Product States (pt. 2)

CSE 490Q: Quantum Computation
• Not every state is of the form \((p|0> + q|1>) \otimes (s|0> + t|1>)\)

• Those that are are called **product states**
  • can describe the two qubits individually

• All other states are called **entangled states**
  • cannot describe the two qubits individually
  • accurate description requires describing them *together*
Some Entangled States

• An EPR pair is
  • $0.707|00\rangle + 0.707|11\rangle$

• The other Bell states are
  • $0.707|00\rangle - 0.707|11\rangle$
  • $0.707|01\rangle + 0.707|10\rangle$
  • $0.707|01\rangle - 0.707|10\rangle$

• All are “maximally entangled” states
Entangled States

\[(\alpha |0\rangle + \beta |1\rangle) \otimes (\gamma |0\rangle + \delta |1\rangle)\]

- Part of an entangled state cannot be described (by itself) as an ordinary state
  - its behavior (on its own) must be different than that of any one state

- From experiment, we find that it behaves like a probability distribution on states
  - it behaves as if the other side has been measured
  - as we saw, this is essentially forced on Nature by special relativity
Example Entangled State

- Consider the state

\[ \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \]

\[
\begin{bmatrix}
\Pr (0) = \frac{1}{2} \\
\Pr (1) = \frac{3}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
P_0 = \frac{1}{2} \\
P_1 = \frac{1}{2}
\end{bmatrix}
\]

\[
\begin{bmatrix}
|00\rangle \\
|10\rangle
\end{bmatrix} = (\frac{1}{\sqrt{2}} |0\rangle \otimes |1\rangle) + (\frac{1}{\sqrt{2}} |1\rangle \otimes |0\rangle)
\]
Entangled States

• Talking about part of an entangled state does not change it in any way
  • nothing “happens” to it
  • the rest of the state still exists

• It is still an ordinary quantum state on the full set of qubits
  • applying U to our part of a state is applying $U \otimes I$ to the whole state

• We just cannot describe its behavior in the usual manner without all the qubits
  • entangled means it cannot be described as a state all on its own
Pure States are Unentangled

- If a state is entangled, it behaves like a probability distribution on states
  - sometimes called a “mixed state”
  - an ordinary quantum state is called a “pure state”, by comparison

- **Contrapositive:** if a qubit is not behaving like a probability distribution, then it is not entangled with *anything else* (in the whole universe)
Pure States Are Unentangled

• Suppose our qubits are entangled with another one...

• Suppose we have $a|x> \otimes |0> + b|y> \otimes |1>$$\frac{\text{w.l.o.g. assume } |x| = 1 = |y|}{\text{first part is } |x> \text{ with probability } |a|^2 \text{ and } |y| \text{ with probability } |b|^2}$

• Our qubits would behave sometimes like $|x>$ and sometimes like $|y>$

• That is not ordinary state behavior unless $|x> = |y>$
  • in that case, this is $a|x> \otimes |0> + b|x> \otimes |1> = |x> \otimes (a|0> + b|1>)$, so this is a product state
If a state is entangled, it behaves like a probability distribution on states.

**Contrapositive:** if a qubit is not behaving like a probability distribution, then it is not entangled with *anything else* (in the whole universe).

Interestingly, maximally entangled states cannot be entangled with anything else.

- this is called “monogamy” of entanglement.
Monogamy of Entanglement

- Maximally entangled states cannot be entangled with anything else
  - this is called “monogamy” of entanglement

- An EPR is a shared resource
  - no one else could have access to it via entanglement
  - we will see numerous examples later on

- Entanglement measure try to quantify the entanglement between parts
  - $0 = \text{product}, 1 = \text{maximally entangled}$
  - but other amounts are also possible
Example of Monogamy

• Might think you could get an EPR pair from a 3-part entangled state

• Consider a GHZ state: $0.707 \, |000\rangle + 0.707 \, |111\rangle$

• Measuring the third bit gives:
  • $|00\rangle$ with probability $1/2$
  • $|11\rangle$ with probability $1/2$
  • this distribution on states is not the same as $0.707 \, |00\rangle + 0.707 \, |11\rangle$
Measurement Problem

• Measurement occurs when your state becomes entangled with the environment (the measuring device)

• Not necessary to think of something happening to the state
Product Operations

- If $U$ and $V$ are unitaries, then $U \otimes V$ is a 2-qubit unitary unitaries of this form
  - CNOT operation
  - AND operation (made reversible)
- Not all 2-qubit unitaries of this form
- We need multi-qubit operations to do useful things
Non-Product Operations

• If $U$ and $V$ are unitaries, then $U \otimes V$ is a 2-qubit unitary

• Not all 2-qubit unitaries of this form
  • CNOT operation
  • AND operation (made reversible)

• We need multi-qubit operations to do useful things
• Allowing any non-product operation is too powerful

• Any algorithm is implemented in a single operation
  • just multiply together the individual matrices

• Can solve the halting problem!
  • let code(P) be the Java code for a program
  • define H by \( H|\text{code}(P),0\rangle = |\text{code}(P),1\rangle \) if P halts on code(P)
    and \( |\text{code}(P),0\rangle \) otherwise
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Restricted Operations

• Fix the first problem by requiring operations to act on only $O(1)$ bits
  • just 2 is enough
  • but we may allow 3 or 4 bit operations for convenience

• One step is applying $I \otimes \ldots \otimes I \otimes U \otimes I \otimes \ldots \otimes I$
  • where U is 2x2 or 4x4
Restricted Operations

• Fix the first problem by requiring operations to act on only $O(1)$ bits

• Physics also requires this — the principle of “locality”
  • we cannot get 1b qubits close enough to act on them simultaneously

• Some complaints about quantum mechanics relate to this
  • it does not appear to be local: “spooky action at a distance”
  • but operations (changes to states) are local
  • entanglement causes our descriptions of states to be non-local