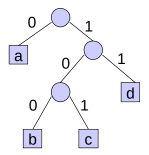
CSE 490 GZ
Introduction to Data Compression
Winter 2002

Huffman Coding

Huffman Coding

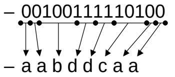
- Huffman (1951)
- Uses frequencies of symbols in a string to build a variable rate prefix code.
  - Each symbol is mapped to a binary string.
  - More frequent symbols have shorter codes.
  - No code is a prefix of another.
- Example:

a	0
b	100
c	101
d	11



Variable Rate Code Example

- Example: a 0, b 100, c 101, d 11
- Coding:
  - aabddcaa = 16 bits
  - 0 0 100 11 11 101 0 0= 14 bits
- Prefix code ensures unique decodability.
  - 00100111110100



  - a a b d d c a a

Cost of a Huffman Tree

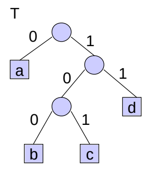
- Let $p_1, p_2, \ldots, p_m$ be the probabilities for the symbols $a_1, a_2, \ldots, a_m$, respectively.
- Define the cost of the Huffman tree T to be

$$C(T) = \sum_{i=1}^{m} p_i r_i$$

where r_i is the length of the path from the root to a_i.
- C(T) is the expected length of the code of a symbol coded by the tree T. C(T) is the bit rate of the code.

Example of Cost

- Example: a 1/2, b 1/8, c 1/8, d 1/4



$$C(T) = \underbrace{1 \times 1/2}_{a} + \underbrace{3 \times 1/8}_{b} + \underbrace{3 \times 1/8}_{c} + \underbrace{2 \times 1/4}_{d} = 1.75$$

Huffman Tree

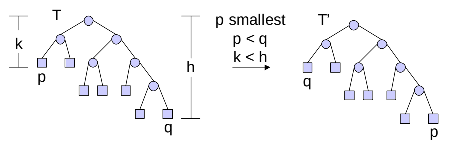
- Input: Probabilities $p_1, p_2, \ldots, p_m$ for symbols $a_1, a_2, \ldots, a_m$, respectively.
- Output: A tree that minimizes the average number of bits (bit rate) to code a symbol. That is, minimizes

$$HC(T) = \sum_{i=1}^{m} p_i r_i \quad \text{bit rate}$$

where r_i is the length of the path from the root to a_i. This is the Huffman tree or Huffman code

Optimality Principle 1

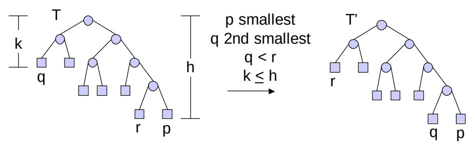
- In an Huffman tree a lowest probability symbol has maximum distance from the root.
  - If not exchanging a lowest probability symbol with one at maximum distance will lower the cost.



$C(T') = C(T) + hp - hq + kq - kp = C(T) - (h-k)(q-p) < C(T)$

Optimality Principle 2

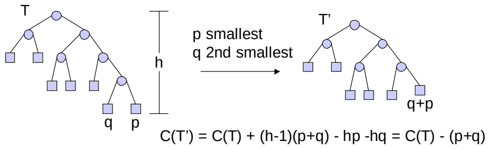
- The second lowest probability is a sibling of the the smallest in some Huffman tree.
  - If not, we can move it there not raising the cost.



$C(T') = C(T) + hq - hr + kr - kq = C(T) - (h-k)(r-q) \leq C(T)$

Optimality Principle 3

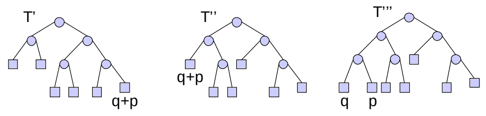
- Assuming we have a Huffman tree T whose two lowest probability symbols are siblings at maximum depth, they can be replaced by a new symbol whose probability is the sum of their probabilities.
  - The resulting tree is optimal for the new symbol set.



$C(T') = C(T) + (h-1)(p+q) - hp - hq = C(T) - (p+q)$

Optimality Principle 3 (cont')

- If T' were not optimal then we could find a lower cost tree T''. This will lead to a lower cost tree T''' for the original alphabet.



$C(T''') = C(T'') + p + q < C(T') + p + q = C(T)$ which is a contradiction

Recursive Huffman Tree Algorithm

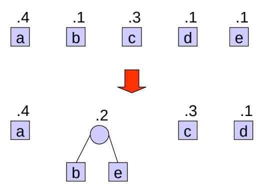
1. If there is just one symbol, a tree with one node is optimal. Otherwise
2. Find the two lowest probability symbols with probabilities p and q respectively.
3. Replace these with a new symbol with probability p + q.
4. Solve the problem recursively for new symbols.
5. Replace the leaf with the new symbol with an internal node with two children with the old symbols.

Iterative Huffman Tree Algorithm

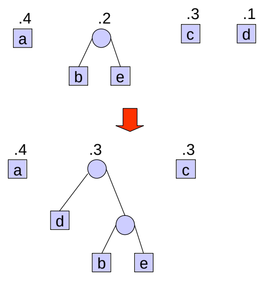
```
form a node for each symbol $a_i$ with weight $p_i$;
insert the nodes in a min priority queue ordered by probability;
while the priority queue has more than one element do
    min1 := delete-min;
    min2 := delete-min;
    create a new node n;
    n.weight := min1.weight + min2.weight;
    n.left := min1;
    n.right := min2;
    insert(n)
return the last node in the priority queue.
```

Example of Huffman Tree Algorithm (1)

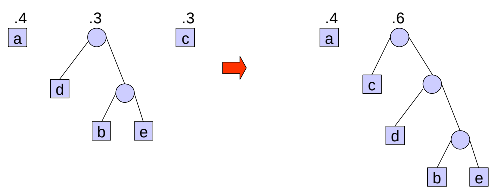
- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1



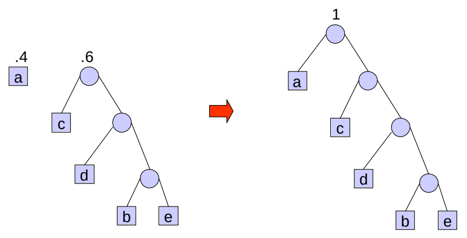
Example of Huffman Tree Algorithm (2)



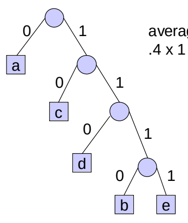
Example of Huffman Tree Algorithm (3)



Example of Huffman Tree Algorithm (4)



Huffman Code



average number of bits per symbol is
.4 x 1 + .1 x 4 + .3 x 2 + .1 x 3 + .1 x 4 = 2.1

a	0
b	1110
c	10
d	110
e	1111

Optimal Huffman Code vs. Entropy

- P(a) =.4, P(b)=.1, P(c)=.3, P(d)=.1, P(e)=.1

Entropy

$$H = -(.4 \times \log_2(.4) + .1 \times \log_2(.1) + .3 \times \log_2(.3) + .1 \times \log_2(.1) + .1 \times \log_2(.1))$$
$$= 2.05 \text{ bits per symbol}$$

Huffman Code

$$HC = .4 \times 1 + .1 \times 4 + .3 \times 2 + .1 \times 3 + .1 \times 4$$
$$= 2.1 \text{ bits per symbol}$$

pretty good!

In Class Exercise

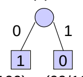
- P(a) = 1/2, P(b) = 1/4, P(c) = 1/8, P(d) = 1/16, P(e) = 1/16
- Compute the Huffman tree and its bit rate.
- Compute the Entropy
- Compare
- Hint: For the tree change probabilities to be integers: a:8, b:4, c:2, d:1, e:1. Normalize at the end.

Quality of the Huffman Code

- The Huffman code is within one bit of the entropy lower bound.

$$H \le HC \le H + 1$$

- Huffman code does not work well with a two symbol alphabet.
  - Example: P(0) = 1/100, P(1) = 99/100
  - HC = 1 bits/symbol



  - H = -((1/100)*$\log_2$(1/100) + (99/100)$\log_2$(99/100)) = .08 bits/symbol

Powers of Two

- If all the probabilities are powers of two then

$$HC = H$$

- Proof by induction on the number of symbols.

Let $p_1 \le p_2 \le \ldots \le p_n$ be the probabilities that add up to 1

If n = 1 then HC = H (both are zero).

If n > 1 then $p_1 = p_2 = 2^{-k}$ for some k, otherwise the sum cannot add up to 1.

Combine the first two symbols into a new symbol of probability $2^{-k} + 2^{-k} = 2^{-k+1}$.

Powers of Two (Cont.)

By the induction hypothesis

$$\begin{aligned}
HC(p_1+p_2,p_3,\ldots,p_n) &= H(p_1+p_2,p_3,\ldots,p_n)\\
&= -(p_1+p_2)\log_2(p_1+p_2) - \sum_{i=3}^{n} p_i \log_2(p_i)\\
&= -2^{-k+1}\log_2(2^{-k+1}) - \sum_{i=3}^{n} p_i \log_2(p_i)\\
&= -2^{-k+1}(\log_2(2^{-k})+1) - \sum_{i=3}^{n} p_i \log_2(p_i)\\
&= -2^{-k}\log_2(2^{-k}) - 2^{-k}\log_2(2^{-k}) - \sum_{i=3}^{n} p_i \log_2(p_i) - 2^{-k} - 2^{-k}\\
&= -\sum_{i=1}^{n} p_i \log_2(p_i) - (p_1+p_2)\\
&= H(p_1,p_2,\ldots,p_n) - (p_1+p_2)
\end{aligned}$$

Powers of Two (Cont.)

By the previous page,

$$HC(p_1+p_2,p_3,\ldots,p_n) = H(p_1,p_2,\ldots,p_n) - (p_1+p_2)$$

By the properties of Huffman trees (principle 3),

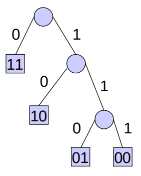
$$HC(p_1,p_2,\ldots,p_n) = HC(p_1+p_2,p_3,\ldots,p_n) + (p_1+p_2)$$

Hence,

$$HC(p_1,p_2,\ldots,p_n) = H(p_1,p_2,\ldots,p_n)$$

Extending the Alphabet

- Assuming independence P(ab) = P(a)P(b), so we can lump symbols together.
- Example: P(0) = 1/100, P(1) = 99/100
  - P(00) = 1/10000, P(01) = P(10) = 99/10000, P(11) = 9801/10000.



HC = 1.03 bits/symbol (2 bit symbol) = .515 bits/bit

Still not that close to H = .08 bits/bit

Quality of Extended Alphabet

- Suppose we extend the alphabet to symbols of length k then

$$H \le HC \le H + 1/k$$

- Pros and Cons of Extending the alphabet
  - + Better compression
  - - 2^k symbols
  - - padding needed to make the length of the input divisible by k

Huffman Codes with Context

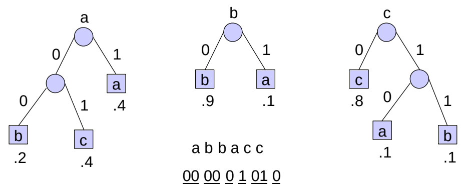
- Suppose we add a one symbol context. That is in compressing a string $x_1x_2\ldots x_n$ we want to take into account x_{k-1} when encoding x_k.
  – New model, so entropy based on just independent probabilities of the symbols doesn't hold. The new entropy model (2nd order entropy) has for each symbol a probability for each other symbol following it.
  – Example: {a,b,c}

		next		
		a	b	c
prev	a	.4	.2	.4
	b	.1	.9	0
	c	.1	.1	.8

Multiple Codes

		next		
		a	b	c
prev	a	.4	.2	.4
	b	.1	.9	0
	c	.1	.1	.8

Code for first symbol	
a	00
b	01
c	10



a b b a c c

<u>00</u> <u>00</u> <u>0</u> <u>1</u> <u>01</u> <u>0</u>

Complexity of Huffman Code Design

- Time to design Huffman Code is O(n log n) where n is the number of symbols.
  – Each step consists of a constant number of priority queue operations (2 deletemin's and 1 insert)

Approaches to Huffman Codes

1. Frequencies computed for each input
   – Must transmit the Huffman code or frequencies as well as the compressed input
   – Requires two passes
2. Fixed Huffman tree designed from training data
   – Do not have to transmit the Huffman tree because it is known to the decoder.
   – H.263 video coder
3. Adaptive Huffman code
   – One pass
   – Huffman tree changes as frequencies change