CSE 484 / CSE M 584: Computer Security and Privacy

Cryptography
[Finish Asymmetric Cryptography]

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Admin

• Last day of crypto; then web security

• Want more crypto?
  – CSE 490C (Rachel Lin):
    https://courses.cs.washington.edu/courses/cse490c/19au/
  – Stanford Coursera (Dan Boneh):
    https://www.coursera.org/learn/crypto
Diffie-Hellman Key Exchange

• Alice and Bob never met and share no secrets

• **Public info: p and g**
  – p is a large prime, g is a **generator** of $\mathbb{Z}_p^*$
    • $\mathbb{Z}_p^* = \{1, 2 \ldots p-1\}; \forall a \in \mathbb{Z}_p^* \ \exists i$ such that $a=g^i \mod p$
    • **Modular arithmetic:** numbers “wrap around” after they reach p

```
Alice
```

```
Bob
```

Pick secret, random X

Pick secret, random Y

$g^x \mod p$

$g^y \mod p$

Compute $k=(g^y)^x=g^{xy} \mod p$

Compute $k=(g^x)^y=g^{xy} \mod p$
**Diffie-Hellman: Conceptually**

- **Common paint:** \( p \) and \( g \)
- **Secret colors:** \( x \) and \( y \)
- **Send over public transport:**
  - \( g^x \mod p \)
  - \( g^y \mod p \)
- **Common secret:** \( g^{xy} \mod p \)

([from Wikipedia])
Why is Diffie-Hellman Secure?

• **Discrete Logarithm (DL) problem:**
  given \( g^x \mod p \), it’s hard to extract \( x \)
  – There is no known efficient algorithm for doing this
  – This is **not** enough for Diffie-Hellman to be secure!

• **Computational Diffie-Hellman (CDH) problem:**
  given \( g^x \) and \( g^y \), it’s hard to compute \( g^{xy} \mod p \)
  – … unless you know \( x \) or \( y \), in which case it’s easy

• **Decisional Diffie-Hellman (DDH) problem:**
  given \( g^x \) and \( g^y \), it’s hard to tell the difference between
  \( g^{xy} \mod p \) and \( g^r \mod p \) where \( r \) is random
Properties of Diffie-Hellman

• Assuming DDH problem is hard (depends on choice of parameters!), Diffie-Hellman protocol is a secure key establishment protocol against passive attackers
  – Common recommendation:
    • Choose \( p = 2q + 1 \), where \( q \) is also a large prime
    • Choose \( g \) that generates a subgroup of order \( q \) in \( \mathbb{Z}_p^* \)
  – Eavesdropper can’t tell the difference between the established key and a random value
  – In practice, often hash \( g^{xy} \mod p \), and use the hash as the key
  – Can use the new key for symmetric cryptography

• Diffie-Hellman protocol (by itself) does not provide authentication (against active attackers)
  – Person in the middle attack (also called “man in the middle attack”)

Person In The Middle Attack
More on Diffie-Hellman Key Exchange

• Important Note:
  – We have discussed discrete logs modulo integers
  – Significant advantages in using elliptic curve groups
    • Groups with some similar mathematical properties (i.e., are “groups”) but have better security and performance (size) properties
Public Key Encryption
Requirements for Public Key Encryption

• **Key generation:** computationally easy to generate a pair (public key $PK$, private key $SK$)

• **Encryption:** given plaintext $M$ and public key $PK$, easy to compute ciphertext $C = E_{PK}(M)$

• **Decryption:** given ciphertext $C = E_{PK}(M)$ and private key $SK$, easy to compute plaintext $M$
  – Infeasible to learn anything about $M$ from $C$ without $SK$
  – Trapdoor function: $\text{Decrypt}(SK, E_{PK}(M)) = M$
Some Number Theory Facts

- Euler totient function $\varphi(n)$ ($n \geq 1$) is the number of integers in the $[1,n]$ interval that are relatively prime to $n$
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p - 1$
  - Note that $\varphi(ab) = \varphi(a) \varphi(b)$
RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

• Key generation:
  – Generate large primes p, q
    • Say, 1024 bits each (need primality testing, too)
  – Compute n=pq and \( \phi(n) = (p-1)(q-1) \)
  – Choose small e, relatively prime to \( \phi(n) \)
    • Typically, \( e=3 \) or \( e=2^{16}+1=65537 \)
  – Compute unique d such that \( ed \equiv 1 \mod \phi(n) \)
    • Modular inverse: \( d \equiv e^{-1} \mod \phi(n) \)
  – Public key = (e,n); private key = (d,n)

• Encryption of m: \( c = m^e \mod n \)
• Decryption of c: \( c^d \mod n = (m^e)^d \mod n = m \)
Why is RSA Secure?

• **RSA problem:** given $c$, $n=pq$, and $e$ such that $\gcd(e, \varphi(n))=1$, find $m$ such that $m^e = c \mod n$
  
  – In other words, recover $m$ from ciphertext $c$ and public key $(n,e)$ by taking $e$th root of $c$ modulo $n$
  
  – There is no known efficient algorithm for doing this

• **Factoring problem:** given positive integer $n$, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$

• If factoring is easy, then RSA problem is easy (knowing factors means you can compute $d=\text{inverse of } e \mod (p-1)(q-1)$)
  
  – It may be possible to break RSA without factoring $n$ – but if it is, we don’t know how
RSA Encryption Caveats

• Encrypted message needs to be interpreted as an integer less than n

• Don’t use RSA directly for privacy – output is deterministic! Need to pre-process input somehow

• Plain RSA also does not provide integrity
  – Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting M, encrypt $M \oplus G(r) \oplus H(M \oplus G(r))$
  – r is random and fresh, G and H are hash functions
Digital Signatures: Basic Idea

Given: Everybody knows Bob’s public key
Only Bob knows the corresponding private key

Goal: Bob sends a “digitally signed” message

1. To compute a signature, must know the private key
2. To verify a signature, only the public key is needed
RSA Signatures

• Public key is \((n,e)\), private key is \((n,d)\)
• To sign message \(m\): \(s = m^d \mod n\)
  – Signing & decryption are same underlying operation in RSA
  – It’s infeasible to compute \(s\) on \(m\) if you don’t know \(d\)
• To verify signature \(s\) on message \(m\):
  verify that \(s^e \mod n = (m^d)^e \mod n = m\)
  – Just like encryption (for RSA primitive)
  – Anyone who knows \(n\) and \(e\) (public key) can verify signatures produced with \(d\) (private key)
• In practice, also need padding & hashing
  – Standard padding/hashing schemes exist for RSA signatures
DSS Signatures

• Digital Signature Standard (DSS)
• Public key: \((p, q, g, y=g^x \text{ mod } p)\), private key: \(x\)
• Security of DSS requires hardness of discrete log
  – If could solve discrete logarithm problem, would extract \(x\) (private key) from \(g^x \text{ mod } p\) (public key)

• Again: We’ve discussed discrete logs modulo integers; significant advantages to using elliptic curve groups instead.
Cryptography Summary

• Goal: **Privacy**
  – Symmetric keys:
    • One-time pad, Stream ciphers
    • Block ciphers (e.g., DES, AES) → modes: EBC, CBC, CTR
  – Public key crypto (e.g., Diffie-Hellman, RSA)

• Goal: **Integrity**
  – MACs, often using hash functions (e.g., SHA-256)

• Goal: **Privacy and Integrity**
  – Encrypt-then-MAC

• Goal: **Authenticity (and Integrity)**
  – Digital signatures (e.g., RSA, DSS)