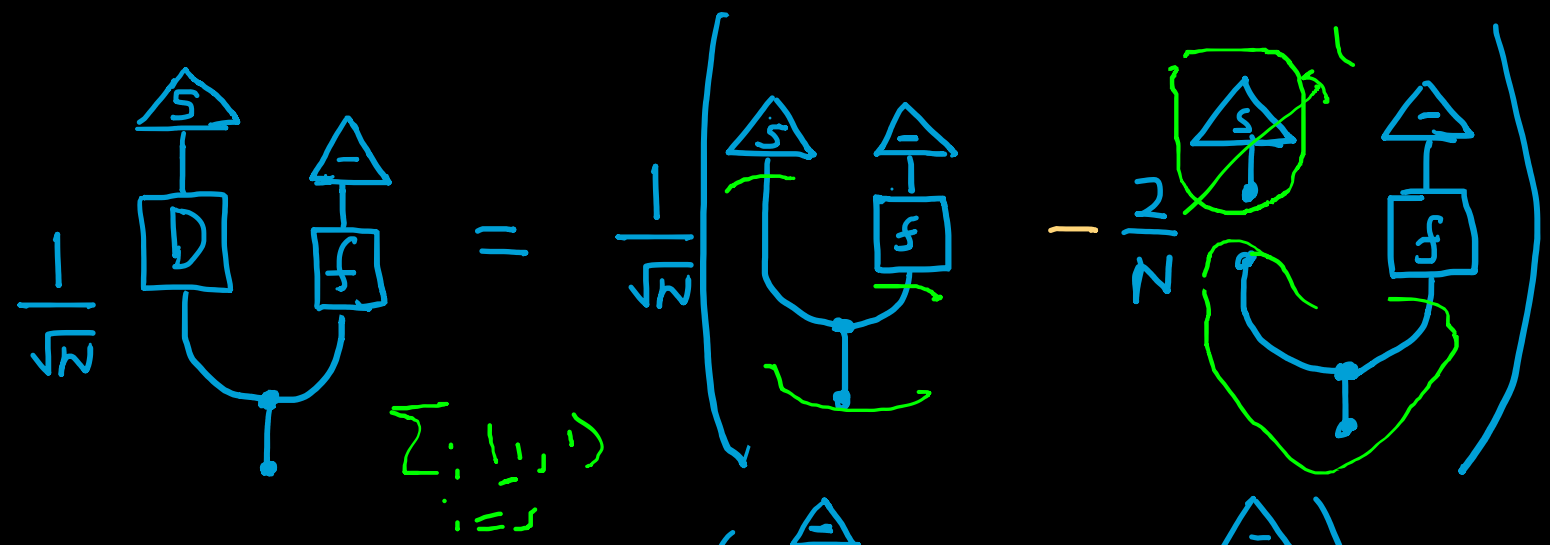


(We are now adding and subtracting diagrams.)

- Again, the scale factor (2/N) accounts for ψ vs erase / prepare

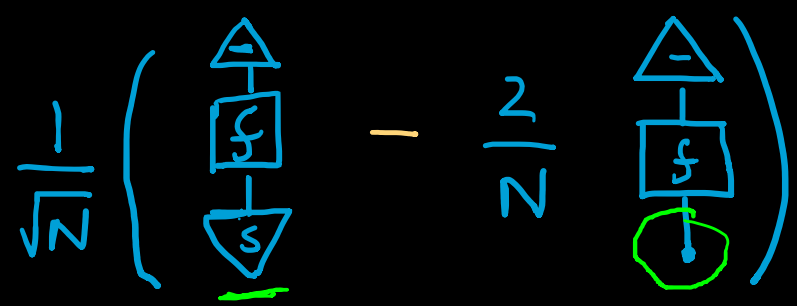
Grover with Diagrams

- Substituting this in, we have



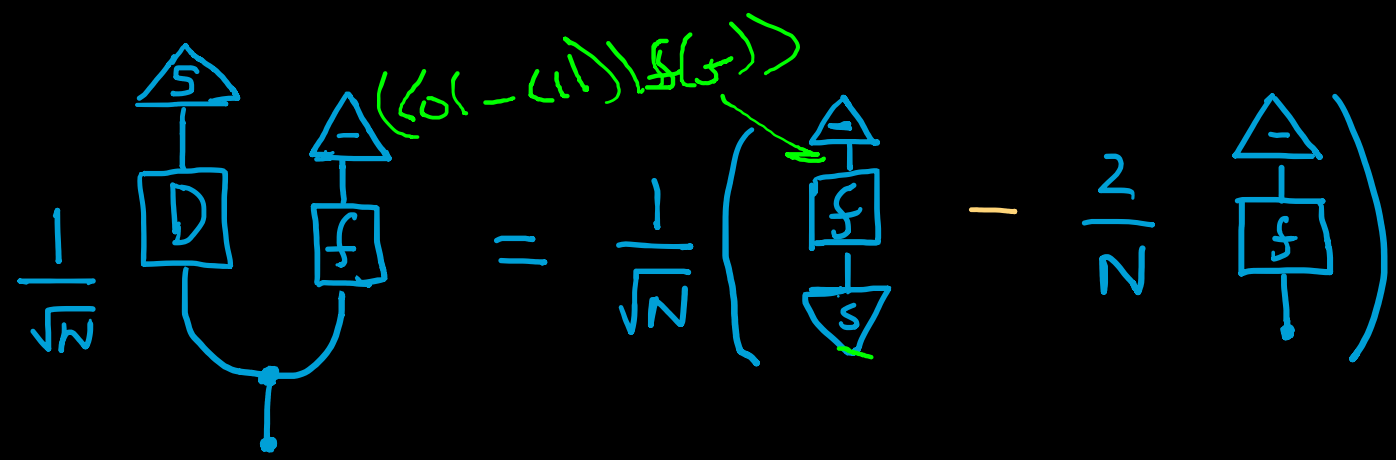
using $\langle s|\psi\rangle = 1$

$\langle s|\sum_i |i\rangle = 1$



Grover with Diagrams

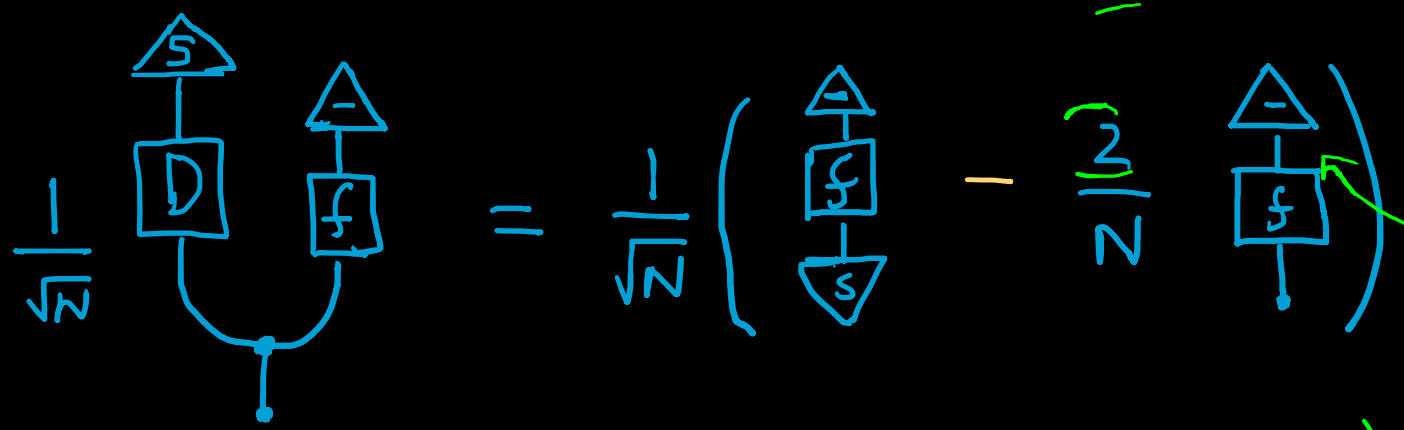
- Substituting this in, we have



- First term is <f(s)|-> = $\underbrace{-1^{f(s)}}$.

Grover with Diagrams

- Substituting this in, we have



$(|0\rangle\langle 1|)\cdot = \frac{\#0s}{N} - \frac{\#1s}{N}$

- Last term is 2 times the fraction of 0s minus 1s.

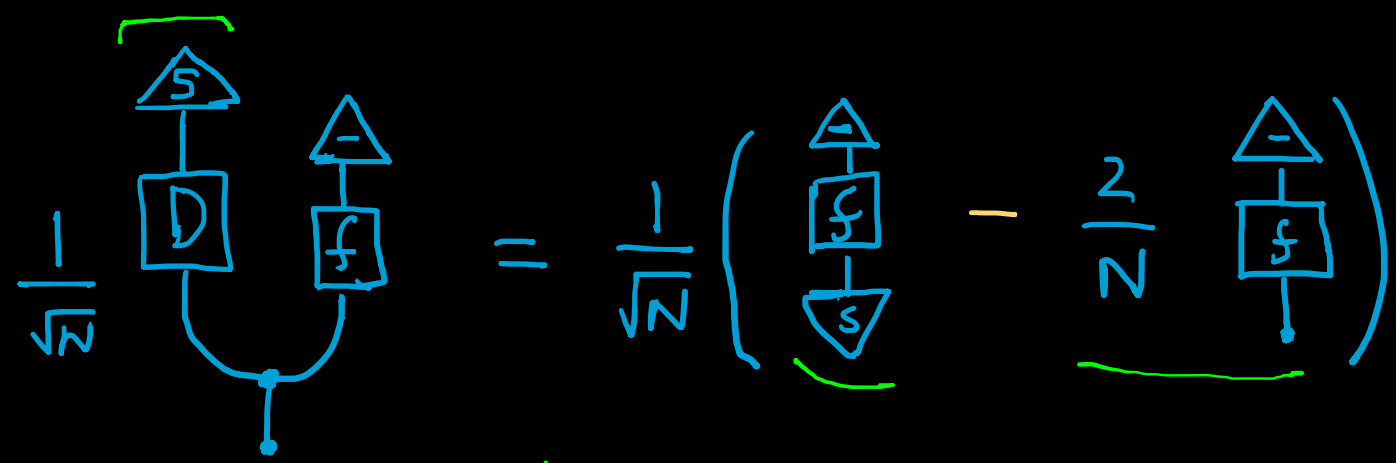
$f\left(\frac{1}{N}\sum_z |z\rangle\right) = \frac{1}{N}\sum_z f(z)$

$= \frac{\#0s}{N}|0\rangle + \frac{\#1s}{N}|1\rangle$

- If f has 1/4 1s, then we get 2(3/4 – 1/4) = 2(2/4) = 1

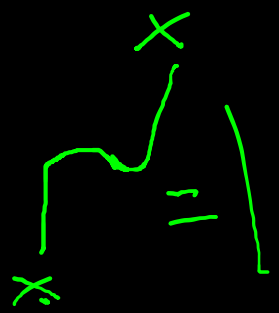
Grover with Diagrams

- Substituting this in, we have



- Value in parentheses is $-1^{f(s)} - 1$
- This is 0 if f(s) = 0, so the measurement never succeeds.
- I.e., we only measure s such that f(s) = 1.

Summary & Future Work



- Diagrammatic tools can be useful for understanding quantum processes
  - can reveal things hidden in the linear algebra
- We can make them more powerful if we go further into the math weeds.
- However, there is not any one notation that works perfectly for all situations.
- My thoughts on what is missing:
  - these methods only have equivalences
  - thinking back to CSE 311, we only get so far with equivalences
  - to go farther, we need inference mechanisms as well