## CSE 490 GZ
## Introduction to Data Compression
Winter 2004

Huffman Coding

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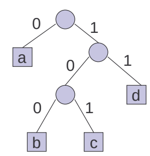
## Huffman Coding

- Huffman (1951)
- Uses frequencies of symbols in a string to build a variable rate prefix code.
  - Each symbol is mapped to a binary string.
  - More frequent symbols have shorter codes.
  - No code is a prefix of another.
- Example:

a 0
b 100
c 101
d 11



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## Variable Rate Code Example

- Example: a 0, b 100, c 101, d 11
- Coding:
  - aabddcaa = 16 bits
  - 0 0 100 11 11 101 0 0= 14 bits
- Prefix code ensures unique decodability.
  - 00100111110100
  - a a b d d c a a



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## Cost of a Huffman Tree

- Let $p_1, p_2, \ldots, p_m$ be the probabilities for the symbols $a_1, a_2, \ldots, a_m$, respectively.
- Define the cost of the Huffman tree T to be

$$C(T) = \sum_{i=1}^{m} p_i r_i$$

where $r_i$ is the length of the path from the root to $a_i$.
- C(T) is the expected length of the code of a symbol coded by the tree T. C(T) is the bit rate of the code.



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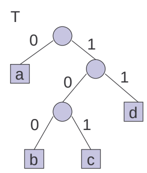
## Example of Cost

- Example: a 1/2, b 1/8, c 1/8, d 1/4




C(T) = 1 x 1/2 + 3 x 1/8 + 3 x 1/8 + 2 x 1/4 = 1.75
a b c d



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## Huffman Tree

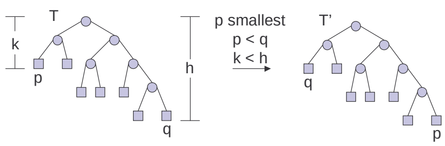
- Input: Probabilities $p_1, p_2, \ldots, p_m$ for symbols $a_1, a_2, \ldots, a_m$, respectively.
- Output: A tree that minimizes the average number of bits (bit rate) to code a symbol. That is, minimizes

$$HC(T) = \sum_{i=1}^{m} p_i r_i \quad \text{bit rate}$$

where $r_i$ is the length of the path from the root to $a_i$. This is the Huffman tree or Huffman code

## Optimality Principle 1

- In an Huffman tree a lowest probability symbol has maximum distance from the root.
  - If not exchanging a lowest probability symbol with one at maximum distance will lower the cost.

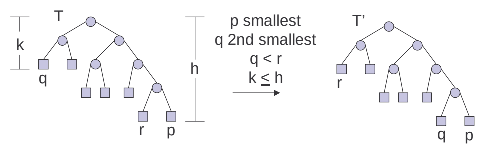



C(T') = C(T) + hp - hq + kq - kp = C(T) - (h-k)(q-p) < C(T)



## Optimality Principle 2

- The second lowest probability is a sibling of the the smallest in some Huffman tree.
  - If not, we can move it there not raising the cost.

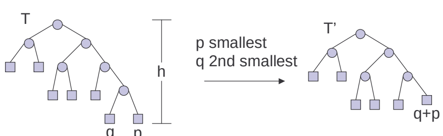



C(T') = C(T) + hq - hr + kr - kq = C(T) - (h-k)(r-q) ≤ C(T)



## Optimality Principle 3

- Assuming we have a Huffman tree T whose two lowest probability symbols are siblings at maximum depth, they can be replaced by a new symbol whose probability is the sum of their probabilities.
  - The resulting tree is optimal for the new symbol set.

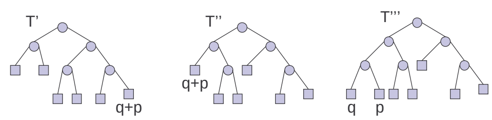



C(T') = C(T) + (h-1)(p+q) - hp -hq = C(T) - (p+q)



## Optimality Principle 3 (cont')

- If T' were not optimal then we could find a lower cost tree T''. This will lead to a lower cost tree T''' for the original alphabet.




C(T''') = C(T'') + p + q < C(T') + p + q = C(T) which is a contradiction



## Recursive Huffman Tree Algorithm

1. If there is just one symbol, a tree with one node is optimal. Otherwise
2. Find the two lowest probability symbols with probabilities p and q respectively.
3. Replace these with a new symbol with probability p + q.
4. Solve the problem recursively for new symbols.
5. Replace the leaf with the new symbol with an internal node with two children with the old symbols.

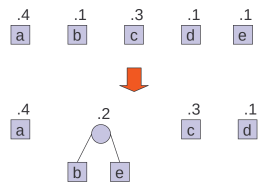


## Iterative Huffman Tree Algorithm

```
form a node for each symbol a_i with weight p_i;
insert the nodes in a min priority queue ordered by probability;
while the priority queue has more than one element do
   min1 := delete-min;
   min2 := delete-min;
   create a new node n;
   n.weight := min1.weight + min2.weight;
   n.left := min1;
   n.right := min2;
   insert(n)
return the last node in the priority queue.
```

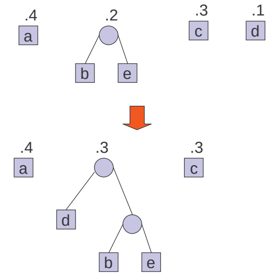
## Example of Huffman Tree Algorithm (1)

- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1



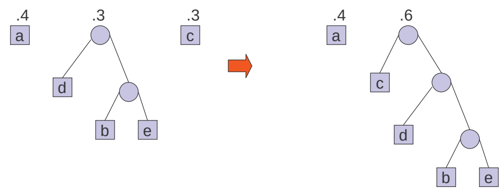



## Example of Huffman Tree Algorithm (2)



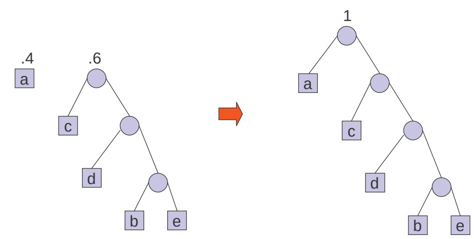



## Example of Huffman Tree Algorithm (3)



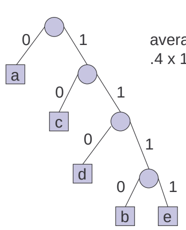



## Example of Huffman Tree Algorithm (4)






## Huffman Code




average number of bits per symbol is
.4 x 1 + .1 x 4 + .3 x 2 + .1 x 3 + .1 x 4 = 2.1

| | |
|---|---|
| a | 0 |
| b | 1110 |
| c | 10 |
| d | 110 |
| e | 1111 |



## Optimal Huffman Code vs. Entropy

- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1

### Entropy

$$H = -(.4 \times \log_2(.4) + .1 \times \log_2(.1) + .3 \times \log_2(.3) + .1 \times \log_2(.1) + .1 \times \log_2(.1))$$
= 2.05 bits per symbol

### Huffman Code

HC = .4 x 1 + .1 x 4 + .3 x 2 + .1 x 3 + .1 x 4
= 2.1 bits per symbol
pretty good!

## In Class Exercise

- P(a) = 1/2, P(b) = 1/4, P(c) = 1/8, P(d) = 1/16, P(e) = 1/16
- Compute the Huffman tree and its bit rate.
- Compute the Entropy
- Compare
- Hint: For the tree change probabilities to be integers: a:8, b:4, c:2, d:1, e:1. Normalize at the end.

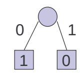


## Quality of the Huffman Code

- The Huffman code is within one bit of the entropy lower bound.

$$H \le HC \le H+1$$

- Huffman code does not work well with a two symbol alphabet.
  - Example: P(0) = 1/100, P(1) = 99/100
  - HC = 1 bits/symbol




  - H = $-((1/100)*\log_2(1/100) + (99/100)\log_2(99/100))$ = .08 bits/symbol



## Powers of Two

- If all the probabilities are powers of two then

$$HC = H$$

- Proof by induction on the number of symbols.

Let $p_1 \le p_2 \le \ldots \le p_n$ be the probabilities that add up to 1

If n = 1 then HC = H (both are zero).

If n > 1 then $p_1 = p_2 = 2^{-k}$ for some k, otherwise the sum cannot add up to 1.

Combine the first two symbols into a new symbol of probability $2^{-k} + 2^{-k} = 2^{-k+1}$.



## Powers of Two (Cont.)

By the induction hypothesis

$$\begin{aligned}
&HC(p_1+p_2,p_3,\ldots,p_n) = H(p_1+p_2,p_3,\ldots,p_n) \\
&= -(p_1+p_2)\log_2(p_1+p_2) - \sum_{i=3}^{n} p_i\log_2(p_i) \\
&= -2^{-k+1}\log_2(2^{-k+1}) - \sum_{i=3}^{n} p_i\log_2(p_i) \\
&= -2^{-k+1}(\log_2(2^{-k})+1) - \sum_{i=3}^{n} p_i\log_2(p_i) \\
&= -2^{-k}\log_2(2^{-k}) - 2^{-k}\log_2(2^{-k}) - \sum_{i=3}^{n} p_i\log_2(p_i) - 2^{-k} - 2^{-k} \\
&= -\sum_{i=1}^{n} p_i\log_2(p_i) - (p_1+p_2) \\
&= H(p_1,p_2,\ldots,p_n) - (p_1+p_2)
\end{aligned}$$



## Powers of Two (Cont.)

By the previous page,

$$HC(p_1+p_2,p_3,\ldots,p_n) = H(p_1,p_2,\ldots,p_n) - (p_1+p_2)$$

By the properties of Huffman trees (principle 3),

$$HC(p_1,p_2,\ldots,p_n) = HC(p_1+p_2,p_3,\ldots,p_n) + (p_1+p_2)$$

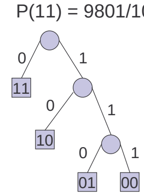
Hence,

$$HC(p_1,p_2,\ldots,p_n) = H(p_1,p_2,\ldots,p_n)$$



## Extending the Alphabet

- Assuming independence P(ab) = P(a)P(b), so we can lump symbols together.
- Example: P(0) = 1/100, P(1) = 99/100
  - P(00) = 1/10000, P(01) = P(10) = 99/10000, P(11) = 9801/10000.




HC = 1.03 bits/symbol (2 bit symbol)
= .515 bits/bit

Still not that close to H = .08 bits/bit

# Quality of Extended Alphabet

- Suppose we extend the alphabet to symbols of length k then

$$H \leq HC \leq H + 1/k$$

- Pros and Cons of Extending the alphabet
  - \+ Better compression
  - \- $2^k$ symbols
  - \- padding needed to make the length of the input divisible by k



# Huffman Codes with Context

- Suppose we add a one symbol context. That is in compressing a string $x_1x_2...x_n$ we want to take into account $x_{k-1}$ when encoding $x_k$.
  – New model, so entropy based on just independent probabilities of the symbols doesn't hold. The new entropy model (2nd order entropy) has for each symbol a probability for each other symbol following it.
  – Example: {a,b,c}

next

| prev | a | b | c |
|---|---|---|---|
| a | .4 | .2 | .4 |
| b | .1 | .9 | 0 |
| c | .1 | .1 | .8 |

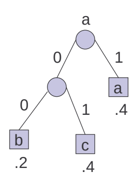


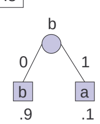
# Multiple Codes

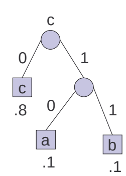
next

| prev | a | b | c |
|---|---|---|---|
| a | .4 | .2 | .4 |
| b | .1 | .9 | 0 |
| c | .1 | .1 | .8 |

| Code for first symbol |
|---|
| a 00 |
| b 01 |
| c 10 |




a b b a c c

<u>00</u> <u>00</u> <u>0</u> <u>1</u> <u>01</u> <u>0</u>



# Complexity of Huffman Code Design

- Time to design Huffman Code is O(n log n) where n is the number of symbols.
  – Each step consists of a constant number of priority queue operations (2 deletemin's and 1 insert)



# Approaches to Huffman Codes

1. Frequencies computed for each input
   – Must transmit the Huffman code or frequencies as well as the compressed input
   – Requires two passes
2. Fixed Huffman tree designed from training data
   – Do not have to transmit the Huffman tree because it is known to the decoder.
   – H.263 video coder
3. Adaptive Huffman code
   – One pass
   – Huffman tree changes as frequencies change