CSE 484 / CSE M 584: Computer Security and Privacy

Cryptography
[Asymmetric Cryptography]

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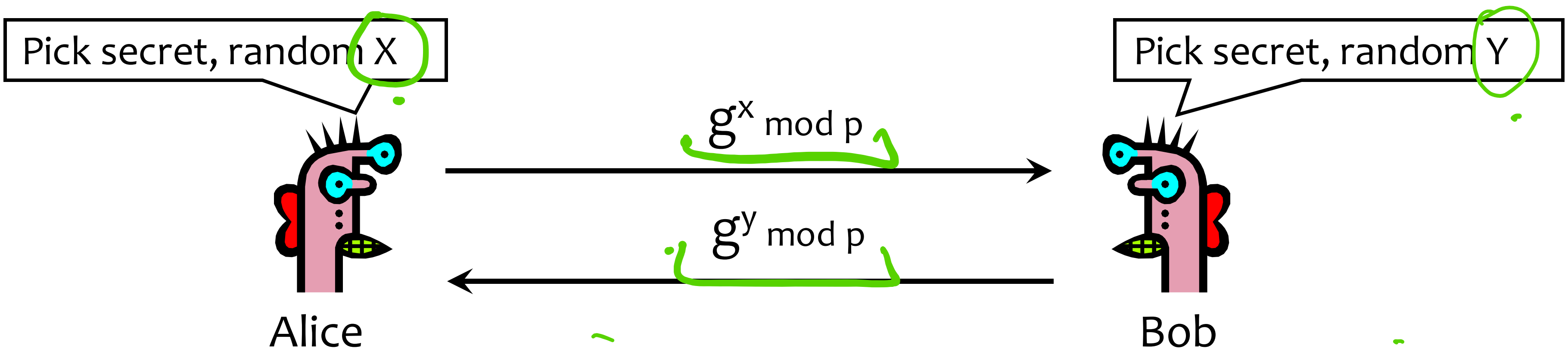
Diffie-Hellman Key Exchange

- Alice and Bob never met and share no secrets
- Public info: p and g
  - p is a large prime, g is a **generator** of Z_p^*
    - $Z_p^*=\{1, 2 \ldots p\text{-}1\}$; a Z_p^* i such that $a=g^i$ mod p
    - Modular arithmetic: numbers "wrap around" after they reach p



Compute $k=(g^y)^x=g^{xy}$ mod p

Compute $k=(g^x)^y=g^{xy}$ mod p

Example Diffie Hellman Computation

public

$\boxed{p = 11 \quad g = 2}$

Alice

$x = 9$

Bob

$y = 4$

$g^x \bmod p$

$2^9 \bmod 11$

$\boxed{6}^4 \bmod 11$

$= 9 = K$

$g^y \bmod p$

$2^4 \bmod 11$

$\boxed{5}^9 \bmod 11$

$= 9$

$K = hash(9)$

$g^{xy} \bmod p$

$2^{9 \cdot 4} \bmod 11 = 9$

$hash(9) = K$

abc123 = AES key

Why is Diffie-Hellman Secure?

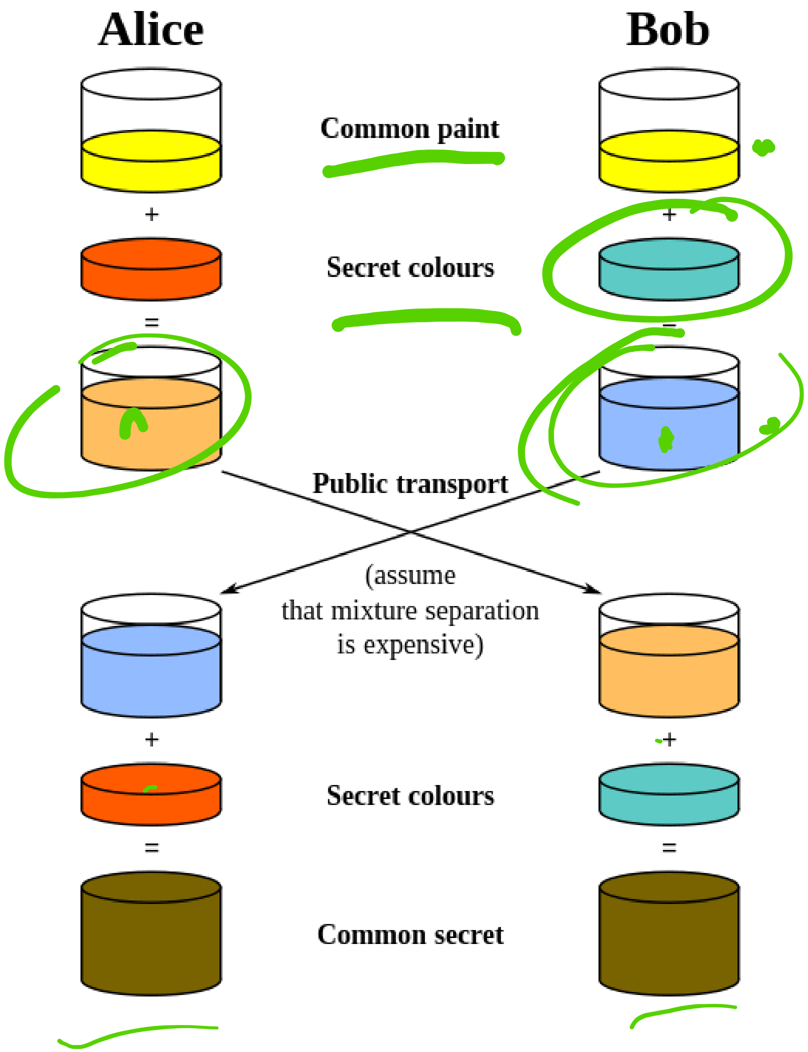
$5 = 2^{9} \bmod 11$

- Discrete Logarithm (DL) problem:
  given $g^x \bmod p$, it's hard to extract x
  - There is no known <u>efficient</u> algorithm for doing this
  - This is <u>not</u> enough for Diffie-Hellman to be secure!
- Computational Diffie-Hellman (CDH) problem:
  given g^x and g^y, it's hard to compute $g^{xy} \bmod p$

  5 6
  - … unless you know x or y, in which case it's easy
- Decisional Diffie-Hellman (DDH) problem:
  given g^x and g^y, it's hard to tell the difference between $g^{xy} \bmod p$ and $g^r \bmod p$ where r is random

"looks random"

Diffie-Hellman: Conceptually



Common paint: p and g

Secret colors: x and y

Send over public transport:
g^x mod p
g^y mod p

Common secret: g^{xy} mod p

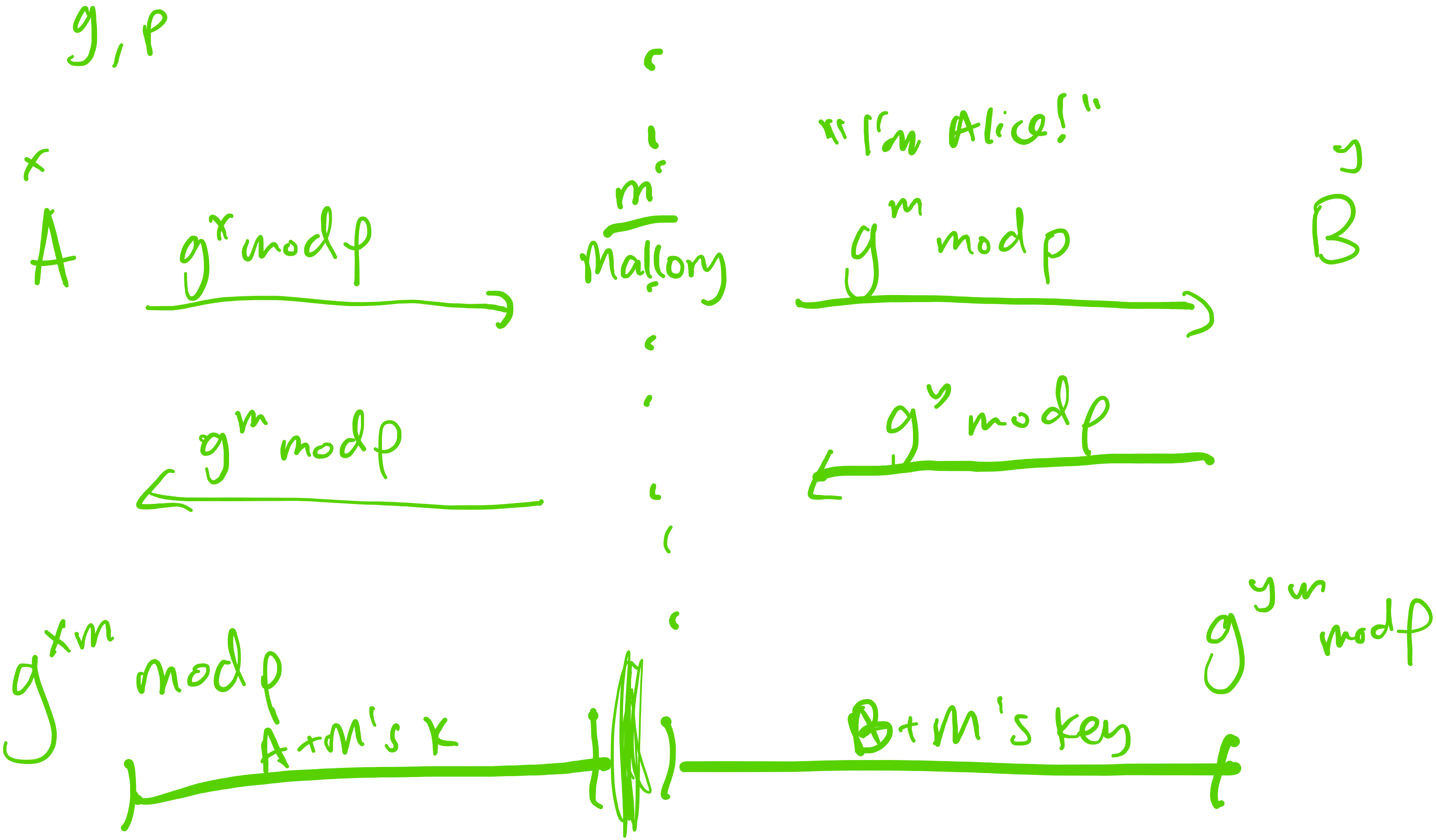
[from Wikipedia]

Diffie-Hellman Caveats

- Assuming DDH problem is hard (depends on choice of parameters!), Diffie-Hellman protocol is a secure key establishment protocol against passive attackers
  - Common recommendation:
    - Choose p=2q+1, where q is also a large prime
    - Choose g that generates a subgroup of order q in Z_p*
  - Eavesdropper can't tell the difference between the established key and a random value
  - In practice, often hash g^{xy} *mod p*, and use the hash as the key
  - Can use the new key for symmetric cryptography
- Diffie-Hellman protocol (by itself) does not provide authentication (against active attackers)
  - Person in the middle attack (also called "man in the middle attack")

"MITM"

Person In The Middle Attack



More on Diffie-Hellman Key Exchange

- Important Note:
  - We have discussed discrete logs modulo integers
  - Significant advantages in using elliptic curve groups
    - Groups with some similar mathematical properties (i.e., are "groups") but have better security and performance (size) properties

Stepping Back: Asymmetric Crypto

- We've just seen session key establishment
  - Can then use shared key for symmetric crypto
- Next: public key encryption
  - For confidentiality
- Then: digital signatures
  - For authenticity

block ciphers (AES-CBC)

hash | MAC

A

A_{pub}, A_{priv}

Encrypt(msg, B_{pub}) = C

B

$\boxed{B_{pub}}$, B_{priv}

Decrypt(C, B_{priv})

Requirements for Public Key Encryption

- Key generation: computationally easy to generate a pair (public key PK, private key SK)
- Encryption: given plaintext M and public key PK, easy to compute ciphertext $C=E_{PK}(M)$
- Decryption: given ciphertext $C=E_{PK}(M)$ and private key SK, easy to compute plaintext M
  - Infeasible to learn anything about M from C without SK
  - Trapdoor function: Decrypt(SK,Encrypt(PK,M))=M

Some Number Theory Facts

- Euler totient function $\overset{\text{phi}}{\varphi(n)}$ ($n \geq 1$) is the number of integers in the [1,n] interval that are relatively prime to n
  - Two numbers are relatively prime if their greatest common divisor (gcd) is 1
  - Easy to compute for primes: $\varphi(p) = p-1$
  - Note that $\varphi(ab) = \varphi(a)\,\varphi(b)$ if a & b are relatively prime

p, q primes

$\varphi(pq) = \varphi(p)\,\varphi(q)$

RSA Cryptosystem [Rivest, Shamir, Adleman 1977]

- Key generation:
  - Generate large primes p, q
    - Say, 1024 bits each (need primality testing, too)
  - Compute **n**=pq and Π(**n**)=(p-1)(q-1) $= \varphi(pq) = \varphi(p)\varphi(q)$
  - Choose small **e**, relatively prime to Π(n)
    - Typically, **e=3** or **e=2^{16}+1=65537**
  - Compute unique **d** such that ed ≡ 1 mod Π(n)
    - Modular inverse: d ≡ e^{-1} mod Π(n)
  - Public key = (e,n); private key = (d,n)
- Encryption of m: c = m^e mod n
- Decryption of c: c^d mod n = $(m^e)^d$ mod n = m

(p-1)(q-1)

How to compute?

brute force
d=1

- extended Euclidean alg

- Wolfram Alpha

Why RSA Decryption Works (FYI)

$e*d=1 \bmod phi(n)$, thus $e*d=1+k*phi(n)$ for some k

Let m be any integer in Z_n* (not all of Z_n)

$c^d \bmod n = (m^e)^d \bmod n = m^{1+k\cdot\phi(n)} \bmod n$

$= (m \bmod n) * (m^{k\cdot\phi(n)} \bmod n)$

Recall: Euler's thm: if a in Z_n*, then $a^{phi(n)}=1 \bmod n$

$c^d \bmod n = (m \bmod n) * (1 \bmod n)$

$= m \bmod n$

Proof omitted: True for all m in Z_n, not just m in Z_n*

Why RSA Decryption Works (FYI)

- Decryption of c: $c^d \bmod n = (m^e \bmod n)^d \bmod n = (m^e)^d \bmod n = m$
- Recall $\mathbf{n}=pq$ and $\varphi(\mathbf{n})=(p-1)(q-1)$ and $ed \equiv 1 \bmod \varphi(n)$

- Chinese Remainder Theorem: To show $m^{ed} \bmod n \equiv m \bmod n$, sufficient to show:
  - $m^{ed} \bmod p \equiv m \bmod p$
  - $m^{ed} \bmod q \equiv m \bmod q$

- If $m \equiv 0 \bmod p \rightarrow m^{ed} \equiv 0 \bmod p$

- Else $m^{ed} = m^{ed-1}m = m^{k(q-1)(p-1)}m = m^{h(p-1)}m$ for some k, and h=k(q-1). Why? Recall how d was chosen and the definition of mod.
- Fermat Little Theorem: $m^{(p-1)h}m \equiv 1^h m \bmod p \equiv m \bmod p$

Why is RSA Secure?

$ed \equiv 1 \bmod \varphi(n)$

- RSA problem: given c, n=~~pq~~, and e such that $\gcd(e, \varphi(n))=1$, find m such that $m^e=c \bmod n$
  - In other words, recover m from ciphertext c and public key (n,e) by taking e^{th} root of c modulo n
  - There is no known efficient algorithm for doing this

$\varphi(n) = \varphi(p)\varphi(q)$

- Factoring problem: given positive integer n, find primes $p_1, \ldots, p_k$ such that $n=p_1^{e_1}p_2^{e_2}\ldots p_k^{e_k}$

- If factoring is easy, then RSA problem is easy (knowing factors means you can compute d = inverse of e mod (p-1)(q-1))
  - It may be possible to break RSA without factoring n -- but if it is, we don't know how

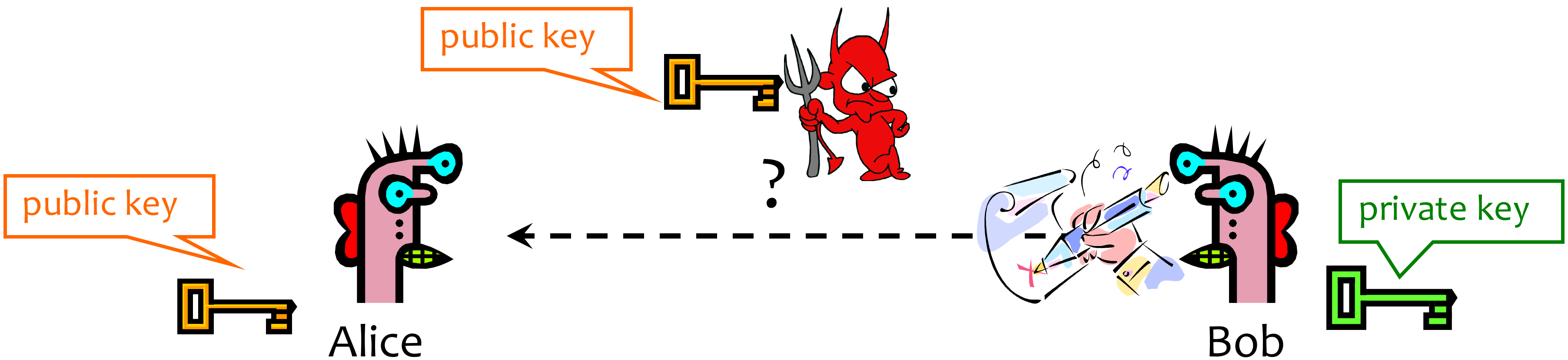
msg tag
msg' tag'
MAC: hash(msg, key)

RSA Encryption Caveats

- Encrypted message needs to be interpreted as an integer less than n
- Don’t use RSA **directly** for privacy – output is deterministic! Need to pre-process input somehow
- Plain RSA also does not provide integrity
  - Can tamper with encrypted messages

In practice, OAEP is used: instead of encrypting M, encrypt M□G(r) ; r□H(M□G(r))
- r is random and fresh, G and H are hash functions

Digital Signatures: Basic Idea



Given: Everybody knows Bob's public key
Only Bob knows the corresponding private key

Goal: Bob sends a "digitally signed" message

1. To compute a signature, must know the private key
2. To verify a signature, only the public key is needed

RSA Signatures

- Public key is **(n,e)**, private key is **(n,d)**
- To sign message m: $s = m^d \bmod n$
  - Signing & decryption are same **underlying** operation in RSA
  - It's infeasible to compute **s** on **m** if you don't know **d**
- To verify signature s on message m:

  verify that $s^e \bmod n = (m^d)^e \bmod n = m$
  - Just like encryption (for RSA primitive)
  - Anyone who knows **n** and **e** (public key) can verify signatures produced with d (private key)
- In practice, also need padding & hashing
  - Standard padding/hashing schemes exist for RSA signatures

DSS Signatures

- Digital Signature Standard (DSS)
  - U.S. government standard (1991, most recent rev. 2013)
- Public key: $(p, q, g, y=g^x \bmod p)$, private key: x
- Security of DSS requires hardness of discrete log
  - If could solve discrete logarithm problem, would extract x (private key) from $g^x \bmod p$ (public key)

- Again: We've discussed discrete logs modulo integers; significant advantages to using elliptic curve groups instead.

Cryptography Summary

- Goal: Privacy
  - Symmetric keys:
    - One-time pad, Stream ciphers
    - Block ciphers (e.g., DES, AES) → modes: EBC, CBC, CTR
  - Public key crypto (e.g., Diffie-Hellman, RSA)
- Goal: Integrity
  - MACs, often using hash functions (e.g, SHA-256)
- Goal: Privacy and Integrity
  - Encrypt-then-MAC
- Goal: Authenticity (and Integrity)
  - Digital signatures (e.g., RSA, DSS)

Want More Crypto?

- Some suggestions:
  - CSE 490C (Rachel Lin):
    https://courses.cs.washington.edu/courses/cse490c/20au/
  - Stanford Coursera (Dan Boneh):
    https://www.coursera.org/learn/crypto