CSE 484 : Computer Security and Privacy

Asymmetric Cryptography

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Administrivia

- Lab 1 due on Friday
- HW2 out today, due two weeks from Friday (May 14)

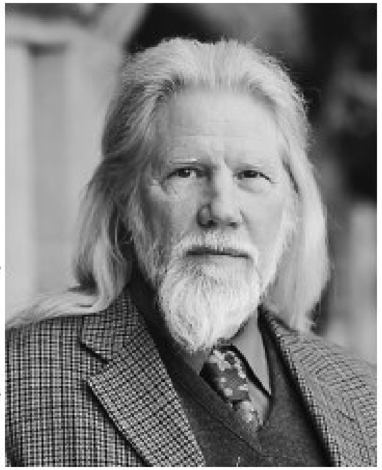
Session Key Establishment

Modular Arithmetic

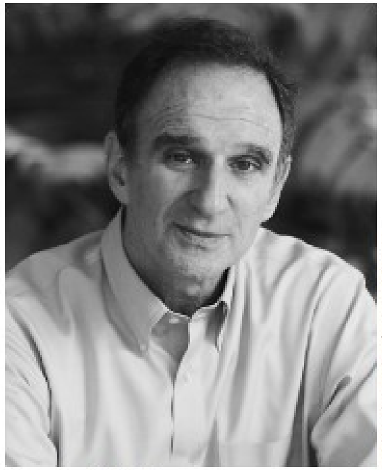
- Given g and prime p, compute: g^1 mod p, g^2 mod p, … g^{100} mod p
  - For p=11, g=10
    - 10^1 mod 11 = 10, 10^2 mod 11 = 1, 10^3 mod 11 = 10, …
    - Produces cyclic group {10, 1} (order=2)
  - For p=11, g=7
    - 7^1 mod 11 = 7, 7^2 mod 11 = 5, 7^3 mod 11 = 2, …
    - Produces cyclic group {7,5,2,3,10,4,6,9,8,1} (order = 10)
    - g=7 is a "generator" of Z_{11}^*

Diffie-Hellman Protocol (1976)

Diffie and Hellman Receive 2015 Turing Award



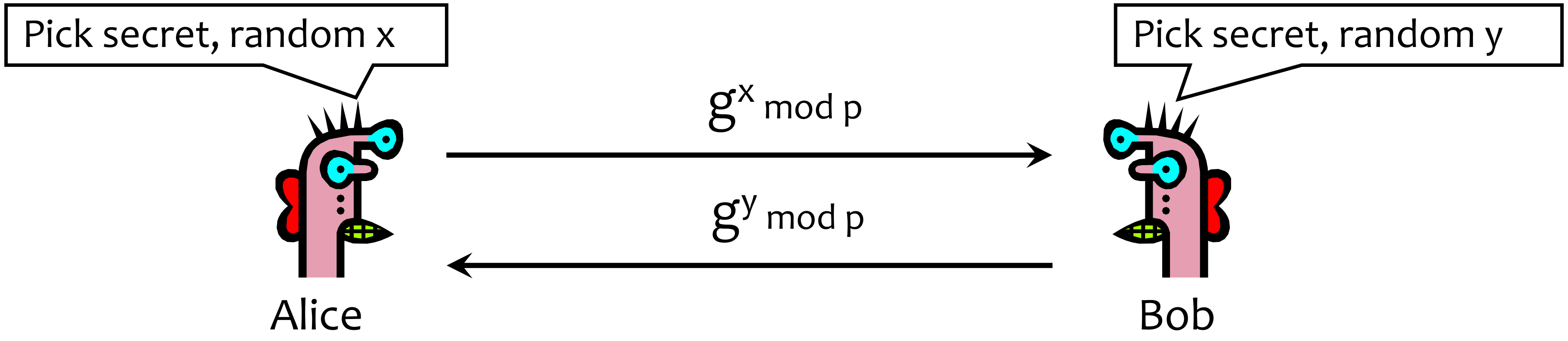
Whitfield Diffie



Martin E. Hellman

Diffie-Hellman Protocol (1976)

- Alice and Bob never met and share no secrets
- <u>Public</u> info: p and g
  - p is a large prime, g is a **generator** of Z_p*
    - Z_p*={1, 2 … p-1}; a Z_p* i such that a=g^i mod p
    - <u>Modular arithmetic</u>: numbers "wrap around" after they reach p



Compute k=$(g^y)^x$=g^{xy} mod p

Compute k=$(g^x)^y$=g^{xy} mod p

Example Diffie Hellman Computation

Why is Diffie-Hellman Secure?

- Discrete Logarithm (DL) problem:

  given g^x *mod p*, it's hard to extract x

  - There is no known efficient algorithm for doing this
  - This is not enough for Diffie-Hellman to be secure!

- Computational Diffie-Hellman (CDH) problem:

  given g^x and g^y, it's hard to compute g^{xy} *mod p*

  - … unless you know x or y, in which case it's easy

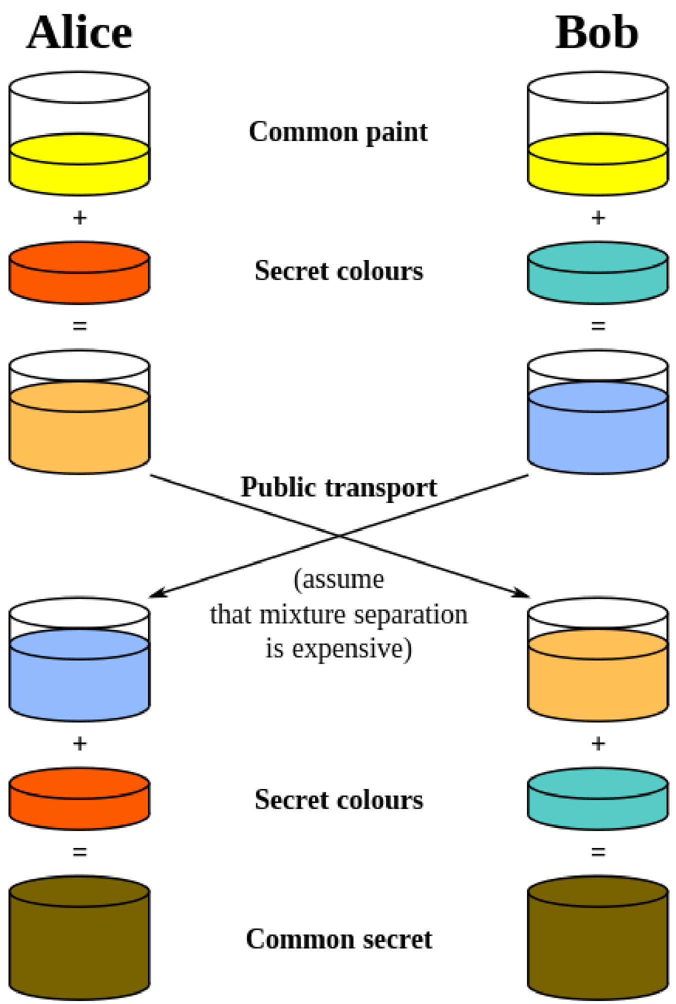
- Decisional Diffie-Hellman (DDH) problem:

  given g^x and g^y, it's hard to tell the difference between g^{xy} *mod p* and g^r *mod p* where r is random

More on Diffie-Hellman Key Exchange

- Important Note:
  - We have discussed discrete logs modulo integers
  - Significant advantages in using elliptic curve groups
    - Groups with some similar mathematical properties (i.e., are “groups”) but have better security and performance (size) properties

Diffie-Hellman: Conceptually



Common paint: p and g

Secret colors: x and y

Send over public transport:
$g^x \bmod p$
$g^y \bmod p$

Common secret: $g^{xy} \bmod p$

[from Wikipedia]

Diffie-Hellman Caveats

- Assuming DDH problem is hard (depends on choice of parameters!), Diffie-Hellman protocol is a secure key establishment protocol against passive attackers
  - Common recommendation:
    - Choose p=2q+1, where q is also a large prime
    - Choose g that generates a subgroup of order q in Z_p*
    - DDH is hard in this group
  - Eavesdropper can't tell the difference between the established key and a random value
  - In practice, often hash $g^{xy} \bmod p$, and use the hash as the key
  - Can use the new key for symmetric cryptography
- Diffie-Hellman protocol (by itself) does not provide authentication (against active attackers)
  - Person in the middle attack (also called "man in the middle attack")

Example from Earlier

- Given g and prime p, compute: $g^1 \bmod p$, $g^2 \bmod p$, … $g^{100} \bmod p$
  - For p=11, g=10
    - $10^1 \bmod 11 = 10$, $10^2 \bmod 11 = 1$, $10^3 \bmod 11 = 10$, …
    - Produces cyclic group {10, 1} (order=2)
  - For p=11, g=7
    - $7^1 \bmod 11 = 7$, $7^2 \bmod 11 = 5$, $7^3 \bmod 11 = 2$, …
    - Produces cyclic group {7,5,2,3,10,4,6,9,8,1} (order = 10)
    - g=7 is a "generator" of Z_{11}^*
  - For p=11, g=3
    - $3^1 \bmod 11 = 3$, $3^2 \bmod 11 = 9$, $3^3 \bmod 11 = 5$, …
    - Produces cyclic group {3,9,5,4,1} (order = 5) (5 is a prime)
    - g=3 generates a group of prime order

Stepping Back: Asymmetric Crypto

- We've just seen session key establishment
  - Can then use shared key for symmetric crypto
- Next: public key encryption
  - For confidentiality
- Then: digital signatures
  - For authenticity

Requirements for Public Key Encryption

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to learn anything about M from C without SK
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts

- Euler totient function $\varphi(n)$ $(n \geq 1)$ is the number of integers in the [1,n] interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p-1$
  - Note that $\varphi(ab) = \varphi(a)\,\varphi(b)$ if a & b are relatively prime

RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 2048 bits each (need primality testing, too)
  - Compute **n**=pq and **φ(n)**=(p-1)(q-1)
  - Choose small **e**, relatively prime to $\varphi(n)$
    - Typically, **e=3** or **$e=2^{16}+1=65537$**
  - Compute unique **d** such that $ed \equiv 1 \bmod \varphi(n)$
    - Modular inverse: $d \equiv e^{-1} \bmod \varphi(n)$
  - Public key = (e,n); private key = (d,n)
- Encryption of m: $c = m^e \bmod n$
- Decryption of c: $c^d \bmod n = (m^e)^d \bmod n = m$

How to compute?

Why is RSA Secure?

- RSA problem: given c, n=pq, and e such that $\gcd(e, \varphi(n))=1$, find m such that $m^e = c \bmod n$
  - In other words, recover m from ciphertext c and public key (n,e) by taking e^{th} root of c modulo n
  - There is no known efficient algorithm for doing this *without* knowing p and q
- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}$

- If factoring is easy, then RSA problem is easy (knowing factors means you can compute d = inverse of e mod (p-1)(q-1))
  - It may be possible to break RSA without factoring n -- but if it is, we don't know how

RSA Encryption Caveats

- Encrypted message needs to be interpreted as an integer less than n
- Don't use RSA **directly** for privacy – output is deterministic! Need to pre-process input somehow
- Plain RSA also does not provide integrity
  - Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting M, encrypt

$M \oplus G(r) \,||\, r \oplus H(M \oplus G(r))$

- r is random and fresh, G and H are hash functions

RSA OAEP

$M \oplus G(r) \,||\, r \oplus H(M \oplus G(r))$