Product States (pt. 2)

CSE 490Q: Quantum Computation

Product States

- Not every state is of the form $(p|0\rangle + q|1\rangle) \otimes (s|0\rangle + t|1\rangle)$
- Those that are are called product states
  - can describe the two qubits individually
- All other states are called entangled states
  - cannot describe the two qubits individually
  - accurate description requires describing them *together*

Some Entangled States

- An EPR pair is $\frac{1}{\sqrt{2}}$
  - 0.707|00> + 0.707|11>
- The other Bell states are
  - 0.707|00> - 0.707|11>
  - 0.707|01> + 0.707|10>
  - 0.707|01> - 0.707|10>
- All are “maximally entangled” states

Entangled States

$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$

- Part of an entangled state cannot be described (by itself) as an ordinary state
  - its behavior (on its own) must be different than that of any one state

- From experiment, we find that it behaves like a probability distribution on states
  - it behaves as if the other side has been measured
  - as we saw, this is essentially forced on Nature by special relativity

Example Entangled State

- Consider the state $\boxed{\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle}$.

$$= |0\rangle \otimes \left(\frac{1}{\sqrt{2}}|1\rangle\right) + |1\rangle \otimes \left(\frac{1}{\sqrt{2}}|0\rangle\right).$$

$$\left[\begin{array}{ll} Pr(0) = \frac{1}{2} & Pr(1) = \frac{1}{2} \\ \hookrightarrow |1\rangle & \hookrightarrow |0\rangle \end{array}\right]$$

$$= \left(\frac{1}{\sqrt{2}}|0\rangle\right) \otimes |1\rangle + \left(\frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$$

$$\left[\begin{array}{ll} Pr(1) = \frac{1}{2} & Pr(0) = \frac{1}{2} \\ \hookrightarrow |0\rangle & \hookrightarrow |1\rangle \end{array}\right].$$

Entangled States

- Talking about part of an entangled state does not change it in any way
  - nothing “happens” to it
  - the rest of the state still exists

- It is still an ordinary quantum state on the full set of qubits
  - applying U to our part of a state is applying $U \otimes I$ to the whole state

- We just cannot *describe* its behavior in the usual manner without all the qubits
  - entangled means it cannot be described as a state all on its own

Pure States are Unentangled

- If a state is entangled, it behaves like a probability distribution on states
  - sometimes called a "mixed state"
  - an ordinary quantum state is called a "pure state", by comparison

- **Contrapositive**: if a qubit is not behaving like a probability distribution, then it is not entangled with *anything else* (in the whole universe)

Pure States Are Unentangled

- Suppose our qubits are entangled with another one…

Lab Andro

- Suppose we have $a|x\rangle \otimes |0\rangle + b|y\rangle \otimes |1\rangle$
  - w.l.o.g. assume $|x| = 1 = |y|$
  - first part is $|x\rangle$ with probability $|a|^2$ and $|y|$ with probability $|b|^2$

- Our qubits would behave sometimes like $|x\rangle$ and sometimes like $|y\rangle$

- That is not ordinary state behavior unless $\boxed{|x\rangle = |y\rangle}$
  - in that case, this is $a|x\rangle \otimes |0\rangle + b|x\rangle \otimes |1\rangle = |x\rangle \otimes (a|0\rangle + b|1\rangle)$, so this is a product state

Monogamy of Entanglement

- If a state is entangled, it behaves like a probability distribution on states
- **Contrapositive**: if a qubit is not behaving like a probability distribution, then it is not entangled with *anything else* (in the whole universe)
- Interestingly, maximally entangled states cannot be entangled with anything else
  - this is called "monogamy" of entanglement

Monogamy of Entanglement

- Maximally entangled states cannot be entangled with anything else
  - this is called "monogamy" of entanglement

- An EPR is a shared resource $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$
  - no one else could have access to it via entanglement
  - we will see numerous examples later on

- Entanglement measure try to quantify the entanglement between parts
  - 0 = product, 1 = maximally entangled
  - but other amounts are also possible

Example of Monogamy

$$\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \quad \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

- Might think you could get an EPR pair from a 3-part entangled state
- Consider a GHZ state: 0.707 |000> + 0.707 |111>

$$\frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

- Measuring the third bit gives:
  - |00> with probability 1/2
  - |11> with probability 1/2
  - this distribution on states is not the same as 0.707 |00> + 0.707 |11>

$$\left(\frac{1}{\sqrt{2}}|00\rangle\right)\otimes|0\rangle + \frac{1}{\sqrt{2}}\left(|11\rangle\right)\otimes|1\rangle$$

$$\boxed{\begin{array}{ll} \Pr\ 1/2 & \Pr\ 1/2 \\ |00\rangle & |11\rangle \end{array}}$$

$$\neq$$

$$\boxed{\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle}$$

Measurement Problem

- Measurement occurs when your state becomes entangled with the environment (the measuring device)

- Not necessary to think of something **happening to** the state

(basis of entanglement)

Product Operations

- If U and V are unitaries, then U $\otimes$ V is a 2-qubit unitary
- Not all 2-qubit unitaries of this form
  - CNOT operation
  - AND operation (made reversible)

$|a, b, 0\rangle$

$\mapsto |a, b, ab\rangle$

- We need multi-qubit operations to do useful things

Non-Product Operations

- If U and V are unitaries, then $U \otimes V$ is a 2-qubit unitary
- Not all 2-qubit unitaries of this form
  - CNOT operation
  - AND operation (made reversible)
- We need multi-qubit operations to do useful things

Unrestricted Operations

- Allowing any non-product operation is too powerful

n qubits

$\boxed{2^n \times 2^n}$

- Any algorithm is implemented in a single operation $O(1)$
  - just multiply together the individual matrices

$U_k \cdots U_1 U_2 U_1 = \boxed{U}$

- Can solve the halting problem!
  - let code(P) be the Java code for a program P
  - define H by H|code(P),0> = |code(P),1> if P halts on code(P)
    and |code(P),0> otherwise

Unrestricted Operations

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Restricted Operations

- Fix the first problem by requiring operations to act on only O(1) bits
  - just 2 is enough
  - but we may allow 3 or 4 bit operations for convenience
- One step is applying $I \otimes \ldots \otimes I \otimes U \otimes I \otimes \ldots \otimes I$
  - where U is 2x2 or 4x4

Restricted Operations

- Fix the first problem by requiring operations to act on only O(1) bits
- Physics also requires this — the principle of "locality"
  - we cannot get 1b qubits close enough to act on them simultaneously
- Some complaints about quantum mechanics relate to this
  - it does not appear to be local: "spooky action at a distance"
  - but *operations* (changes to states) are local
  - entanglement causes our *descriptions* of states to be non-local