CSE 484 / CSE M 584: Computer Security and Privacy

Cryptography
[Finish Asymmetric Cryptography]

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Thanks to Dan Boneh, Dieter Gollmann, Dan Halperin, Yoshi Kohno, Ada Lerner, John Manferdelli, John Mitchell, Vitaly Shmatikov, Bennet Yee, and many others for sample slides and materials ...

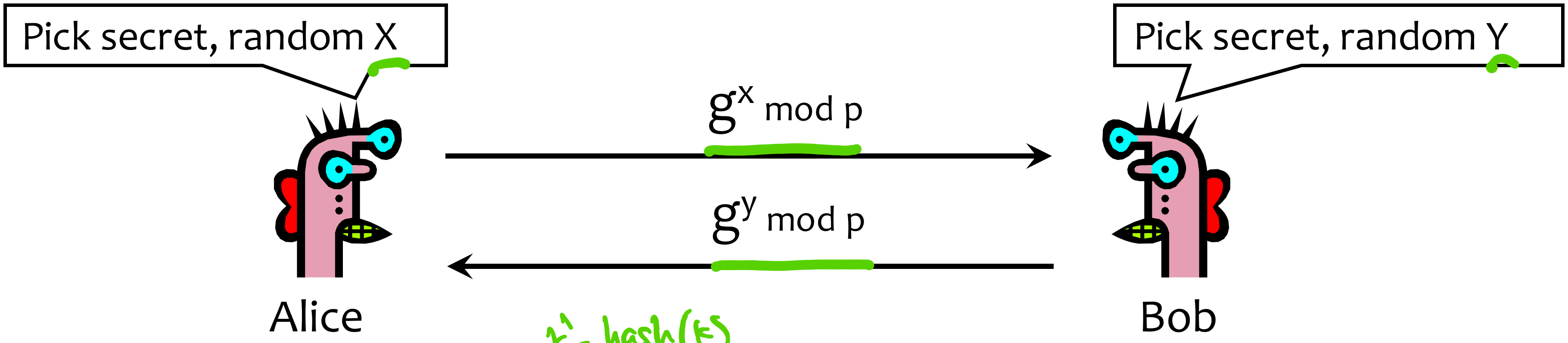
Admin

- Last day of crypto; then web security

- Want more crypto?
  - CSE 490C (Rachel Lin): https://courses.cs.washington.edu/courses/cse490c/19au/
  - Stanford Coursera (Dan Boneh): https://www.coursera.org/learn/crypto

Diffie-Hellman Key Exchange

- Alice and Bob never met and share no secrets
- Public info: p and g
  - p is a large prime, g is a **generator** of Z_p*
    - Z_p*={1, 2 … p-1};   a Z_p*   i such that $a=g^i$ mod p
    - Modular arithmetic: numbers “wrap around” after they reach p



$k' = \text{hash}(k)$

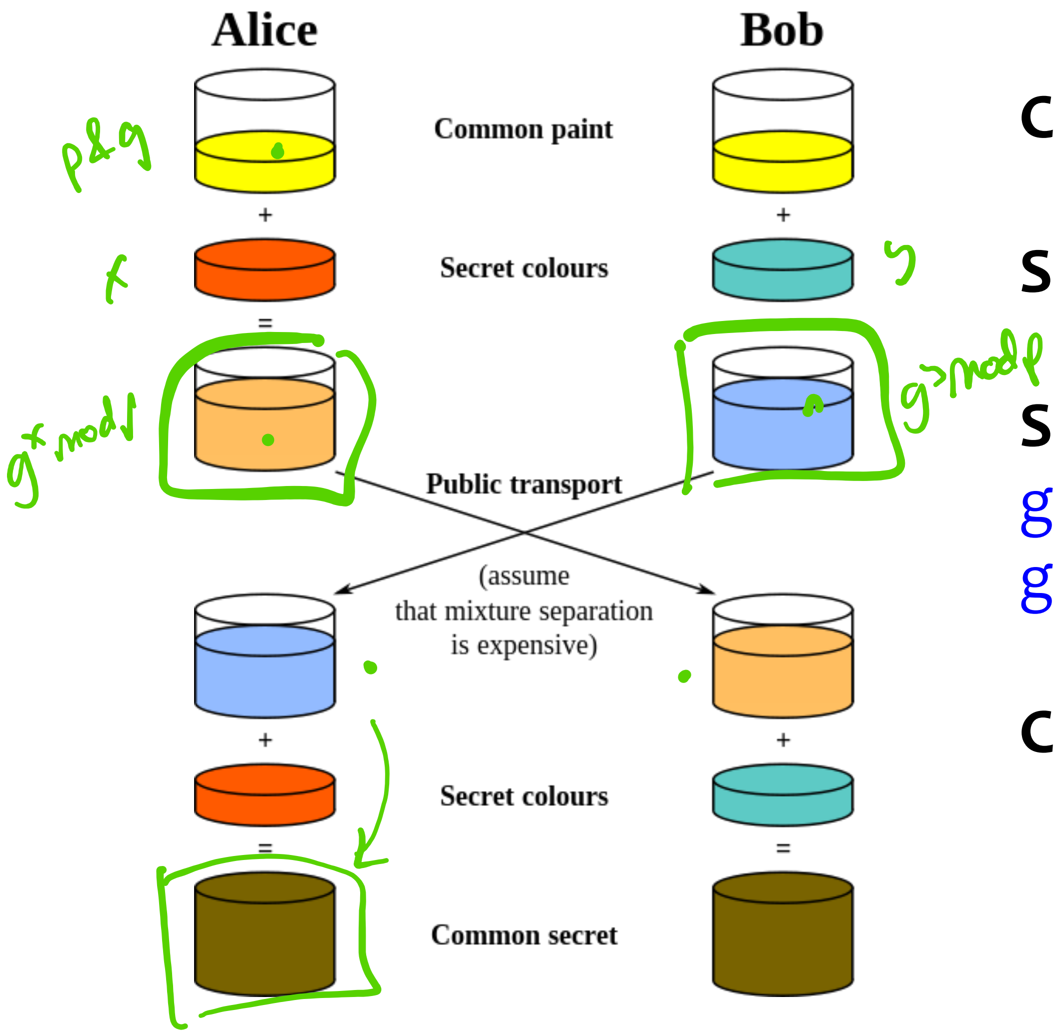
Compute k=$(g^y)^x$=g^{xy} mod p $= k$

mod p

Compute k=$(g^x)^y$=g^{xy} mod p

mod p

Diffie-Hellman: Conceptually



Common paint: p and g

Secret colors: x and y

Send over public transport:
g^x mod p
g^y mod p

Common secret: g^{xy} mod p

[from Wikipedia]

Why is Diffie-Hellman Secure?

- Discrete Logarithm (DL) problem:
  given $\underline{g^x \bmod p}$, it's hard to extract $\underline{x}$
  - There is no known efficient algorithm for doing this
  - This is not enough for Diffie-Hellman to be secure!
- Computational Diffie-Hellman (CDH) problem:
  given $\underline{g^x}$ and $\underline{g^y}$, it's hard to compute $\underline{g^{xy} \bmod p}$ = K
  - … unless you know x or y, in which case it's easy
- Decisional Diffie-Hellman (DDH) problem:
  given $\underline{g^x}$ and $\underline{g^y}$, it's hard to tell the difference between
  $\underbrace{g^{xy} \bmod p}_{K}$ and $\underbrace{g^r \bmod p}_{}$ where r is random

Properties of Diffie-Hellman

- Assuming DDH problem is hard (depends on choice of parameters!), Diffie-Hellman protocol is a secure key establishment protocol against passive attackers
  - Common recommendation:
    - Choose p=2q+1, where q is also a large prime
    - Choose g that generates a subgroup of order q in Z_p*
  - Eavesdropper can't tell the difference between the established key and a random value
  - In practice, often hash g^{xy} *mod* p, and use the hash as the key
  - Can use the new key for symmetric cryptography
- Diffie-Hellman protocol (by itself) does not provide authentication (against active attackers)
  - Person in the middle attack (also called "man in the middle attack")

MITM

Person In The Middle Attack

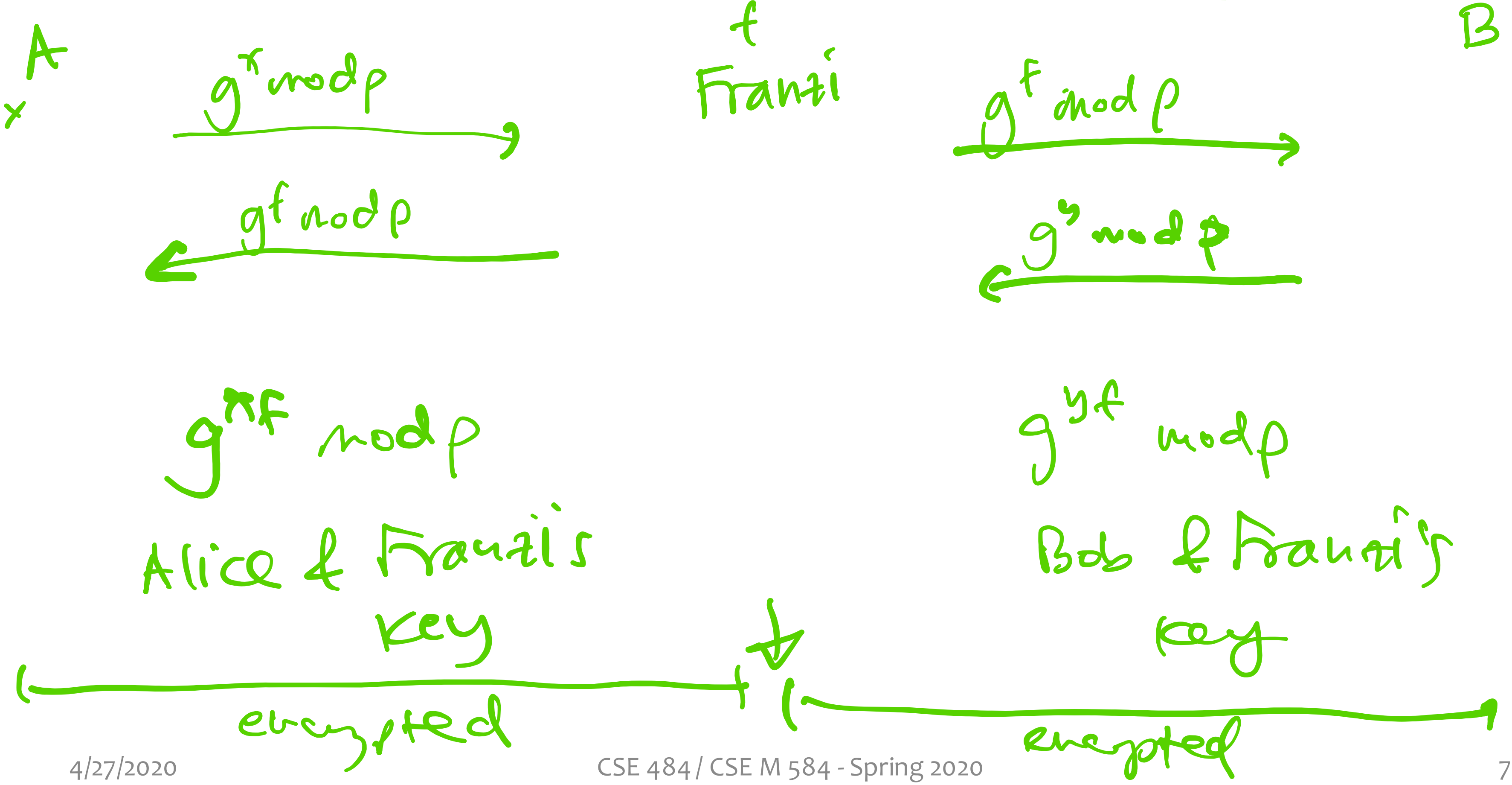
g, p

target:

HTTP - plaintext

HTTPS - encrypted

... key dist?



More on Diffie-Hellman Key Exchange

$g^{x} \bmod p$

- Important Note:
  - We have discussed discrete logs modulo integers
  - Significant advantages in using elliptic curve groups
    - Groups with some similar mathematical properties (i.e., are "groups") but have better security and performance (size) properties

Public Key Encryption

Requirements for Public Key Encryption

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to learn anything about M from C without SK
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

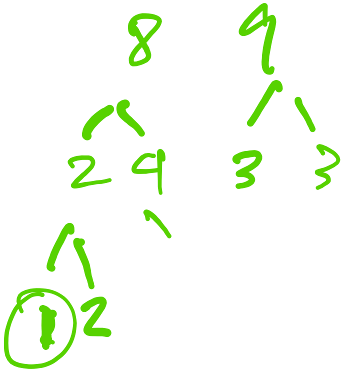
RSA

RSA

Some Number Theory Facts

phi $\phi(n)$

- Euler totient function $\varphi(n)$ $(n \geq 1)$ is the number of integers in the $[1,n]$ interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p-1$
  - Note that $\boxed{\varphi(ab) = \varphi(a)\,\varphi(b)}$



Important: Difficult to compute d unless you know $\varphi(n)$.

- extended Euclid. alg.
- wolfram alpha ☺
- brute force

RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q — secret
    - Say, 1024 bits each (need primality testing, too)
  - Compute **n**=pq and $\prod$(**n**)=(p-1)(q-1)
  - Choose small **e**, relatively prime to $\prod$(n)
    - Typically, **e=3** or **e=2^{16}+1=65537**
  - Compute unique **d** such that ed ≡ 1 mod $\prod$(n)
    - Modular inverse: d ≡ e^{-1} mod $\prod$(n)
  - Public key = (e,n); private key = (d,n)
- Encryption of m: c = m^e mod n
- Decryption of c: c^d mod n = $(m^e)^d$ mod n = m

How to compute?

$\varphi(p) = p-1$

$\varphi(pq) = \varphi(p)\varphi(q)$

Why is RSA Secure?

- RSA problem: given c, $n=\underline{pq}$, and $\overset{\text{public}}{e}$ such that $\gcd(e, \varphi(n))=1$, find m such that $m^{e}\underline{=c} \bmod n$
  - In other words, recover m from ciphertext c and public key (n,e) by taking eth root of c modulo n
  - There is no known efficient algorithm for doing this
- Factoring problem: given positive integer n, find primes $p_1, \dots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$

- If factoring is easy, then RSA problem is easy (knowing factors means you can compute $\underline{d}$ = inverse of e mod (p-1)(q-1))
  - It may be possible to break RSA without factoring n -- but if it is, we don't know how

Why RSA Decryption Works (FYI)

$e \cdot d = 1 \bmod \phi(n)$, thus $e \cdot d = 1 + k \cdot \phi(n)$ for some k

Let m be any integer in Z_n* (not all of Z_n)

$c^d \bmod n = (m^e)^d \bmod n = m^{1+k \cdot \varphi(n)} \bmod n$

$= (m \bmod n) * (m^{k \cdot \varphi(n)} \bmod n)$

Recall: Euler's theorem: if $a \in Z_n$*, then $a^{\phi(n)} = 1 \bmod n$

$c^d \bmod n = (m \bmod n) * (1 \bmod n)$

$= m \bmod n$

Proof omitted: True for all m in Z_n, not just m in Z_n*

Why RSA Decryption Works (FYI)

- Decryption of c: $c^d \bmod n = (m^e \bmod n)^d \bmod n = (m^e)^d \bmod n = m$
- Recall **n**=pq and φ(**n**)=(p-1)(q-1) and $ed \equiv 1 \bmod \varphi(n)$

- Chinese Remainder Theorem: To show $m^{ed} \bmod n \equiv m \bmod n$, sufficient to show:
  - $m^{ed} \bmod p \equiv m \bmod p$
  - $m^{ed} \bmod q \equiv m \bmod q$

- If $m \equiv 0 \bmod p$ → $m^{ed} \equiv 0 \bmod p$

- Else $m^{ed} = m^{ed-1}m = m^{k(q-1)(p-1)}m = m^{h(p-1)}m$ for some k, and h=k(q-1). Why? Recall how d was chosen and the definition of mod.
- Fermat Little Theorem: $m^{(p-1)h}\, m \equiv 1^h m \bmod p \equiv m \bmod p$

RSA Encryption Caveats

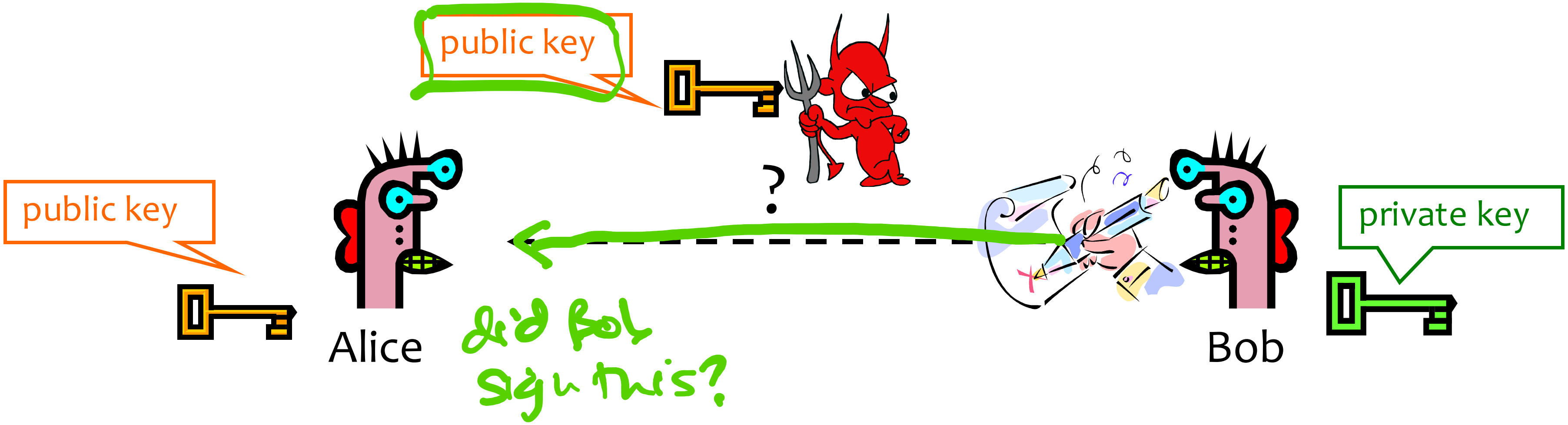
16 3 12 one character mod small n

- Encrypted message needs to be interpreted as an integer less than n $m^e \bmod n$ → smaller than n
- Don’t use RSA **directly** for privacy – output is deterministic! Need to pre-process input somehow
- Plain RSA also does not provide integrity
  - Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting M, encrypt M□G(r) ; r□H(M□G(r))

- r is random and fresh, G and H are hash functions

Digital Signatures: Basic Idea



Given: Everybody knows Bob's public key
Only Bob knows the corresponding private key

Goal: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, only the public key is needed

RSA Signatures

Btw, can also do sigs w/ discrete log based protocols

e(mod n)

(m, s)

- Public key is **(n,e)**, private key is **(n,d)**
- To sign message m: s = m^d mod n
  - Signing & decryption are same **underlying** operation in RSA
  - It's infeasible to compute **s** on **m** if you don't know **d**
- To verify signature s on message m:

  verify that s^e mod n = $(m^d)^e$ mod n = m
  - Just like encryption (for RSA primitive)
  - Anyone who knows **n** and **e** (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures

DSS Signatures

- Digital Signature Standard (DSS)
  - U.S. government standard (1991, most recent rev. 2013)
- Public key: (p, q, g, $y=g^x$ mod p), private key: x
- Security of DSS requires hardness of discrete log
  - If could solve discrete logarithm problem, would extract x (private key) from g^x mod p (public key)

- Again: We've discussed discrete logs modulo integers; significant advantages to using elliptic curve groups instead.

Cryptography Summary

- Goal: Privacy
  - Symmetric keys:
    - One-time pad, Stream ciphers
    - Block ciphers (e.g., DES, AES) → modes: EBC, CBC, CTR
  - Public key crypto (e.g., Diffie-Hellman, RSA)
- Goal: Integrity
  - MACs, often using hash functions (e.g, SHA-256)
- Goal: Privacy and Integrity
  - Encrypt-then-MAC
- Goal: Authenticity (and Integrity)
  - Digital signatures (e.g., RSA, DSS)