Homework 8
CSE 490q

Due: Fri, Dec 11th by 11pm PST

1. In the Deutsch-Josza problem, the oracle f was assumed to be either constant ($f \equiv 0$ or $f \equiv 1$) or balanced, meaning that exactly half of the inputs give the value 1. Suppose that f were not perfectly balanced, but only approximately balanced, giving a value of 1 on k extra values (beyond the expected half of the inputs).

   (a) Assuming that f takes n bits as input (so there are 2^n possible inputs), what is the probability that measurement at the end of the Deutsch-Josza circuit gives a result of 0^n?

   Hint: you should not need to redo any of the analysis from lecture except the final measurement probability calculation. Note, however, that the slides have an error in that calculation, writing $2^{n/2}$ when it should say 2^{n-1}.

   (b) What is the largest that k can be while the algorithm still fails less than 1/3 of the time? What is this amount as a percentage of the inputs?

2. In this problem, we will look at the how to implement the quantum Fourier transform $\mathcal{F}$ used, for example, in our phase estimation algorithm.

   (a) Write out the matrix for the quantum Fourier transform on 1 qubit, $\mathcal{F}_1$.

   (b) What is other (more familiar) name for the matrix from part (a)?

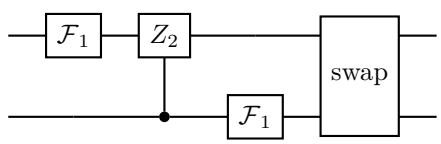
   (c) Write out the matrix for the quantum Fourier transform on 2 qubits, $\mathcal{F}_2$. The formula for $\mathcal{F}$ assumes that the states are numbers $0, 1, \ldots, 2^n - 1$, but recall that we can identify numbers with their corresponding binary representation: $00 = 0$, $01 = 1$, $10 = 2$, $11 = 3$.

   (d) Let Z_k be the matrix defined by

   $$Z_k := \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}$$

   What is the other (more familiar) name for Z_1? What is the matrix for Z_2?

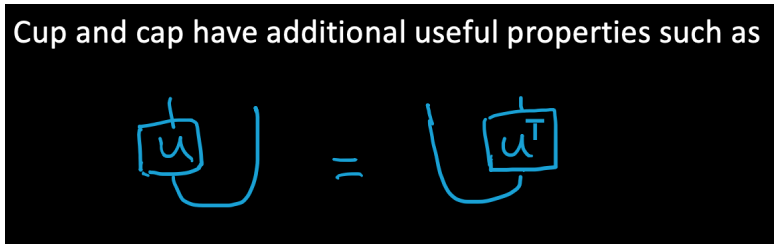
   (e) Verify that the following circuit implements $\mathcal{F}_2$:

   Hint: it is sufficient to calculate the result of the circuit for $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and verify that they match those from the matrix in part (b).

In part (e), above, we showed that $\mathcal{F}_2 = (I \otimes \mathcal{F}_1)[CZ_2]_{2,1}(\mathcal{F}_1 \otimes I)$ followed by a reversal of the bits. In general, it can be shown that $\mathcal{F}_n = (I \otimes \mathcal{F}_1)[CZ_n]_{n,1}[CZ_{n-1}]_{n,2} \cdots [CZ_2]_{n,n-1}(\mathcal{F}_{n-1} \otimes I)$ followed by a reversal of all the qubits. (See the textbook for a proof.) This recursive formula makes it fairly easy to draw circuits for $\mathcal{F}_3$, $\mathcal{F}_4$, and so on.

3. Lecture 19, on diagrammatic reasoning, included the following claim[1]:



Prove, from the linear algebra definitions, that this claim is true assuming that the wire represents a single qubit and

$$U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Hint: recall that the cup shape is shorthand for a "prepare" followed by a "copy", both operations that were defined via linear algebra in the lecture slides.

4. In lecture, we discussed matrices formed by multiplying one of $\{I, X, Z, XZ\}$ by one of $\{1, -1, i, -i\}$. These sixteen matrices are called the "Pauli group". Show that the inverse of any matrix in the Pauli group is another matrix in the Pauli group.

   Hint: note that all of these matrices are unitary.

Suppose that we want to measure our state $|\phi\rangle$ to see if we get the result x (e.g., x could be 010 if this is a 3-qubit state). In lecture, we learned that the probability of this occuring is $|c|^2$, where c is the coefficient of $|x\rangle$ in the vector $|\phi\rangle$. Since $c = \langle x|\phi\rangle$, we can also write this as

$$\begin{aligned} |c|^2 &= c\bar{c} \\ &= \langle x|\phi\rangle \overline{(\langle x|\phi\rangle)} \\ &= \langle x|\phi\rangle \langle\phi|x\rangle \\ &= \mathrm{tr}(\langle x|\phi\rangle \langle\phi|x\rangle) \\ &= \mathrm{tr}(|\phi\rangle\langle\phi| \, |x\rangle\langle x|) \end{aligned}$$

where we have used the fact that $\mathrm{tr}(a) = a$, when a is just a number, and the cyclical property of the trace (i.e., that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$), proven in lecture.

5. Suppose that we have a mixed state that is state $|\phi\rangle$ with probability p and state $|\xi\rangle$ with probability $1-p$. Define the matrix $\rho = p\,|\phi\rangle\langle\phi| + (1-p)\,|\xi\rangle\langle\xi|$. Prove that $\mathrm{tr}(\rho\,|x\rangle\langle x|)$ is the probability that we get the result x when measuring this mixed state.

   Hint: use the properties of trace, in particular, the fact that it is a linear mapping.

A matrix like ρ above is called *density matrix*. It is an alternative way of representing a mixed state. A pure state like $|\phi\rangle$ has a corresponding density matrix $|\phi\rangle\langle\phi|$ with rank 1.

6. Consider a mixed qubit state that is $|\phi\rangle$ or some orthogonal state $|\phi^\perp\rangle$, each with probability 1/2. These are called "completely mixed" states. Prove that the density matrices corresponding to *all* completely mixed qubit states are the same. Identify that density matrix.

   Hint: define $V = |\phi\rangle\langle 0| + |\phi^\perp\rangle\langle 1|$, representing a change of basis from $|0\rangle$ and $|1\rangle$ to $|\phi\rangle$ and $|\phi^\perp\rangle$, respectively. Write $|\phi\rangle = V\,|0\rangle$ and $|\phi^\perp\rangle = V\,|1\rangle$ and then calculate the density matrix.

[1] Actually, the slides said $U^\dagger$ rather than U^T, but the latter is correct.

Problem 6 and the calculation prior to it showed that we can calculate a measurement probability for a pure or mixed state using just the density matrix. If our density matrix is ρ, then the probability of measuring x is given by $\mathrm{tr}(\rho\,|x\rangle\langle x|)$.

It is also possible to calculate the density matrix representing the mixed state that results from applying a unitary U to an initial mixed state. In our example above, we would have $U\,|\phi\rangle$ with probability p and $U\,|\xi\rangle$ with probability $1-p$, and the corresponding density matrix would be

$$p(U\,|\phi\rangle)(U\,|\phi\rangle)^\dagger + (1-p)(U\,|\xi\rangle)(U\,|\xi\rangle)^\dagger = p\;U\,|\phi\rangle\langle\phi|\,U^\dagger + (1-p)\;U\,|\xi\rangle\langle\xi|\,U^\dagger = U\rho U^\dagger$$

Hence, if we start in a mixed state with density matrix ρ, then the mixed state we get after applying U to our state has a density matrix of $U\rho U^\dagger$.

7. Explain why all completely mixed qubit states are indistinguishable from one another.

   Hint: Use your answers to Problem 5 and 6 along with the comments, just above, about how unitary operations change density matrices.