Diagrammatic Reasoning (pt. 2)

CSE 490Q: Quantum Computation
Diagrammatic Reasoning

- Using diagrams to analyze and understand quantum processes
  - not the same as circuits
  - ignore more details of the computation

- Cup and cap
  - not simple descriptions of any circuit element
  - very strong, useful mathematical properties
  - make it easy to describe the trace of a matrix

- More examples today
• We can describe the Uf gate more nicely as follows:

\begin{align*}
\text{U}_{f} &= \begin{cases}
0 & \text{if } f(x) = 0 \\
1 & \text{if } f(x) = 1
\end{cases} \\
\end{align*}

where “+” takes \(|a,b\rangle\) to \(|a+b\rangle\)

• Note: this describes behavior not how it is computed
  • the “+” gate is not even unitary!

Analyzing an Oracle

\begin{itemize}
  \item \(x\) \(b\) \(\oplus f(x)\)
  \item \(U_f\) = \(+\)
  \item \(|x, b\rangle\rangle, (x, f(x), b)\rangle\rangle, (x, x, b)\rangle\rangle, (x, b)\rangle\rangle\)
\end{itemize}
"+" has its own special properties including:

\[
\begin{align*}
(a - b)(1 - c) &= a_1b_1 + b_1c_1 - c_1b_1 \\
(a_1 + b_1)(a_0 - b_0) &= -a_1b_1 - a_0b_0 + a_1b_0 + a_0b_1 \\
(a_1 - a_0)(b_1 + b_0) &= (a_1 - b_1)(a_0 - b_0) + (a_0 + b_0)(a_1 + b_1) \\
(a_1 - a_0)(b_1 - b_0) &= (a_1 - b_1)(a_0 - b_0) - (a_0 + b_0)(a_1 + b_1)
\end{align*}
\]
• We can describe the Uf gate more nicely as follows:

where “+” takes $|a,b\rangle$ to $|a+b\rangle$

• Note: this describes behavior not how it is computed
  • the “+” gate is not even unitary!
Problem: Determine whether f is constant or balanced

• The circuit we used to solve it was

• Let’s do the analysis this time using diagrams...
Suppose that $f$ is constant ($f(x) = b$ for all $x$, where $b = 0$ or $1$)

erases the input and replaces it with $b$
• Substituting this into the diagram above

• $<-|b> = (0<| - 1|)|b> = \pm 1$ is a global phase, so it has no effect

• The circuit is then just $H^\otimes n H^\otimes n = I$, so we always measure $\oplus$
Suppose that $f$ is balanced (half 0s, half 1s).

This time, we will use the fact that we know $U_f$ is applied to $|\psi\rangle$, an equal superposition over all possible values.

Then, we have

$$2^n = 2^n$$

$$f\left(\frac{1}{\sqrt{2^n}} \sum_{a=0}^{2^n-1} |a\rangle\right) = \frac{1}{\sqrt{2^n}} \sum_{a=0}^{2^n-1} f(a)$$

$$= \frac{1}{\sqrt{2^n}} \left( \frac{n}{2} |0\rangle + \frac{n}{2} |1\rangle \right) = \frac{\sqrt{n}}{2}(10 + 11) = \frac{\sqrt{n}}{2} 14$$
Substituting this into the diagram above:

- The input to $U_f$ is $\psi$ when $x = 0^n$
- This shows that the circuit, which attempts to measure $\psi$, always fails.
- I.e., we never get that measurement outcome ($0^n$)
• The input to $U_f$ is $\psi$ when $x = 0^n$.
• The circuit always tries to measure $\psi$.

• Our analysis showed that this never succeeds.

Deutsch-Josza with Diagrams
• If we use an input of $\psi$, then
  • we never measure $0^n$ when $f$ is balanced
  • we always measure $0^n$ when $f$ is constant
• We will consider the case when Grover only requires 1 iteration.

• This occurs when \( \frac{1}{4} \) of all the inputs are 1. (Or if we have only 2 qubits.)
  • the algorithm from before generalizes to any constant fraction of 1s
  • (redo the analysis... only the number of iterations changes)
This diagram describes a measurement result of “s”

(constant factor is the scale difference $\psi$ between and “prepare”)

$$D = 2\langle \psi \rangle |\psi \rangle - I$$
• D operation is “yellow part”, given by $2|\psi><\psi| - I$

(We are now adding and subtracting diagrams.)

• Again, the scale factor $(2/N)$ accounts for $\psi$ vs erase / prepare
Substituting this in, we have

\[
\frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} \left( \sum_i |\sum_i \rangle \right) - \frac{2}{N}
\]

using \( \langle s | \psi \rangle = 1 \)
Substituting this in, we have

\[ \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} \left( \langle 0 | - | 1 \rangle - \frac{2}{N} \right) \]

First term is \( \langle f(s) | - \rangle = -1^{f(s)} \).
Substituting this in, we have

\[ \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} \left( - \frac{2}{N} \right) \]

Last term is 2 times the fraction of 0s minus 1s.

If \( f \) has 1/4 1s, then we get \( 2(\frac{3}{4} - \frac{1}{4}) = 2(\frac{2}{4}) = 1 \)
Substituting this in, we have

\[
\frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} - \frac{2}{N}
\]

Value in parentheses is \(-1^{f(s)} - 1\)

This is 0 if \(f(s) = 0\), so the measurement never succeeds.

I.e., we only measure \(s\) such that \(f(s) = 1\).
Summary & Future Work

- Diagrammatic tools can be useful for understanding quantum processes
  - can reveal things hidden in the linear algebra

- We can make them more powerful if we go further into the math weeds.
  - However, there is not any one notation that works perfectly for all situations.

- My thoughts on what is missing:
  - these methods only have equivalences
  - thinking back to CSE 311, we only get so far with equivalences
  - to go farther, we need inference mechanisms as well